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ON THE NUMERICAL SOLUTION OF THE MULTIDIMENSIONAL SINGULAR INTEGRALS AND INTEGRAL EQUATIONS, USED IN THE THEORY OF LINEAR VISCOELASTICITY

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ABSTRACT. In the present report, we investigate the formulation, for the numerical evaluation of the multidimensional singular integrals and integral equations, used in the theory of linear viscoelasticity. Some simple formulas are given for the numerical solution of the general case of the multidimensional singular integrals. Moreover a numerical technique is also established for the numerical solution of some special cases of the multidimensional singular integrals like the two - and three - dimensional singular integrals. An application is given to the determination of the fracture behaviour of a thick, hollow circular cylinder of viscoelastic material restrained by an enclosing thin elastic ring and subjected to a uniform pressure.

KEYS WORDS AND PHRASES. Numerical integration, Integration interval, Multidimensional singular integrals and integral equations, Linear Viscoelasticity, 2-D and 3-D singular integrals.
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1. INTRODUCTION.

In the present report we consider the multidimensional singular integrals and integral equations, with many applications in the theory of elasticity, plasticity and viscoelasticity. Tricomi [1], [2] has written the first important work on multi-dimensional singular integrals and investigated double singular integrals of the following form:

$$I(x_0,y_0) = \int_S w(x,y) \frac{f(x_0,y_0,\theta)}{r^2} u(x,y)\,ds \tag{1.1}$$

where
$$(x-x_0) + i(y-y_0) = r\,e^{i\theta} \tag{1.2}$$

and $w(x,y)$ is the weight function for various quadrature rules.

If the density function $u(x_0,y_0)$ in eqn. (1.1) is a bounded and Hölder - continuous function in s and also if the characteristic function $f(x_0,y_0,\theta)$ is bounded and for a fixed pole $X(x_0,y_0)$ is continuous with respect to θ, then according to

Tricomi, the necessary and sufficient condition for the existence of the singular integral I in the principal value sense, is that its characteristic should satisfy the following condition:

$$\int_0^{2\pi} f(x,y,\theta)\, d\theta = 0 \tag{1.3}$$

The next important work on multi-dimensional singular integrals was done by Giraud [3] - [5]. He investigated integrals taken over a closed Liapounov manifold Γ of any dimension m. This manifold is broken up into a finite number of mutually overlapping parts, each one of which has a one-to-one mapping on a region of an m-dimensional Euclidean space.

He investigated singular integrals of the following form:

$$\int_\Gamma K(x,y)\, u(y)\, d\Gamma_y \tag{1.4}$$

where the function u(y) satisfies a Lipschitz condition with positive index.

Giraud also studied the following singular integral equation:

$$u(x) - \mu \int_\Gamma K(x,y)\, u(y)\, d\Gamma_y = f(x) \tag{1.5}$$

with a kernel, the singular part $K_1(x,y)$ of which has the following special form:

$$K_1(x,y) = \sum_{a=1}^{m} C_a(x)\ (x_a-y_a) \left[\sum_{\beta,\gamma=1}^{m} A_{\beta\gamma}(x_\beta-y_\beta)\ (x_\gamma-y_\gamma) \right]^{\frac{-m+1}{2}} \tag{1.6}$$

where $C_a(x)$ are certain given functions.

If we operate on both sides of eqn. (1.5) with:

$$u(x) + \mu \int_\Gamma H(x,y;\mu)\, u(y)\, d\Gamma_y \tag{1.7}$$

where $H(x,y;\mu)$ is any singular kernel then we shall get the following equation:

$$\begin{aligned} &[I + \mu^2\ \phi(x,\mu)]\, u(x) + \mu \int_\Gamma [H(x,y;\mu) - K(x,y) \\ &- \mu \int_\Gamma H(x,z;\mu)\ K(z,y)\ d\Gamma_z]\ u(y)\ d\Gamma_y = \\ &= f(x) + \mu \int_\Gamma H(x,y;\mu)\ f(y)\ d\Gamma_y \end{aligned} \tag{1.8}$$

where $\phi(x,\mu)$ is a function, completely determined by the kernels K and H and the manifold Γ.

