Internat. J. Math. & Math. Sci.
Vol. 10 No. 4 (1987) 825-827

AN INVERSE PROBLEM FOR HELMHOLTZ'S EQUATION

A.G. RAMM

Mathematics Department
Kansas State University
Manhattan, KS 66506

(Received March 20, 1987)

ABSTRACT. The refraction coefficient in Helmholtz's equation is found from the knowledge of a family of the solutions to this equation on two lines.

KEYS WORDS AND PHRASES. Helmholtz equation, inverse problem, Born approximation.
1980 AMS SUBJECT CLASSIFICATION CODE. 35R30, 35J05.

1. INTRODUCTION.

Let

$$[\nabla^2+k^2+k^2v(x)]u = -\delta(x-y) \ , \quad k > 0 \tag{1.1}$$

where $x = (x_1,x_3)$, $y = (y_1,y_3)$, $v = v(x_1,x_3)$, $u = u(x_1,x_3,y_1,y_3,k)$.

Assume that

$$v(x) = 0 \text{ for } x_1 \geqq a \text{ or } x_1 \leqq -a, \text{ or } x_3 \geqq 0 \text{ or } x_3 \leqq -R, \ v\varepsilon L^2 \tag{1.2}$$

Here $R > 0$ is an arbitrary large fixed number. Write (1.1) as

$$u = g + k^2 \int gvudz, \quad g := (i/4)H_0^{(1)}(k|x-y|) \tag{1.3}$$

where the integral is taken over the support of v and $H_0^{(1)}$ is the Hankel function.

The problem is: find $v(k)$ *from the knowledge of* $u(-a,x_3,a,y_3,k)$ *for all* $-\infty < x_3,y_3 < \infty$ *and* $0 < k < k_0$, *where* $k_0 > 0$ *is an arbitrary small number.*

2. SOLUTION.

Let $L_a = \{x: \ x_1 = a, \ x_3 \ \varepsilon R^1\}$, $R^1 = (-\infty,\infty)$. We use the method given in [1], [2]. It follows from (1.3) that

$$f(x_3,y_3,k) := k^{-2}(u-g) = \int gvgdz + o(k) \text{ as } k\to 0, \ x\varepsilon L_{-a}, \ y\varepsilon L_a. \tag{2.1}$$

Let us take the Fourier transform of (2.1) in x_3 and y_3, define $\tilde{f}(\lambda,\mu) := (2\pi)^{-2} \iint_{-\infty}^{\infty} \exp(-i\lambda x_3-i\mu y_3)f(x_3,y_3)dx_3dy_3$, and use the formula

$$(2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-i\lambda x_3)g(x,z)dx_3 = i(4\pi)^{-1}\exp\left\{-i\lambda z_3+i(a+z_1)(k^2-\lambda^2)^{\frac{1}{2}}\right\}/(k^2-\lambda^2)^{\frac{1}{2}} \tag{2.2}$$

where $x = (-a,x_3)$, the radical $(k^2+io-\lambda^2)^{\frac{1}{2}} > 0$ for $\lambda^2 < k^2$ and is defined by analytic continuation for all complex λ on the complex λ-plane with the cut $(-k,k)$, $k > 0$, so that

$$(k^2-\lambda^2)^{\frac{1}{2}} = i\ (\lambda^2-k^2)^{\frac{1}{2}} \quad \text{if} \quad k^2 < \lambda^2 . \tag{2.3}$$

The result is

$$\tilde{f}(\lambda,\mu) = \int dz v(z) h(\lambda,\mu,z,k) + o(k) \tag{2.4}$$

where for $k^2 > \lambda^2$, $k^2 > \mu^2$, and $r(\lambda) := (k^2-\lambda^2)^{\frac{1}{2}}$ one has

$$h := -(16\pi^2)^{-1}\exp\{-i(\lambda+\mu)z_3 + i(a+z_1)r(\lambda) + i(a-z_1)r(\mu)\}r^{-1}(\lambda)r^{-1}(\mu) \tag{2.5}$$

and for $k^2 < \mu^2$ and $k^2 < \lambda^2$ one uses (2.3).

