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S-ASYMPTOTIC EXPANSION OF DISTRIBUTIONS

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ABSTRACT. This paper contains first a definition of the asymptotic expansion at infinity of distributions belonging to $\mathcal{D}'(R^n)$, named S-asymptotic expansion, as also its properties and application to partial differential equations.

KEYS WORDS AND PHRASES. Convex cone, distribution, behaviour of a distribution at infinity, asymptotic expansion.
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1. INTRODUCTION.

The basic idea of the asymptotic behaviour at infinity of a distribution one can find already in the book of L. Schwartz [1]. To these days many mathematicians tried to find a good definition of the asymptotic behaviour of a distribution. We shall mention only "equivalence at infinity" explored by Lavoine and Misra [2] and the "quasiasymptotic" elaborated by Vladimirov and his pupils [3]. Brichkov [4] introduced the asymptotic expansion of tempered distributions as a useful mathematical tool in quantum field theory. His investigations and definitions were turned just towards these applications. In [4] one can find cited literature in which asymptotic expansion technique, introduced by Brichkov, was used in the quantum field theory. This is a reason to study S-asymptotic expansion.

2. DEFINITION OF THE S-ASYMPTOTIC EXPANSION.

In the classical analysis we say that the sequence $\{\psi_n(t)\}$ of numerical functions is asymptotic if and only if $\psi_{n+1}(t) = o(\psi_n(t))$, $t \to \infty$. The formal series $\sum_{n\geq 1} u_n(t)$ is an asymptotic expansion of the function $u(t)$ related to the asymptotic sequence $\{\psi_n(t)\}$ if

$$u(t) - \sum_{n=1}^{k} u_n(t) = o(\psi_k(t)), \ t \to \infty \tag{2.1}$$

for every $k \in N$ and we write

$$u(t) \sim \sum_{n=1}^{\infty} u_n(t) \mid \{\psi_n(t)\}, \ t \to \infty \tag{2.2}$$

When for every $n \in N$ $u_n(t) = c_n\psi_n(t)$, c_n are complex numbers, expansion (2.2) is unique, that means the numbers c_n can be determined in only one way.

In this text Γ will be a convex cone with vertex at zero belonging to R^n and $\Sigma(\Gamma)$ the set of all real valued and positive functions $c(h)$, $h \in \Gamma$. Notations for the spaces of distributions are as in the books of Schwartz [1].

DEFINITION 1. The distribution $T \in \mathcal{D}'$ has the S-asymptotic expansion related to the asymptotic sequence $\{c_n(h)\} \subset \Sigma(\Gamma)$, we write it

$$T(t+h) \overset{S}{\sim} \sum_{n=1}^{\infty} U_n(t,h) \mid \{c_n(h)\}, \quad \|h\| \to \infty, \ h \in \Gamma \tag{2.3}$$

where $U_n(t,h) \in \mathcal{D}'$ for $n \in N$ and $h \in \Gamma$, if for every $\rho \in \mathcal{D}$

$$<T(t+h), \rho(t)> \sim \sum_{n=4}^{\infty} <U_n(t,h), \rho(t)> \mid \{c_n(h)\}, \quad \|h\| \to \infty, \ h \in \Gamma \tag{2.4}$$

REMARK. 1) In the special case $U_n(t,h) = u_n(t)c_n(h)$, $u_n \in \mathcal{D}$, $n \in N$, we shall write

$$T(t+h) \overset{S}{\sim} \sum_{n=1}^{\infty} u_n(t)\, c_n(h) \ , \quad \|h\| \to \infty \ , \ h \in \tag{2.5}$$

and the given S-asymptotic expansion is unique.

2) To define the S-asymptotic expansion in $\mathcal{S}'(R^n)$, we have only to suppose that in relation (2.4) T and U_n are in $\mathcal{S}'$ and ρ in $\mathcal{S}$.

Brichkov's general definition is slightly different [5].