Further investigations were done by Mikhlin [6] - [9] who proved that a singular operator of the type:

$$a\, u(x) + \int_{E_m} \frac{f(\theta)}{r^m}\, u(y)\, dy \tag{1.9}$$

where: $r = |y - x|$ and: $\theta = \frac{y - x}{r}$ (1.10ab)

(E_m is an Euclidean space of m dimensions)

is bounded in the Hilbert - Euclidean space $L_2(E_m)$ if the symbol of the operator is bounded and its norm doesn't exceed the maximum of the modulus of the symbol.

Furthermore, Mikhlin has studied the following integral equation:

$$a(x)\ u(x) + \int_{E_m} \frac{f(x,\theta)}{r^m}\ u(y)\ dy \tag{1.11}$$

If the symbol satisfies certain demands with regard to smoothness, then a multi-dimensional singular integral equation permits regularization, if and only if, the modulus of its symbol has a positive lower band. The theorem about the regularization can be extended also to systems of singular equations and likewise for the case where the singular integral entering the equation is taken over any closed Liapounov manifold.

The basic problem studied by Calderon and Zygmund [10] - [13] are singular integrals of the form:

$$\int_{E_m} \frac{f(\theta)}{r^m}\ u(y)\ dy \tag{1.12}$$

They investigated integral (1.12) in the Lebesgue-Euclidean spaces $L_p(E_m)$ where $1 < p < \infty$, $p \neq 2$. They proved that the integral (1.12) is bounded in $L_p(E_m)$ if $f(\theta)$ satisfies the Dini condition:

$$\int_0^1 \frac{\omega(t)}{t}\ dt < \infty \tag{1.13}$$

where $\omega(t)$ is the modulus of continuity of the characteristic $f(\theta)$. Moreover Calderon and Zygmund investigated the compounding of singular operators of the type:

$$K\ u = a\ u(x) + \int_{E_m} \frac{f(\theta)}{r^m}\ u(y)\ dy \tag{1.14}$$

Over the past years some papers have been published by using singular integral equation methods in elasticity, plasticity and fracture mechanics theory for isotropic and anisotropic solids [14] - [33].

On the other hand, some scientists have studied problems of the classical theory of viscoelasticity, followed classical lines [34] - [39], while Rizzo and Shippy [40] have used the Boundary Integral Equation Method (B.I.E.M.). In this report the Singular Integral Operators Method (S.I.O.M.) which has been used by the present author to the solution of elasticity and plasticity problems [21], [22] shall be extended to the solution of viscoelasticity problems.

2. INTRODUCTORY FORMULAE OF THE MULTIDIMENSIONAL SINGULAR INTEGRALS.

Let us consider the following multidimensional singular integral: [6]-[9]

$$I(x) = \int_{E_m} \frac{f(x,\theta)}{r^m} u(y)\, dy \tag{2.1}$$

where x,y are points in the space E_m and:

$$r = |y - x| \quad \text{and} \quad \theta = \frac{y - x}{r} \tag{2.2ab}$$

Furthermore, the point x is the pole, the function $f(x,\theta)$ the characteristic, and the function $u(y)$ the density of the singular integral (2.1).

Thus, let us consider the following assumptions:

(a) In any bounded part of the space E_m the density $u(x) \varepsilon$ Lip a, a > 0

(b) at infinity $u(x) = 0\,(|x|^{-k})$, $k > 0$

(c) The characteristic is bounded and for a fixed pole x is continuous with respect to θ.

If these three assumptions are valid, then according to Mikhlin [6], [7] for the existence of the singular integral (2.1) it is necessary and sufficient that:

$$\int_{s} f(x,\theta)\, ds = 0 \tag{2.3}$$

where S is the unit sphere over which θ moves.

