SOME APPLICATIONS OF LEGENDRE NUMBERS

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ABSTRACT.   The associated Legendre functions are defined using the Legendre numbers.
From these the associated Legendre polynomials are obtained and the derivatives of
these polynomials at x = 0 are derived by using properties of the Legendre numbers.
These derivatives are then used to expand the associated Legendre polynomials and x^n
in series of Legendre polynomials. Other applications include evaluating certain in-
tegrals, expressing polynomials as linear combinations of Legendre polynomials, and
expressing linear combinations of Legendre polynomials as polynomials. A connection
between Legendre and Pascal numbers is also given.

KEY WORDS AND PHRASES. Associated Legendre functions and polynomials, Legendre poly-
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dre polynomials, Legendre and Pascal numbers.
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1. INTRODUCTION.

     The Legendre numbers were introduced and many of their elementary properties were
developed in [1]. We apply these properties to a variety of problems and the use of
Legendre numbers may provide somewhat simpler solutions to the problems.

2. DERIVATIVES OF ASSOCIATED LEGENDRE POLYNOMIALS AT x = 0.

     For n and s non-negative integers the associated Legendre functions are
defined as usual by

$$P_n^s(x) = (1 - x^2)^{s/2} D^s P_n(x),\qquad (2.1)$$

where $P_n(x)$ is a Legendre polynomial and $D^s \equiv \dfrac{d^s}{dx^s}$. Since $P_n(x)$ can be expressed,
see [1], by using the Legendre numbers, P_n^i, (where $P_n^i = P_n^i(x)]_{x=0}$), as

$$P_n(x) = \sum_{i=0}^{n} \frac{P_n^i x^i}{i!} ,\qquad (2.2)$$

Equation (2.1) becomes

$$P_n^s(x) = (1 - x^2)^{s/2} D^s \sum_{i=0}^{n} \frac{P_n^i x^i}{i!} \tag{2.3}$$

It is clear that $P_n^s(x)$ is a polynomial of degree n for s even. Thus, let $s = 2m$ and recall, see [1], that $P_n^i = 0$ for n and i of different parity. Omitting these null terms from (2.3), one has

$$P_n^{2m}(x) = \begin{cases} (1 - x^2)^m D^{2m} \sum_{i=0}^{\frac{n}{2}} \frac{P_n^{2i} x^{2i}}{(2i)!} \,, & n \text{ even} \\[4mm] (1 - x^2)^m D^{2m} \sum_{i=0}^{\frac{n-1}{2}} \frac{P_n^{2i+1} x^{2i+1}}{(2i+1)!} \,, & n \text{ odd.} \end{cases} \tag{2.4}$$

Taking the indicated derivatives in (2.4) gives

$$P_n^{2m}(x) = \begin{cases} (1 - x^2)^m \sum_{i=0}^{\frac{n-2m}{2}} \frac{P_n^{2m+2i} x^{2i}}{(2i)!} \,, & n \text{ even} \\[4mm] (1 - x^2)^m \sum_{i=0}^{\frac{n-2m-1}{2}} \frac{P_n^{2m+2i+1} x^{2i+1}}{(2i+1)!} \,, & n \text{ odd.} \end{cases} \tag{2.5}$$

To obtain $D^k P_n^{2m}(x)$, Leibniz' Theorem is used. For $0 \le k \le n$, one has from (2.5)

$$D^k P_n^{2m}(x) = \begin{cases} \sum_{i=0}^{k} \left[\binom{k}{i} D^{k-i}(1 - x^2)^m D^i \sum_{i=0}^{\frac{n-2m}{2}} \frac{P_n^{2m+2i} x^{2i}}{(2i)!} \right] \,, & n \text{ even} \\[6mm] \sum_{i=0}^{k} \left[\binom{k}{i} D^{k-i}(1-x^2)^m D^i \sum_{i=0}^{\frac{n-2m-1}{2}} \frac{P_n^{2m+2i+1} x^{2i+1}}{(2i+1)!} \right] \,, & n \text{ odd.} \end{cases} \tag{2.6}$$

From this equation one sees that for n even, $D^k P_n^{2m}(x)]_{x=0} = 0$ for k odd, since in each term the first factor is 0 for i even and the second is 0 for i odd. For n odd, $D^k P_n^{2m}(x)]_{x=0} = 0$ for k even, since in each term the first factor is 0 for i odd and the second is 0 for i even.