DEFINITION 1'. The distribution $g \in \mathcal{S}'$ has the asymptotic expansion related to the asymptotic sequence $\{\psi_n(t)\}$ on the ray $\{\lambda h_o, \lambda > 0\}$, $h_o \in R^n$

$$g(\lambda h_o - t) \sim \sum_{n=1}^{\infty} \hat{c}_n(t,\lambda) \mid \{\psi_n(\lambda)\} \ , \ \lambda \in R, \ \lambda \to \infty \tag{2.6}$$

where $\hat{c}_n(t,\lambda) \in \mathcal{S}'$ for $\lambda \geq \lambda_o > 0$, if for every $\phi \in \mathcal{S}$

$$<g(\lambda h_o - t), \phi(t)> \sim \sum_{n=1}^{\infty} <\hat{c}_n(t,\lambda), \phi(t)> \mid \{\psi_n(\lambda)\}, \ \lambda \to \infty \tag{2.7}$$

Relation 2.6 can be transformed in

$$f(x)\, e^{i\lambda \ll x, h_o \gg} \sim \sum_{n=1}^{\infty} c_n(x,\lambda) \mid \{\psi_n(\lambda)\}, \quad \lambda \to \infty \tag{2.8}$$

by the Fourier transform, if we take $f(x) = F^{-1}[g(t)]$; $\rho(x) = F^{-1}[\phi(t)]$ and $F[\hat{c}_n(t,\lambda)] = (2\pi)^n c_n(x,\lambda)$. We denote by $F[\rho]$ the Fourier transform of ρ and by $F^{-1}[g]$ the inverse Fourier transform of g. Also, for $x,t \in R^n$ $\ll x,t \gg = \sum_{i=1}^{n} x_i t_i$.

In his papers Brichkov considered only the asymptotic expansinons (2.8) and in one dimensional case. We shall study the asymptotic expansion not in $\mathcal{S}'(R)$ but in the whole $\mathcal{D}'(R^n)$, not only on a ray but on a cone in R^n. Our results enlarge Brichkov's to be valued for the elements of $\mathcal{D}'(R^n)$ (Corollary 1), they are proved with less suppositions (Propositions 5 and 6) or give new properties of the S-asymptotic.

A distribution belonging to $\mathcal{S}'$ can have S-asymptotic expansion in $\mathcal{D}'$ without having the same S-asymptotic expansion in $\mathcal{S}'$. Such an example is the regular distribution $\tilde{f}$ defined by the function

$$f(t) = H(t) \exp(1/(1+t^2)) \exp(-t) \ , \quad t \in R$$

where

$$H(t) = 1,\ t \geq 0 \quad \text{and} \quad H(t) = 0, \quad t < 0.$$

It is easy to prove that for $h \in R_+$

$$\tilde{f}(t+h) \overset{s}{\sim} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} (1+(t+h)^2)^{1-n} \exp(-t-h) \,|\, \{e^{-h} h^{2(1-n)}\},\ h \to \infty.$$

But

$$U_n(t,h) = (1+(t+h)^2)^{1-n} \exp(-t-h), \quad n \in N, \quad h > 0$$

do not belong to $\mathcal{S}'$.

The regular distribution $\tilde{g}$ defined by the function

$$g(t) = \exp(1+(1+t^2)) \exp(t), \quad t \in R$$

belongs to $\mathcal{D}'$ but it is not in $\mathcal{S}'$. It has S-asymptotic expansion in $\mathcal{D}'$:

$$\tilde{g}(t+h) \overset{s}{\sim} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} (1+(t+h)^2)^{1-n} \exp(t+h) \,|\, \{e^{h} h^{2(1-n)}\},\ h \to \infty$$

where $\Gamma = R_+$.