Thus, from (2.1) we have the following formula:

$$\int_{E_m} \frac{f(x,\theta)}{r^m} u(y)\, dy = \int_{r>1} \frac{f(x,\theta)}{r^m} u(y)\, dy$$

$$+ \int_{r<1} \frac{f(x,\theta)}{r^m} [u(y) - u(x)]\, dy + u(x) \int_{r<1} \frac{f(x,\theta)}{r^m}\, dy \tag{2.4}$$

The first two integrals on the right-hand side converge absolutely, but for the third integral if we introduce spherical coordinates with a centre at x, then we'll get:

$$\int_{r<1} \frac{f(x,\theta)}{r^m}\, dy = \lim_{\varepsilon\to 0} \int_{\varepsilon<r<1} \frac{f(x,\theta)}{r^m}\, dy =$$

$$= \lim_{\varepsilon\to 0} \ln \frac{I}{\varepsilon} \int_{S} f(x,\theta)\, ds \tag{2.5}$$

if condition (2.3) is satisfied.

But if condition (2.3) is not satisfied, then the singular integral (2.1) can be expressed in terms of absolutely convergent integrals by the formula:

$$\int_{E_m} \frac{f(x,\theta)}{r^m} u\,(y)\, dy = \int_{r>1} \frac{f(x,\theta)}{r^m} u\,(y)\, dy + \int_{r<1} \frac{f(x,\theta)}{r^m} [u(y) - u(x)]dy \tag{2.6}$$

3. BASIC FORMULAE OF THE MULTIDIMENSIONAL SINGULAR INTEGRAL EQUATIONS.

Let us consider the following multidimensional integral equation:

$$A(x)\, u(x) + \int_{E_m} \frac{f(x,\theta)}{r^m}\, u(y)\, dy + B\, u = 0 \tag{3.1}$$

The integral equation (3.1) is singular if the following assumptions are valid:

(a) The coefficient $A(x)$ satisfies the inequality:

$$|A(y) - A(x)| \le C_1\, r^{\lambda}\, [(1+x^2)\,(1+y^2)]^{-\frac{\lambda}{2}} \tag{3.2}$$

(b) The characteristic $f(x,\theta)$ satisfies the conditions of section 2 and also the inequality:

$$|f(y,\theta) - f(x,\theta)| \le C_2\, r^{\mu}\, [(1+x^2)\,(1+y^2)]^{-\frac{\mu}{2}} \tag{3.3}$$

(c) The operator B is completely continuous in $L_p(E_m)$ for a certain p in the interval $1 < p < \infty$.

In the assumptions (3.2) and (3.3) C_1, C_2, λ, μ, denote positive constants. Moreover, let us consider the following singular integral equation in simplest form:

$$I(x,y,\theta) = A(x)\, u(x) + \int_{E_m} \frac{f(x,\theta)}{r^m}\, u(y)\, dy \tag{3.4}$$

Thus, by using some series of m-dimensional spherical functions, of order n, one obtains:

$$f(x,\theta) = \sum_{n=1}^{\infty} \Lambda_{n,m}\,(x,\theta) = \sum_{n=1}^{\infty} \Lambda_{n,m}\,(x,\theta_1,\theta_2,\ldots,\theta_{m-1}) \tag{3.5}$$

By using (3.5), equation (3.4) takes the following form:

$$I(x,y,\theta) = A(x) + \sum_{n=1}^{\infty} B_{n,m}\, \Lambda_{n,m}\,(x,\theta) \tag{3.6}$$

where:

$$B_{n,m} = \frac{i^n\, \pi^{\frac{m}{2}}\, \Gamma(\frac{n}{2})}{\Gamma(\frac{m+n}{2})} \tag{3.7}$$

By using Stirling's formula we obtain the following assumption:

$$|B_{n,m}| \le C\, n^{-\frac{m}{2}} \tag{3.8}$$

where C is a constant.