One observes that for n even,

$$D^i \sum_{i=0}^{\frac{n-2m}{2}} \frac{P_n^{2m+2i} x^{2i}}{(2i)!} \Bigg]_{x=0} = \begin{cases} 0, & i \text{ odd} \\ 0, & i > n - 2m \\ P_n^{2m+i}, & i \text{ even}, \ i \le n - 2m, \end{cases} \tag{2.7}$$

and for n odd,

$$D^i \sum_{i=0}^{\frac{n-2m-1}{2}} \frac{P_n^{2m+2i+1} x^{2i+1}}{(2i+1)!} \Bigg]_{x=0} = \begin{cases} 0, & i \text{ even} \\ 0, & i > n - 2m \\ P_n^{2m+i}, & i \text{ odd}, \ i \le n - 2m \end{cases} \tag{2.8}$$

Further,

$$D^{k-i}(1-x^2)^m]_{x=0} = \begin{cases} 0, & k-i \text{ odd} \\ 0, & k-i > 2m \\ D^{k-i}\sum_{j=0}^{m}(-1)^j\binom{m}{j}x^{2j}]_{x=0}, & k-i \text{ even, } k-i \le 2m. \end{cases} \qquad (2.9)$$

The derivative on the right in (2.9) can be expressed using factorial notation as

$$\sum_{j=\frac{k-i}{2}}^{m}(-1)^j\binom{m}{j}(2j)^{(k-i)}x^{2j-k+i}\Big]_{x=0}, \quad k-i \text{ even, } k-i \le 2m,$$

which reduces to the first term

$$(-1)^{\frac{k-i}{2}}\begin{bmatrix} m \\ \frac{k-i}{2} \end{bmatrix}(k-i)!, \quad k-i \text{ even, } \frac{k-i}{2} \le m,$$

for $x = 0$. Now (2.9) can be expressed as

$$D^{k-i}(1-x^2)^m]_{x=0} = \begin{cases} 0, & k-i \text{ odd} \\ 0, & k-i > 2m \\ (-1)^{\frac{k-i}{2}}\begin{bmatrix} m \\ \frac{k-i}{2} \end{bmatrix}(k-i)!, & k-i \text{ even, } \frac{k-i}{2} \le m. \end{cases} \qquad (2.10)$$

Using Equations (2.7), (2.8), and (2.10) in (2.5) and the observations following (2.6), one sees that

$$D^k P_n^{2m}(x)]_{x=0} = \begin{cases} 0, & n \text{ even, } k \text{ odd} \\ 0, & n \text{ odd, } k \text{ even} \\ \displaystyle\sum_{i=0}^{k}\frac{(-1)^{\frac{k-i}{2}}k!m!P_n^{2m+i}}{i!(\frac{k-i}{2})!(m-\frac{k-i}{2})!}, & n \text{ and } k \text{ of the same parity}, \end{cases} \qquad (2.11)$$

where a term in the series above is 0 if $k-i$ is odd or if $m < \frac{k-i}{2}$. Also, recall that $P_n^{2m+i} = 0$ for $i > n - 2m$, for i odd and n even, and for i even and n odd.

Equation (2.11) provides a formula for evaluating $D^k P_n^{2m}(x)]_{x=0}$. Of course, the answers obtained by (2.11) agree with those obtained by other methods -- and can be easily verified for small integers k, m, and n.