3. PROPERTIES OF THE S-ASYMPTOTIC EXPANSION.

PROPOSITION 1. Let $S \in \mathcal{E}'$ and $T \in \mathcal{D}'$. If

$$T(t+h) \overset{s}{\sim} \sum_{n=1}^{\infty} U_n(t,h) \,|\, \{c_n(h)\},\ \|h\| \to \infty,\ h \in \Gamma$$

then the convolution

$$(S*T)(t+h) \overset{s}{\sim} \sum_{n=1}^{\infty} (S * U_n)(t,h) \,|\, \{c_n(h)\} \quad \|h\| \to \infty \quad h \in \Gamma \tag{3.1}$$

PROOF. We know that

$$\langle (S*T)(t+h), \rho(t)\rangle - \sum_{n=1}^{k} \langle (S*U_n)(t,h), \rho(t)\rangle = \langle S*[T(t+h) - \sum_{n=1}^{k} U_n(t,h)], \rho(t)\rangle .$$

It remains only to use the continuity of the convolution.

COROLLARY 1. If

$$T(t+h) \overset{s}{\sim} \sum_{n=1}^{\infty} U_n(t,h) \,|\, \{c_n(h)\} \quad \|h\| \to \infty,\ h \in \Gamma$$

then

$$T^{(k)}(t+h) \overset{s}{\sim} \sum_{n=1}^{\infty} U_n^{(k)}(t,h) \,|\, \{c_n(h)\},\ \|h\| \to \infty,\ h \in \Gamma \tag{3.2}$$

where $T^{(k)} = (D_{t_1}^{k_1} \dots D_{t_n}^{k_n}) T$, $k = (k_1, \dots, k_n) \in N_o^n$, $N_o = N \cup \{0\}$.

PROOF. We have only to take $S = \delta^{(k)}$ in Proposition 1.

REMARK. Proposition 1. is valued as well if we suppose that $T \in \mathcal{S}'$ and $S \in \mathcal{O}_c'$.

PROPOSITION 2. Let f, $U_n(t,h)$ and $V_n(t)$, $n \in N$ and $h \in \Gamma$, be the local integrable functions such that for every compact set $K \subset R^n$

$$f(t+h) \sim \sum_{n=1}^{\infty} U_n(t,h) \,|\, \{c_n(h)\},\ \|h\| \to \infty,\ h \in \Gamma,\ t \in K$$

and

$$|f(t+h) - \sum_{n=1}^{k} U_n(t,h)| / c_k(h) \leq V_k(t), \ t \in K, \ h \in \Gamma$$

and $||h|| \geq r(k,K)$, then for the regular distribution $\tilde{f}$ defined by f we have

$$\tilde{f}(t+h) \overset{S}{\sim} \sum_{n=1}^{\infty} \tilde{U}_n(t,h) \mid \{c_n(h)\}, \ ||h|| \to \infty, \ h \in \Gamma .$$

PROOF. The proof is a consequence of the Lebesgue's theorem.

PROPOSITION 3. Suppose that T_1 and T_2 belong to $\mathcal{D}'$ and equal over the open set Ω which has the property: for every $r > 0$ there exists a β_o such that the ball $B(0,r) = \{x \in R^n, \ ||x|| \leq r\}$ is in $\{\Omega\text{-}h, \ h \in \Gamma, \ ||h|| \geq \beta_o\}$. If

$$T_1(t+h) \overset{S}{\sim} \sum_{n=1}^{\infty} U_n(t,h) \mid \{c_n(h)\}, \ ||h|| \to \infty, \ h \in \Gamma$$

then

$$T_2(t+h) \overset{S}{\sim} \sum_{n=1}^{\infty} U_n(t,h) \mid \{c_n(h)\}, \ ||h|| \to \infty, \ h \in \Gamma$$

as well.

PROOF. We have only to prove that for every $c_k(h)$

$$\lim_{||h|| \to \infty, h \in \Gamma} < [T_1(t+h) - T_2(t+h)] \ / \ c_k(h) \ , \ \rho(t)> \ = 0, \ \rho \in \mathcal{D} \tag{3.3}$$

Let supp $\rho \subset B(0,r)$. The distribution $T_1(t+h) - T_2(t+h)$ equals zero over Ω-h. By the supposition there exists a β_o such that the ball $B(0,r)$ is in $\{\Omega - h, \ h \in \Gamma, \ ||h|| \geq \beta_o\}$. This proves out relation (3.3).