Moreover, let us consider the more simplest form of multidimensional singular integral equation:

$$I = A\, u(x) + \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{E_m} \frac{f(\theta)}{r^m}\, u(y)\, dy \tag{3.9}$$

A simplest method for the numerical evaluation of eqn (3.9) is by using the Fourier transform:

$$I = F^{-1}\ \phi\ F\ u \tag{3.10}$$

where F is a Fourier transform, of the singular operator with symbol $\phi(\theta)$. In (3.10), $\phi(x,\theta)$ is given by the following formula:

$$\phi(x,\theta) = \sum_{m=1}^{\infty} a_n^{(k)}\ (x)\ \Lambda_{n,m}^{(k)}(\theta) \tag{3.11}$$

where $\Lambda_{n,m}^{(k)}$ (θ) are linearly independent spherical functions of order n.

4. SOME SPECIAL CASES FOR THE MULTI-DIMENSIONAL SINGULAR INTEGRALS.

4.1. The 2 - Dimensional Singular Integral

Let us consider the following two-dimensional singular integral: [22], [31], [32], [33]

$$\Phi(x_0,y_0) = \int_S \frac{w(x,y)\ g(x_0,y_0,\theta)}{r^2}\ u\ (x,y)\ dS, \quad (x_0,y_0) \in S \tag{4.1}$$

in which: $$(x-x_0) + i\ (y-y_0) = r\ e^{i\theta} \tag{4.2}$$

If we assume that the boundary of S is described by the following equation:

$$R = R(\theta), \quad 0 < \theta < 2\pi \tag{4.3}$$

then eqn (4.1) may also be written as follows:

$$\Phi = \int_S \frac{g(\theta)}{r^2}\ u\ (r,\theta)\ ds \tag{4.4}$$

in which: $$g(\theta) = g_0\ (x_0,y_0,\theta) \tag{4.5}$$

$$u(r,\theta) = u_0\ (x_0,y_0) \tag{4.6}$$

By using the trapezoidal rule with m abscissae, then it can be concluded that:

$$\Phi = \int_0^{2\pi} G(\theta)\ d\theta \simeq \frac{2\pi}{m} \sum_{i=0}^{m-1} G\left(\frac{2\pi i}{m}\right) \tag{4.7}$$

where: $$G(\theta) = g(\theta) \int_0^{R(\theta)} \frac{u(r,\theta)}{r}\ dr$$

$$\simeq g(\theta)\ \left[\sum_{k=1}^{n} A_k\ u\ (R\ (\theta)\ \rho_k,\theta) + u\ (\rho,\theta)\ \ln\ |R(\theta)|\right] \tag{4.8}$$

where ρ_k and A_k are the abscissae and weights.

Furthermore let us use the following numerical integration rules:

i. The Gauss-Legendre rule.

By using the Gauss-Legendre numerical integration rule then the weight function has the following form:

$$w\ (x,y) = 1 \tag{4.9}$$

In this case the singular integral (4.1) may be numerically evaluated as follows:

$$\Phi \simeq \sum_{k=1}^{m} A_m \frac{[\Phi_A(x_0,y_0,x_m) - \Phi_B(x_0,y_0,x_m)]}{x_m - x_0}$$

$$- 2\,(\Phi_A(x_0,y_0) - \Phi_B(x_0,y_0))H_n(x_0) \tag{4.10}$$

where:

$$\Phi_A \simeq \sum_{k=1}^{n} A_k \frac{g(x_0,y_0,x,y_k)}{y_k - y_0} u(x,y_k)$$

$$- 2g(x_0,y_0,x)\, u(x,y_0)\, H_n(y_0)$$

$$(y_0 \neq y_k,\quad k = 1, 2, \ldots, n) \tag{4.11}$$

$$\Phi_B \simeq \sum_{\substack{k=1\\k\neq m}}^{n} A_k \frac{g(x_0,y_0,x,y_k)}{y_k - y_0} u(x,y_k)$$

$$+ A_m \left.\frac{d[g(x_0,y_0,x,y)\, u(x,y)]}{dy}\right|_{y=y_0} - 2g(x_0,y_0,x)\, u(x,y_0)X_n(y_0)$$

$$(y_0 = y_m,\ k = 1, 2, \ldots, n) \tag{4.12}$$

ii. The Gauss-Radau rule.