3. ASSOCIATED LEGENDRE POLYNOMIALS AND x^n AS SERIES OF LEGENDRE POLYNOMIALS.

It is known that an associated Legendre polynomial can be expressed as a series of Legendre polynomials. Equation (2.11) and a table of Legendre numbers, see Table 1, can be used to provide a formula for the coefficients in the series. To outline the method, let

$$P_n^{2m}(x) \equiv \sum_{i=0}^{n}A_i P_i(x). \qquad (3.1)$$

Take n derivatives to obtain n other identities,

$$D^k P_n^{2m}(x) \equiv \sum_{i=0}^{n}A_i D^i P_i(x), \quad k = 1, 2, 3, \ldots, n.$$

In these $n+1$ identities let $x = 0$. Use (2.11) on the left sides of the resulting $n+1$ identities and recall that $D^k P_i(x)]_{x=0} = P_i^k$, see [1]. The right sides can be

simplified by using $P_i^k = 0$ for $k > i$. The system of $n + 1$ identities in $n + 1$ unknowns can be solved for the A_i's and (3.1) gives the desired expansion. The values of the A_i's can be obtained in the order A_n, A_{n-1}, A_{n-2},...,A_0, by

$$
\begin{cases}
A_n = \dfrac{D^n P_n^{2m}(x)]_{x=0}}{P_n^n} \\[4ex]
A_{n-i} = \dfrac{D^{n-i} P_n^{2m}(x)]_{x=0} - \displaystyle\sum_{k=n-i+1}^{n} A_k P_k^{n-i}}{P_{n-i}^{n-i}} \quad , \ 1 \le i \le n \\[5ex]
\phantom{A_{n-i}} = \dfrac{D^{n-i} P_n^{2m}(x)]_{x=0} - \displaystyle\sum_{k=1}^{[\frac{i}{2}]} A_{n-i+2k} P_{n-i+2k}^{n-i}}{P_{n-i}^{n-i}} \quad , \ 1 \le i \le n \ ,
\end{cases}
\qquad (3.2)
$$

where $[\frac{i}{2}] = \frac{i}{2}$ for i even and $\frac{i-1}{2}$ for i odd.

n \ P_n^m	$P_n = P_n^0$	P_n^1	P_n^2	P_n^3	P_n^4	P_n^5	P_n^6	P_n^7	P_n^8 ...
0	1								
1	0	1							
2	$-\dfrac{1}{2}$	0	3						
3	0	$-\dfrac{3}{2}$	0	15					
4	$\dfrac{3}{8}$	0	$-\dfrac{15}{2}$	0	105				
5	0	$\dfrac{15}{8}$	0	$-\dfrac{105}{2}$	0	945			
6	$-\dfrac{15}{48}$	0	$\dfrac{105}{8}$	0	$-\dfrac{945}{2}$	0	10,395		
7	0	$-\dfrac{105}{48}$	0	$\dfrac{945}{8}$	0	$-\dfrac{10,395}{2}$	0	135,135	
8	$\dfrac{105}{384}$	0	$-\dfrac{945}{48}$	0	$\dfrac{10,395}{8}$	0	$-\dfrac{135,135}{2}$	0	2,027,025
⋮	. . .								

TABLE 1.   LEGENDRE NUMBERS P_n^m

Now, x^n can be expanded in a series of Legendre polynomials in a similar way. Let

$$
x^n \equiv \sum_{i=0}^{n} A_i P_i(x), \qquad (3.3)
$$

and proceed as in the derivation of (3.2) to obtain in order A_n, A_{n-1}, A_{n-2},...,A_0, as

$$
A_n = \dfrac{n!}{P_n^n} \ ,
$$

$$
A_i = \begin{cases}
0, \ i = n - k, \ k \ \text{odd} \\[2ex]
-\dfrac{1}{P_i^i} \displaystyle\sum_{j=1}^{\frac{n-i}{2}} P_{i+2j}^i A_{i+2j}, \ \ i = n - k, \ k \ \text{even}.
\end{cases}
\qquad (3.4)
$$