PROPOSITION 4. Let $S \in \mathcal{D}'$ and for $1 \leq m \leq n$

$$D_{t_m} S(t+h) \overset{S}{\sim} \sum_{i=1}^{\infty} U_i(t,h) \mid \{c_i(h)\}, \ ||h|| \to \infty, \ h \in \Gamma .$$

If the family $\{V_i(t,h), \ i \in N, \ h \in \Gamma\}$ has the properties: $D_{t_m} V_i(t,h) = U_i(t,h)$, $i \in N$, $h \in \Gamma$ and for a $\rho_o \in \mathcal{D}(R)$, $\int_R \rho_o(\tau) d\tau = 1$, and for every $\rho \in \mathcal{D}$, $k \in N$

$$\lim_{||h|| \to \infty, h \in \Gamma} < [S(t+h) - \sum_{i=4}^{k} V_i(t,h)]/c_k(h), \ \rho_o(t_m)\lambda_m(t) > \ = 0$$

where $\lambda_m(t) = \int_R \rho(t_1,\ldots,t_m,\ldots,t_n) \, dt_m$, then

$$S(t+h) \overset{S}{\sim} \sum_{i=1}^{\infty} V_i(t,h) \mid \{c_i(h)\}, \ ||h|| \to \infty, \ h \in \Gamma .$$

PROOF. If $\rho \in \mathcal{D}$ then $\rho(t) = \rho_o(t_m)\lambda_m(t) + \psi(t)$ where $\psi \in \mathcal{D}$ and $\int_R \psi(t_1,\ldots,t_m,\ldots,t_n) \, dt_m = 0$.

Now we have the following equality

$$<[S(t+h) - \sum_{i=1}^{k} V_i(t,h)], \ \rho(t)> \ = \ <[S(t+h) - \sum_{i=1}^{k} V_i(t,h)], \ \rho_o\lambda_m(t)>$$
$$- <[D_{t_m} S(t+h) - \sum_{i=1}^{k} U_i(t+h)], \ \int_{-\infty}^{t_m} \psi(t_1,\ldots,u_m,\ldots,t_n) \, du_m > .$$

It remains only to use the limit in it and Corollary 1.

PROPOSITION 5. Suppose that $S \in \mathcal{D}'$, $\Gamma = \{h \in R^n,\ h = (0,\dots,h_m,\dots,0)\}$, where m is fixed, $1 \leq m \leq n$ and

$$(D_{t_m} S)(t+h) \overset{s}{\sim} \sum_{i=1}^{\infty} U_i(t,h) \mid \{c_i(h)\},\ ||h|| \to \infty,\ h \in \Gamma$$

If there exists $V_i(t,h)$, $D_{h_m} V_i(t,h) = U_i(t,h)$, $i \in N$ and if $c_i(h)$, $i \in N$ are local integrable in h_m and such that

$$\hat{c}_i(h) = \int_1^{h_m} c_i(u)\, du_m \to \infty \quad \text{as} \quad h_m \to \infty$$

then

$$S(t+h) \overset{s}{\sim} \sum_{i=1}^{\infty} V_i(t,h) \mid \{\hat{c}_i(h)\},\ ||h|| \to \infty,\ h \in \Gamma\ .$$

PROOF. By L'Hospital's rule with the Stolz's improvement we have for every $\rho \in \mathcal{D}$ and $k \in N$

$$\lim_{h\to\infty, h\in\Gamma} \frac{\langle S(t+h),\ \rho(t)\rangle - \langle \sum_{i=1}^{k} V_i(t,h),\ \rho(t)\rangle}{\hat{c}_k(h)}$$

$$= \lim_{h\to\infty, h\in\Gamma} \frac{\langle (D_{t_m} S)(t+h),\ \rho(t)\rangle - \langle \sum_{i=1}^{k} U_i(t,h),\ \rho(t)\rangle}{c_k(h)}\ .$$

These five propositions give how is related the S-asymptotic with convolution, derivative, classical expansion and the primitive of a distribution. The next proposition gives the analytical expression of $U_n(t,h) = u_n(t)\ c_n(h)$.