By using the Gauss-Radau numerical integration rule the singular integral (4.2) can be numerically evaluated as follows:

$$\Phi \simeq \frac{2\pi}{m} \sum_{i=0}^{m-1} g\left(\frac{2\pi i}{m}\right) \left[\sum_{j=1}^{n} \frac{w_j}{x_j+1} \left[u\left(\frac{x_j+1}{2}, \frac{2\pi i}{m}\right) - u\left(0, \frac{2\pi i}{m}\right)\right]\right.$$

$$\left. + u\left(0, \frac{2\pi i}{m}\right) \ln \left|R\left(\frac{2\pi i}{m}\right)\right| \right] \tag{4.13}$$

where x_j are the zeros of the Legendre polynomials of order n and w_j the weights of the classical Gauss-Legendre quadrature equation.

iii. The Gauss-Lobatto rule.

In the case of the Gauss-Lobatto numerical integration rule, the integral (4.1) can be evaluated as follows:

$$\Phi \simeq \frac{2\pi}{m} \sum_{i=0}^{m-1} g\left(\frac{2\pi i}{m}\right) \left[\sum_{j=1}^{n} \frac{w_j}{1-x_j^2} \left[\frac{x_j-1}{2} u\left(0, \frac{2\pi i}{m}\right) - \frac{x_j+1}{2} u\left(1, \frac{2\pi i}{m}\right)\right.\right.$$

$$\left. + u\left(\frac{x_j+1}{2}, \frac{2\pi i}{m}\right)\right] + u\left(1, \frac{2\pi i}{m}\right) + u\left(0, \frac{2\pi i}{m}\right) \left[\ell n\ R\left(\frac{2\pi i}{m}\right)^{-1}\right] \Bigg] \tag{4.14}$$

where w_j, x_j are the weights and zeros corresponding to the set of orthogonal polynomials $(P^{(1,0)}(x))$.

4.2. The 3 - Dimensional Singular Integral.

The following integral is a three-dimensional singular integral defined on a three-dimensional finite region V, containing the third-order pole (x^1, y^1, z^1), whose boundary is a closed Lyapounov surface S: [33]

$$I(x^1, y^1, z^1) = \int_V \frac{g(x^1, y^1, z^1, \vartheta, \phi)}{r^3} u(x, y, z)\, dv =$$

$$= \int \frac{g(x^1, y^1, z^1, \vartheta, \phi)}{[(x-x^1)^2 + (y-y^1)^2 + (z-z^1)^2]^{3/2}} u(x,y,z)\, dv \tag{4.15}$$

Furthermore let us introduce the following system of spherical coordinates:

$$x = x^1 + r \sin\vartheta \cos\phi$$

$$y = y^1 + r \sin\vartheta \sin\phi \,, \quad (0 \le \phi \le \pi)$$

$$z = z^1 + r \cos\vartheta \,, \qquad (0 \le \phi \le \pi) \tag{4.16}$$

$$r^2 = (x-x^1)^2 + (y-y^1)^2 + (z-z^1)^2$$

Then, from eqn (4.16) we obtain:

$$I = \lim_{\varepsilon\to 0} \int_0^{\pi} \int_0^{2\pi} \int_{\varepsilon}^{R(\vartheta,\phi)} \sin\vartheta\, f(\vartheta,\phi) \frac{u(r,\vartheta,\phi)}{r}\, d\vartheta\, d\phi\, dr \tag{4.17}$$

Thus, if we integrate eqn (4.17) with respect to ϑ and ϕ by using the trapezoidal rule with abscissae G, D, then we'll get the following formula:

$$I = \frac{2\pi^2}{GD} \sum_{i=1}^{G} \sum_{j=1}^{D} \sin(\vartheta_i)\, \phi(\vartheta_i, \phi_i) \tag{4.18}$$

in which:

$$\vartheta_i = \frac{\pi}{G}(i-1)$$

$$\phi_j = \frac{2\pi\,(j-1)}{D} \tag{4.19}$$

and

$$\phi(\vartheta,\phi) = g(\vartheta,\phi) \int_0^{R(\vartheta,\phi)} \frac{u(r,\vartheta,\phi)}{r}\, dr \tag{4.20}$$

For the numerical evaluation of the integral in (4.20) let us use the following numerical integration rule:

$$\int_0^{R(\vartheta,\phi)} \frac{u(r,\vartheta,\phi)}{r}\, dr \simeq \sum_{k=1}^{L} A_k\, u[R(\vartheta,\phi)\, \rho_k, \vartheta, \phi] + u(0,\vartheta,\phi) \ln[R(\vartheta,\phi)] \tag{4.21}$$

where ρ_k are the abscissas and A_k the weights for the integration interval [0,1].