With these values of the A_i's (3.3) gives the desired expansion, which agrees with the known expansion

$$x^n = \frac{n!}{2^n} \sum_{k=0}^{[\frac{n}{2}]} \frac{(2n-4k+1)P_{n-2k}(x)}{k! \left(\frac{3}{2}\right)_{n-k}}. \tag{3.5}$$

4. SOME INTEGRALS INVOLVING LEGENDRE NUMBERS.

In [1], the result

$$\int_0^1 P_n(x)dx = -\frac{P_{n+1}}{n} \tag{4.1}$$

for n any positive integer is given. Here, two other important integrals are expressed in terms of Legendre numbers. It is known, see [2], that if $m = n$, then

$$\int_0^1 P_n(x)P_m(x)dx = \frac{1}{2n+1}. \tag{4.2}$$

More generally, if m and n are different non-negative integers, Rainville, [2], gives the result

$$(n-m)(n+m+1)\int_a^b P_n(x)P_m(x)dx = (1-x^2)[P_m'(x)P_n(x)-P_m(x)P_n'(x)]\Big]_a^b. \tag{4.3}$$

With $a = 0$, $b = 1$, and the results $P_n^m = \dfrac{d^m(P_n(x))}{dx}\Big]_{x=0}$, $P_1^n = P_m^1 = 0$ for m and n even and $P_n = P_m = 0$ for m and n odd from [1], Equation (4.3) becomes

$$\int_0^1 P_n(x)P_m(x) = \frac{P_m^1 P_n - P_m P_n^1}{(m-n)(m+n+1)}$$

$$= \begin{cases} 0, & m \text{ and } n \text{ of the same parity} \\[2mm] \dfrac{P_m^1 P_n}{(m - n)((m+n+1)}, & m \text{ and } n \text{ of different parity.} \end{cases} \tag{4.4}$$

A third integral can be evaluated as shown by the following computation. With $n \geq 1$,

$$\int_0^1 x^n P_{n-2k}(x)dx = \int_0^1 \left[x^n \sum_{m=0}^{n-2k} \frac{P_{n-2k}^m x^m}{m!} \right] dx \qquad \text{(by 2.2)}$$

$$= \int_0^1 \left(\sum_{m=0}^{n-2k} \frac{P_{n-2k}^m x^{m+n}}{m!} \right) dx$$

$$= \sum_{m=0}^{n-2k} \int_0^1 \frac{P_{n-2k}^m x^{m+n}}{m!} dx$$

$$= \sum_{m=0}^{n-2k} \frac{P_{n-2k}^m x^{m+n+1}}{m!(m+n+1)} \Big]_0^1$$

$$= \sum_{m=0}^{n-2k} \frac{P_{n-2k}^m}{m!(m+n+1)}.$$

Therefore,

$$\int_0^1 x^n P_{n-2k}(x)dx = \sum_{m=0}^{n-2k} \frac{P_{n-2k}^m}{m!(m+n+1)}. \tag{4.5}$$

Since the value of this integral is known one can use this value for the series.

Thus,

$$\sum_{m=0}^{n-2k} \frac{P_{n-2k}^m}{m!\,(m+n+1)} = \frac{n!}{2^n \cdot k!\,\left(\frac{3}{2}\right)_{n-k}} \quad .$$ (4.6)