PROPOSITION 6. Suppose that $T \in \mathcal{D}'$, Γ with nonempty interior,

$$T(t+h) \overset{s}{\sim} \sum_{n=1}^{\infty} u_n(t)\ c_n(h),\quad ||h|| \to \infty,\ h \in \Gamma\ .$$

If $u_m \neq 0$, $m \in N$, then u_m has the form

$$u_m(t) = \sum_{k=1}^{m} P_k^m(t_1,\dots,t_n)\ \exp(\langle\langle a^k, t\rangle\rangle),\ m \in N \tag{3.4}$$

where $a^k = (a_1^k,\dots,a_n^k) \in R^n$ and P_k^m are polynomials, the power of them less of k in every t_i, $i = 1,\dots,n$: $\langle\langle x,t\rangle\rangle = \sum_{i=1}^{n} x_i t_i$.

PROOF. By Definition 1 and our supposition

$$\lim_{||h||\to\infty, h\in\Gamma} T(t+h)/c_1(h) = u_1(t) \neq 0 \tag{3.5}$$

From relation (3.5) follows that u_1 satisfies the equation

$$u_1(t+h_o) = d(h_o)\ u_1(t),\ h_o \in \Gamma \tag{3.6}$$

where

$$d(h_o) = \lim_{||h||\to\infty, h\in\Gamma} c_1(h+h_o)/c_1(h)$$

If h_o is an interior point of Γ and e_k is such element from R^n for which all the coordinates equal zero except the k-th which is 1. Then

$$u_1(t+h_o+\varepsilon e_k) - u_1(t+h_o) = [d(\varepsilon e_k) - d(0)]u_1(t+h_o).$$

Hence the existence of $D_{h_k} d(h)_{h=o} = a_k^1$ and

$$D_{t_k} u_1(t+h_o) = a_k^1 u_1(t+h_o),\ k = 1,\dots,n. \tag{3.7}$$

We know that all the solutions of equation (3.7) are of the form $u_1(t) = C_1\exp(\langle\langle a^1,t\rangle\rangle)$, where C_1 is a constant and $a^1 = (a_1^1,\dots,a_n^1)$.

The following limit gives u_2

$$\lim_{||h||\to\infty, h\in\Gamma} \frac{\langle T(t+h),\rho(t)\rangle - \langle u_1(t),\rho(t)\rangle\, c_1(h)}{c_2(h)} = \langle u_2,\rho\rangle$$

By Corollary 1 follows for $i = 1,\dots,n$

$$\lim_{||h||\to\infty, h\in\Gamma} \frac{\langle (D_{t_i} - a_i^1)T(t+h),\rho(t)\rangle}{c_2(h)} = \langle (D_{t_i} - a_i^1)u_2(t),\rho(t)\rangle$$

Two cases are possible. a) If $(D_{t_i} - a_i^1)\, u_2 = 0$, $i=1,\dots,n$, then $u_2(t) = C_2 \exp(\langle\langle a^1,t\rangle\rangle)$.

b) If $(D_{t_i} - a_i^1)u_2 \neq 0$ for some i, then $(D_{t_i} - a_i^1)u_2(t) = c\exp(\langle\langle a^2,t\rangle\rangle)$ and u_2 has the form $C_2 \exp(\langle\langle a^1,t\rangle\rangle) + P_2^2(t_1,\dots,t_n)\exp(\langle\langle a^2,t\rangle\rangle)$, where P_2^2 is a polynomial of the power less of 2 in every t_i, $i=1,\dots,n$.

In the same way we prove for every u_m.