5. BASIC FORMULAE FOR THE THEORY OF LINEAR VISCOELASTICITY.

Let us consider a linear isotropic viscoelastic solid with the following stress field:

$$\sigma'_{ij} = 2 \int_0^t V_1(F,t-\tau) \frac{\partial \varepsilon'_{ij}}{\partial \tau} (F,\tau)\, d\tau + V_1(F,t)\, \varepsilon'_{ij} (F,0^+) \tag{5.1}$$

$$\sigma_{ii} = 3 \int_0^t V_2(F,t-\tau) \frac{\partial \varepsilon_{ii}}{\partial \tau} (F,\tau)\, d\tau + V_2(F,t)\, \varepsilon_{ii} (F,0^+) \tag{5.2}$$

with: $\varepsilon'_{ij} (F,0^+) = \lim_{t\to 0} \varepsilon_{ij} (F,t), \quad \varepsilon_{ii} (F,0^+) = \lim_{t\to 0} \varepsilon_{ii} (F,t)$ (5.3ab)

where σ'_{ij} and ε'_{ij} are, respectively, deviatoric components of the stress and strain field, V_1 and V_2 are relaxation functions in shear and isotropic compression, respectively, t is the time and F denotes a point in the infinite elastic space where the viscoelastic solid belongs.

By taking the Laplace transform of eqs (5.1) and (5.2), one obtains:

$$\sigma_{ij}^{\prime *} = 2s\, V_1^*(s)\, \varepsilon_{ij}^{\prime *} \tag{5.4}$$

$$\sigma_{ii}^* = 3s\, V_2^*(s)\, \varepsilon_{ii}^* \tag{5.5}$$

Furthermore, the Laplace transformed equation of equilibrium is as following:

$$\sigma_{ij,j}^* = 0 \tag{5.6}$$

Let us use Hooke's law for a linear isotropic, viscoelastic solid:

$$\sigma_{ij}^* (F) = \lambda^* u_{n,n}^*(F)\, \delta_{ij} + \mu^* [u_{i,j}^* (F) + u_{j,i}^* (F)] \tag{5.7}$$

where u_i^* denotes the boundary displacements, λ^* and μ^* the additional Lame elastic constants and δ_{ij} is the delta function of Kronecker.

By applying the Betti-Rayleigh theorem the boundary displacement yields:

$$u_i^* (F) = - \int_L u_j^*(H)\, T_{ij} (H,F)\, dH + \int_L t_j^*(H)\, U_{ij}(H,F)\, dH \tag{5.8}$$

where t_j^* are the boundary tractions on the contour L and H,F are points on the union of the contour L.

In (5.8) the fundamental solutions for the displacements and tractions are given by the following relations:

$$U_{ij} = \frac{\lambda^* + 3\mu^*}{8\pi\mu^*(\lambda^*+2\mu^*)} \left(\frac{1}{r} \right) \left[\delta_{ij} + \left(\frac{\lambda^*+\mu^*}{\lambda^*+3\mu^*} \right) r_{,i}\, r_{,j} \right] \tag{5.9}$$

$$T_{ij} = - \frac{\mu^*}{4\pi(2\mu^*+\lambda^*)} \left(\frac{1}{r^2} \right) \left[\frac{\partial r}{\partial n} \left(\delta_{ij} + \frac{3(\mu^*+\lambda^*)}{\mu^*}\, r_{,i}\, r_{,j} \right) - n_j r_{,i} + n_i r_{,j} \right] \tag{5.10}$$

By combining eqs (5.4), (5.5) and (5.7) one obtains:

$$\lambda^* = -2/3\ s\ V_1^*(s) + s\ V_2^*(s)$$
$$\mu^* = s\ V_1^*(s) \tag{5.11}$$

Finally, for small strain, the transformed components of strain and displacement are related by the formula:

$$2\ \varepsilon_{ij}^* = u_{ij,\ i}^* + u_{ji,\ i}^* \tag{5.12}$$

where u_i^* is given by (5.8).