5. POLYNOMIALS AS LINEAR COMBINATIONS OF LEGENDRE POLYNOMIALS.

Since the Legendre polynomials form a simple set, any polynomial of a single variable can be expressed as a finite series of Legendre polynomials. Using Table 2, this can be done in much the same way any polynomial can be expressed in terms of factorials. Consider the problem of expressing

$$H(x) = 5x^3 - 3x^2 + 4x - 3$$

in terms of Legendre polynomials. By continued subtraction of Legendre polynomials we obtain a zero remainder. Detaching coefficients,

$$
\begin{array}{rrrrr}
H(x) = & 5 & -3 & 4 & -3 \\
2P_3(x) = & 5 & & -3 & \\
\hline
\text{Diff.} = & & -3 & 7 & -3 \\
-2P_2(x) = & & -3 & & 1 \\
\hline
\text{Diff.} = & & & 7 & -4 \\
7P_1(x) = & & & 7 & \\
\hline
\text{Diff.} = & & & & -4 \\
-4P_0(x) = & & & & -4 \\
\hline
\text{Diff.} = & & & & 0. \\
\end{array}
$$

Thus

$$H(x) - 2P_3(x) + 2P_2(x) - 7P_1(x) + 4P_0(x) = 0,$$

from which

$$H(x) = 2P_3(x) - 2P_2(x) + 7P_1(x) - 4P_0(x) .$$ (5.1)

n \ Q_n^m	Q_n^0	Q_n^1	Q_n^2	Q_n^3	Q_n^4	Q_n^5	Q_n^6	Q_n^7	Q_n^8
0	1								
1	0	1							
2	$-\frac{1}{2}$	0	$\frac{3}{2}$						
3	0	$-\frac{3}{2}$	0	$\frac{5}{2}$					
4	$\frac{3}{8}$	0	$-\frac{30}{8}$	0	$\frac{35}{8}$				
5	0	$\frac{15}{8}$	0	$-\frac{70}{8}$	0	$\frac{63}{8}$			
6	$-\frac{5}{16}$	0	$\frac{105}{16}$	0	$-\frac{315}{16}$	0	$\frac{231}{16}$		
7	0	$-\frac{35}{16}$	0	$\frac{315}{16}$	0	$-\frac{693}{16}$	0	$\frac{429}{16}$	
8	$\frac{35}{128}$	0	$-\frac{1260}{128}$	0	$\frac{6930}{128}$	0	$-\frac{12,012}{128}$	0	$\frac{6435}{128}$
.	. . .								
.									
.									

TABLE 2. $Q_n^m = \dfrac{P_n^m}{m!}$

For a second method, set

$$5x^3 - 3x^2 + 4x - 3 \equiv AP_3(x) + BP_2(x) + CP_1(x) + DR_0(x).$$

Take the first three derivatives with respect to x, let x = 0 in each of the four identities, and use $P_n^m(0) = P_n^m$ to obtain the system

$$\begin{cases} -3 = AP_3 + BP_2 + CP_1 + DP_0 \\ 4 = AP_3^1 + BP_2^1 + CP_1^1 \\ -6 = AP_3^2 + BP_2^2 \\ 30 = AP_3^3 \end{cases} \qquad (5.2)$$

Next, use Table 1 and solve for A = 2, B = -2, C = 7, and D = -4 in the order listed to obtain (5.1) again.

More generally, if V(x) is a polynomial of degree n in x, write

$$V(x) = \sum_{i=0}^{n} A_i P_i(x), \qquad (5.3)$$

take n derivatives, let x = 0 in the n + 1 identities, and use $P_n^{(m)}(0) = P_n^m$ to obtain the A_i's in the order i = n to i = 0 as

$$\begin{cases} A_n = \dfrac{V^{(n)}(0)}{P_n^n} \\ A_i = \dfrac{V^{(i)}(0) - \displaystyle\sum_{j=n-i+1}^{n} A_j P_j^{n-i}}{P_{n-i}^{n-i}}, \quad i = n-1, n-2, \ldots, 1, 0 \end{cases} \qquad (5.4)$$

Since $P_n^m = 0$ for m + n odd, the second equation of (5.4) can be expressed as

$$A_i = \dfrac{V^{(i)}(0) - \displaystyle\sum_{j=1}^{\frac{n-i}{2}} A_{i+2j} P_{i+2j}^i}{P_i^i}, \quad i = n-1, n-2, \ldots, 1, 0 \qquad (5.5)$$