PROPOSITION 7. Let $T \in \mathcal{D}'$ and $\Omega \in R^n$ be an open set with the property: for every $r > 0$ there exists a β_r such that the ball $B(h,r) \subset \Omega$ for all $h \in \Gamma$, $||h|| \geq \beta_r$. Suppose

$$T(t+h) \overset{s}{\sim} \sum_{n=1}^{m} U_n(t+h) \mid \{c_1(h),\dots,c_m(h)\},\ ||h|| \to \infty,\ h \in \Gamma$$

for any function $c_m(h)$ from $\Sigma(\Gamma)$, then $T = \sum_{n=1}^{m} U_n$ over Ω.

PROOF. The statement of this Proposition can be obtained from a proposition proved in [6]. However, for completeness, we shall give the proof on the whole.

First we shall prove that if for every $c_m(h) \in \Sigma(\Gamma)$

$$\lim_{||h||\to\infty, h\in\Gamma} \left\langle \frac{T(t+h) - \sum_{n=1}^{m} U_n(t+h)}{c_m(h)},\ \rho(t)\right\rangle = 0 \tag{3.8}$$

then there exists a $\beta(\rho)$ such that

$$\langle [T(t+h) - \sum_{n=1}^{m} U_n(t+h)],\ \rho(t)\rangle = 0,\ h \in \Gamma,\ ||h|| \geq \beta(\rho).$$

Suppose the opposite. We would have a sequence $h_n \in \Gamma$, $||h_n||\to\infty$ such that

$$\langle [T(t+h_n) - \sum_{n=1}^{m} U_n(t+h_n)],\ \rho(t)\rangle = p_n \neq 0,\ n \in N$$

then we choose $c_m(h)$ in such a way that $c_m(h_n) = p_n$ and relation (3.8) would be false.

We denote by $\beta_o(\rho) = \inf\beta(\rho)$. We shall prove that the set $\{\beta_o(\rho), \rho \in \mathcal{D}_K\}$ for every compact set $K \subset R^n$ is bounded. Let us suppose the opposite; then there exists a sequence $\{h_k\}$, $h_k \in \Gamma$, $\|h_k\| \to \infty$ and the sequence $\{\phi_k(t)\}$, $\phi_k \in \mathcal{D}_K$ such that

$$<\bar{T}(t+h_k),\ \phi_p(t)> = A_{k,p} = \begin{cases} a_k \neq 0, & p = k \\ 0, & p < k \end{cases} ;\ \bar{T} = T - \sum_{n=1}^{m} U_n .$$

The construction of the sequence $\{h_k\}$ and ϕ_k can be the following. Let $\phi_k \in \mathcal{D}_K$ be such that $\beta_o(\phi_k)$ is a strict monotone sequence which tends to infinity, then there exist $\{h_k\} \subset \Gamma$ and $\varepsilon_k > 0$, $k \in N$ such that $\beta_o(\phi_{k-1}) + \varepsilon_k \leq \|h_k\| \leq \beta_o(\phi_k) - \varepsilon_k$. Now, we shall construct the sequence $\{\psi_p(t)\}$, $\psi_p \in \mathcal{D}_K$ for which we have

$$<\bar{T}(t+h_k),\ \psi_p(t)> = \begin{cases} 0, & p \neq k \\ a_k, & p = k \end{cases} .$$

Let $\psi_p(t) = \phi_p(t) - \lambda_1^p\phi_1(t) - \ldots - \lambda_{p-1}^p\phi_{p-1}(t)$, $p > 1$. The numbers λ_i^p we can find in such a way that $\psi_p(t)$ satisfies the sought property.