Thus, the components of the stress field for a linear, isotropic, viscoelastic solid can be evaluated by using the numerical technique of Section 4.1.

6. AN APPLICATION OF LINEAR VISCOELASTICITY.

As an application of the previous theory, let us consider a thick, hollow circular cylinder of viscoelastic material restrained by an enclosing thin elastic ring and subjected to a uniform pressure p, applied as a step in time at $t = 0$ (see: Figure 1).

The same problem has been previously solved by Ting [39] by using an exact analytical solution and by Rizzo and Shippy [40] by using the Boundary Integral Equation Method (B.I.E.M.). A comparison will be made between the new Singular Integral Operators Method (S.I.O.M.) introduced in the present report, the theoretical solution [39] and the B.I.E.M. [40].

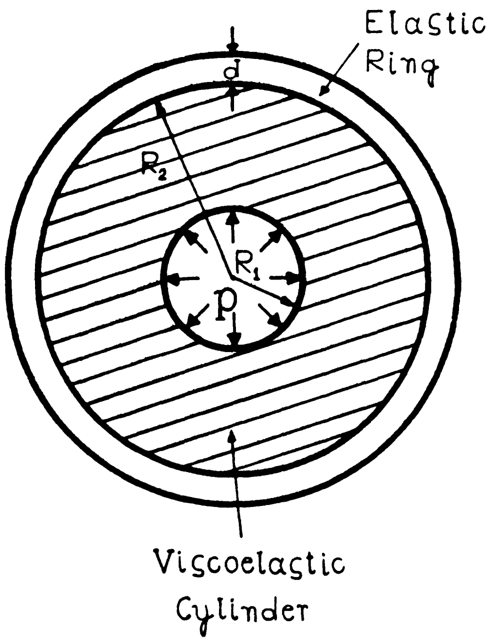

Figure 1: A thick, hollow circular cylinder of viscoelastic solid, restrained by an enclosing thin elastic ring and subjected to a uniform pressure p.

The geometrical sizes of the cylinder are as follows: Ratio of inner to outer radii of the cylinder is $R_2/R_1 = 0.3$ (see:Figure 1).

Moreover we consider that the viscoelastic solid behaves elastically in bulk and as a standard linear solid in shear, so that the relaxation functions V_1 and V_2 are given by the formulas:

$$V_1(t) = V\,[g + (1-g)\,e^{-\lambda t}] \tag{6.1}$$

$$V_2(t) = N \tag{6.2}$$

where g, λ, V and N are constants and we assume that $g = 0.5$ and $V = 0.6\,N$.

Thus, the transformed relaxation functions V_1^* and V_2^* are given by the following relations:

$$V_1^* = (V/s)\,(s + g\lambda) \,/\, (s + \lambda) \tag{6.3}$$

$$V_2^* = N \,/\, s \tag{6.4}$$

Furthermore the thin, elastic ring is characterized by the relation:

$$C = E\,d \,/\, (1 - \nu^2)\,R_2 \tag{6.5}$$

where E is the Young's modulus, ν is the Poisson's ratio, and d its thickness (Fig. 1). Also we assume the value of C / N to be unity: $C / N = 1$.

Figures 2 and 3 show the radial σ_r/p and transverse stress σ_θ/p, respectively, as functions of time.