Table 2 can be used to evaluate a finite series of Legendre polynomials as a polynomial in x. As an example, we evaluate

$$S(x) = P_7(x) - 4P_6(x) - 5P_5(x).$$

Detaching coefficients,

$P_7(x) = 0$	$-\dfrac{35}{16}$	0	$\dfrac{315}{16}$	0	$-\dfrac{693}{16}$	0	$\dfrac{429}{16}$
$-4P_6(x) = \dfrac{5}{4}$	0	$-\dfrac{105}{4}$	0	$\dfrac{315}{4}$	0	$-\dfrac{231}{4}$	
$-5P_5(x) = 0$	$-\dfrac{75}{8}$	0	$\dfrac{350}{8}$	0	$-\dfrac{315}{8}$		
Sum $= \dfrac{5}{4}$	$-\dfrac{185}{16}$	$-\dfrac{105}{4}$	$\dfrac{1015}{16}$	$\dfrac{315}{4}$	$-\dfrac{1323}{16}$	$-\dfrac{231}{4}$	$\dfrac{429}{16}$

Then,

$$S(x) = \frac{5}{4} - \frac{185}{16}x - \frac{105}{4}x^2 + \frac{1015}{16}x^3 + \frac{315}{4}x^4 - \frac{1323}{16}x^5 - \frac{213}{4}x^6 + \frac{429}{16}x^7.$$

6. LEGENDRE AND PASCAL NUMBERS.

Consider Table 3, which gives values for L_n^m. The entries shown are integers, a result that can be easily proved. The alternate diagonals have entries of the form

$$\frac{2^{n+1}P_{n+i}^{n-i}}{(n-i)!} \, , \; i = 0 \; \text{ to } \; n \tag{6.1}$$

reading from upper right to lower left. If $2^n P_n^n / n!$ is factored from

L_n^m \ n	L_n^0	L_n^1	L_n^2	L_n^3	L_n^4	L_n^5	L_n^6	L_n^7	L_n^8
0	1								
1	0	2							
2	-2	0	6						
3	0	-12	0	20					
4	6	0	-60	0	70				
5	0	60	0	-280	0	252			
6	-20	0	420	0	-1260	0	924		
7	0	-280	0	2520	0	-5544	0	3432	
8	70	0	-2520	0	13,860	0	-24,024	0	12,870
·	· · ·								

$$\text{TABLE 3.} \quad L_n^m = 2^n Q_n^m = \frac{2^n P_n^m}{m!}$$

each entry on such a diagonal, the remaining factors are $(-1)^i C(n,i)$. In notation, one has

$$\frac{2^{n+1}P_{n+i}^{n-i}}{(n-i)!} = \frac{(-1)^i 2^n P_n^n C(n,i)}{n!} \, , \; i = 0 \; \text{ to } \; n. \tag{6.2}$$

Equation (6.2) can be simplified to

$$\frac{(-1)^i n! 2^i P_{n+i}^{n-i}}{(n-i)! P_n^n} = C(n,i), \; i = 0 \; \text{ to } \; n, \tag{6.3}$$

which shows a connection between Legendre numbers and Pascal numbers. This result can be easily proved using the general form of the Legendre numbers, P_n^m, given in [1].

REFERENCES

1. HAGGARD, P.W. On Legendre Numbers, International Journal of Mathematics and Mathematical Sciences, Volume 8, Number 2, 1985, 407-411.

2. RAINVILLE, E.D. Special Functions, The Macmillan Company, New York, 1960.

3. COPSON, E.T. An Introduction to the Theory of Functions of a Complex Variable, Oxford University Press, London, 1935.

4. RICHARDSON, C.H. An Introduction to the Calculus of Finite Differences, C. Van Nostrand Company, Inc., New York, 1954.