It is easy to see that $<\bar{T}(t+h_k), \psi_k(t)> = a_k$ and $<\bar{T}(t+h_k), \psi_p(t)> = 0$, $k > p$. For a fixed p and $k < p$ we can find λ_i^p, $i=1,\ldots,p-1$ so that for $k=1,\ldots,p-1$

$$0 = <\bar{T}(t+h_k),\ \psi_p(t)> = A_{k,p} - \lambda_1^p A_{k,1} - \ldots - \lambda_{p-1}^p A_{k,p-1}$$

Hence

$$\lambda_1^p A_{k,1} + \ldots + \lambda_{p-1}^p A_{k,p-1} = A_{k,p},\ k=1,\ldots,p-1,\ p > 1.$$

As $A_{k,k} \neq 0$ for every k, this system has always a solution.

We introduce now a sequence of numbers $\{b_k\}$, $b_k = \sup\{2^k|\psi_k^{(i)}(t)|,\ i < k\}$. Then the function

$$\psi(t) = \sum_{p=1}^{\infty} \psi_p(t)/b_p \in \mathcal{D}_K$$

and this series converges in $\mathcal{D}_K$, thus in $\mathcal{D}$ as well. With this

$$<\bar{T}(t+h_k),\psi(t)> = \sum_{p=1}^{\infty} <\bar{T}(t+h_k),\ \psi_p(t)/b_p> = a_k/b_k$$

If we choose $c(h)$ such that $c(h_k) = a_k/b_k$ then $<[\bar{T}(t+h)/c(h)],\psi(t)>$ does not converge to zero when $\|h\| \to \infty$, $h \in \Gamma$. This is in contradiction with (3.8). Hence, for every compact set K there exists a $\beta_o(K)$ such that $<\bar{T}(t+h),\phi(t)> = 0, \|h\| \geq \beta_o(K)$, $h \in \Gamma$, $\phi \in \mathcal{D}_K$. That means that $\bar{T}(t+h) = 0$ over $B(0,r)$, $\|h\| \geq \beta(r)$, $h \in \Gamma$ and $\bar{T}(t) = 0$ over $B(h,r)$, $\|h\| \geq \beta(r)$, $h \in \Gamma$.

4. APPLICATION OF THE S-ASYMPTOTIC EXPANSION TO PARTIAL DIFFERENTIAL EQUATIONS.

As we mentioned in [4], one can find cited literature in which asymptotic expansion technique (in $\mathcal{S}'$ and in one dimensional case) was used in the quantum field theory. We show how the S-asymptotic expansion in $\mathcal{D}'$ can be applied to solutions of partial differential equations.

PROPOSITION 8. Suppose that E is a fundamental solution of the operator

$$L(D) = \sum_{|\alpha| \geq 0} a_\alpha D^\alpha,\ a_\alpha \in R,\ \alpha \in (N \cup 0)^n;\ L(D) \not\equiv 0 \qquad (4.1)$$

such that

$$E(t+h) \overset{s}{\sim} \sum_{n=1}^{\infty} u_n(t,h) \mid \{c_n(h)\}, \ \|h\| \to \infty, \ h \in \Gamma. \tag{4.2}$$

Then there exists a solution X of the equation

$$L(D)\, X = G, \ G \in \mathcal{E}' \tag{4.3}$$

which has S-asymptotic expansion

$$X(t+h) \overset{s}{\sim} \sum_{n=1}^{\infty} (G * u_n(t,h)) \mid \{c_n(h)\}, \ \|h\| \to \infty, \ h \in \Gamma.$$

PROOF. The well-known Malgrange-Ehrenpreis theorem (see for example [7], p. 212) asserts that there exists a fundamental solution of the operator (4.1) belonging to $\mathcal{D}'$. The solution of equation (4.3) exists and can be expressed by the formula $X = E * G$. To find the S-asymptotic of X we have only to apply Propostition 1.

REMARKS. If we denote by A(L(D),E) the collection of those $T \in \mathcal{D}'$ for which the convolution $E * T$ and $L(D)\delta * E * T$ exist in $\mathcal{D}'$, then the solution $X = E * G$ is unique in the class A(L(D),E) ([7], p. 87).

We can enlarge the space to which belongs G ([7], p. 216).

The fundamental solutions are known for the most important operators L(D).

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