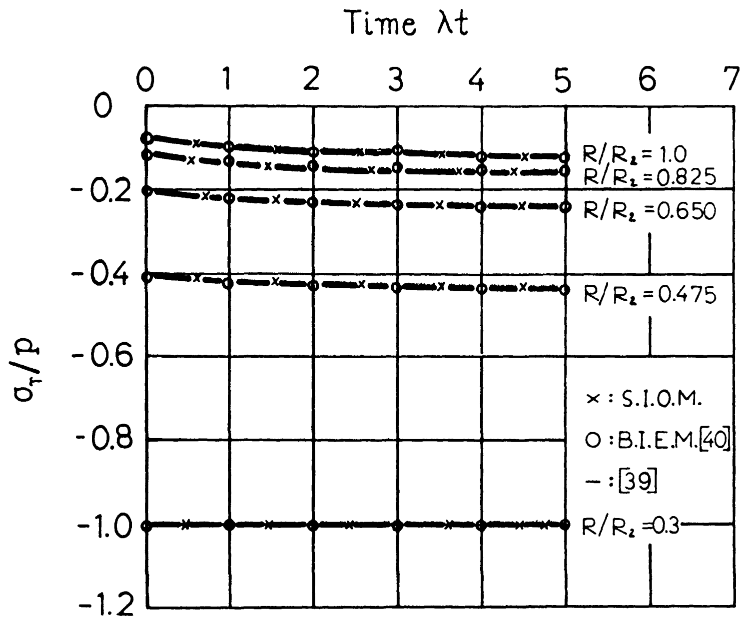

Figure 2: Radial stresses σ_r/p as functions of time for the cylinder of Figure 1.

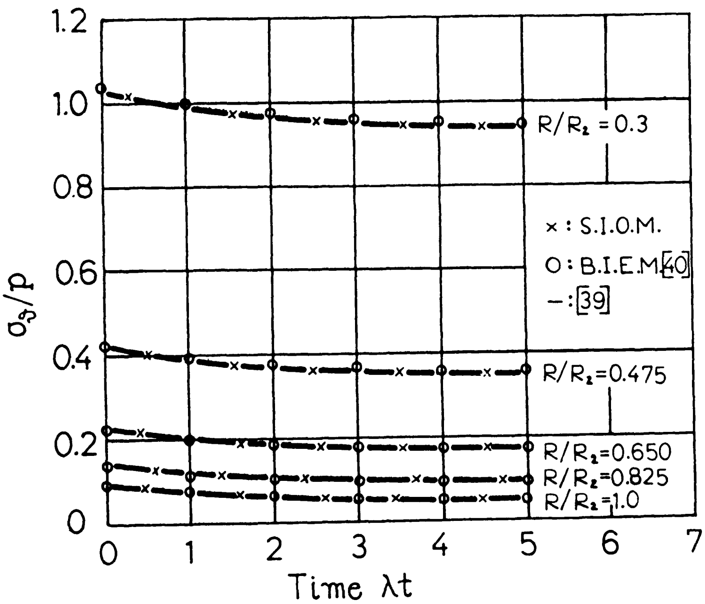

Figure 3: Transverse stresses σ_θ/p as functions of time for the cylinder of Figure 1.

Finally, as it is easily seen from Figures 2 and 3, the numerical results of the S.I.O.M. coincide very well with the theoretical results [39] and the additional numerical results of the B.I.E.M. [40].

7. CONCLUSIONS.

A new numerical technique has been investigated for the numerical evaluation of the multidimensional singular integrals and integral equations used in many fields of mathematical physics.

Especially for the surface singular integrals, some formulas are derived by using the Gauss-Legendre, Lobatto and Radau numerical integration rules. For the construction of such a cubature formula, the two-dimensional singular integral was considered as an iterated one, and the second-order pole involved in this integral was analyzed into a pair of complex poles. Thus, the methods of numerical integration, valid for one-dimensional singular integrals, were extended to the case of the two-dimensional singular integrals. Also, a complete analysis for the numerical evaluation of the three-dimensional singular integrals was also presented.

The technique for the numerical evaluation of the surface singular integrals described in section 4.1 has been used for the determination of the fracture behavior of a linear, viscoelastic, isotropic solid. An application was given to the determination of the radial and transverse stresses in a thick, hollow circular cylinder of viscoelastic solid restrained by an enclosing thin elastic ring and subjected to a uniform pressure.

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