In the Born approximation one drops the term $o(k)$ in (2.4) and solves the resulting linear integral equation for $v(z)$ [2].

In the exact theory one passed to the limit $k \to 0$ in (2.4), obtains a linear integral equation for v and solves this equation analytically [2]. It is not possible to pass to the limit $k \to 0$ in (2.1) because $g(kr) = \alpha(k) + g_0 + O[(kr)^2\ln(k/2)]$ as $k \to 0$, where $g_o := (2\pi)^{-1}\ln(r^{-1})$, $\alpha(k) := -(2\pi)^{-1}\ln(k/2)+i/4-\gamma/(2\pi)$, and $\gamma = 0.5572 ....$ is Euler's constant. Thus $g(kr)$ does not have a finite limit as $k \to 0$. Nevertheless one can pass to the limit $k \to 0$ in (2.4) if $\gamma \neq 0$ or $\mu \neq 0$. The reason is that the term $\alpha(k)$ in (2.1) after the Fourier transform becomes $\alpha(k)\delta(\lambda)\delta(\mu)$, and this term, which contains the factor $\alpha(k) \to \infty$ as $k \to 0$, is zero for $\lambda \neq 0$ or $\mu \neq 0$. Another way to study the limit behavior of the solution to (2.1) is given in [2]. To give the exact theory, pass to the limit $k \to 0$ in (2.4) to get

$$\int v(z)\ \exp(-ipz_3 + qz_1)dz_1dz_3 = \psi(p,q) \tag{2.6}$$

where we used (2.5) and set

$$p := \lambda+\mu,\ q := |\mu| - |\lambda| , \tag{2.7}$$

$$\psi(p,q) := 16\pi^2\tilde{f}(\lambda,\mu)|\lambda|\ |\mu|\ \{\exp a(|\lambda| + |\mu|)\} \tag{2.8}$$

and the right side of (2.8) should be expressed as a function of (p,q) by formulas (2.7).

If $\mu > 0$ and $\lambda > 0$ then the point (p,q) defined by (2.7) runs through $Q_+ = \{p,q: \ |q| < p\ ,\ p > 0\}$.

If $\lambda < 0$ and $\mu < 0$ then (p,q) runs through $Q_- = \{p,q: \ |q| < -p\ ,\ p < 0\}$. If $\psi(p,q)$ is known in Q_+ or Q_- then $v(z)$ can be uniquely recovered from (2.6) by the analytical methods given in [2] p. 270-274, where inversion of the Fourier and Laplace transforms of compactly supported functions from a compact set is given. This inversion problem is ill-posed and its numerical implementation is not a simple matter.

One can use the same ideas to solve equation (2.4) at a fixed $k > 0$ in the Born approximation. The basic equation analogous to (2.4) for the case when $-k < \lambda$, $\mu < k$, is:

$$\int v(z)\ \exp\{-i(pz_3+q_1z_1)\}dz = f(p,q_1) \text{ for } -k < \mu,\ \lambda < k \tag{2.9}$$

where $p = \lambda+\mu$, $q_1 := r(\mu)-r(\lambda)$,

$$F(p,q_1) := -16\pi^2\ \tilde{f}(\lambda,\mu)r(\lambda)r(\mu)\ \exp\{-ia[r(\lambda)+r(\mu)]\} \tag{2.10}$$

and the right side of (2.10) should be expressed as a function of p,q_1.

If (λ,μ) $\{\lambda,\mu:$ $|\lambda| > k$ and $|\mu| > k, \lambda,\mu$ are real$\}$ then the basic equation in the Born approximation is equation (2.6) in which the right side is now given by the formula $\psi = F$, where F is defined by (2.10) and in (2.10) the radicals $r(\lambda)$ and $r(\mu)$ are computed by formula (2.3) for $\lambda^2 > k^2$ and $\mu^2 > k^2$.

Equation (2.9) can also be solved analytically with the prescribed accuracy by the methods given in [2].

The problem considered is of interest in application.

REFERENCES

1. RAMM, A.G. Inverse Scattering for Geophysical Problems, Phys. Letters 99A (1983), 258-260.

2. RAMM, A.G. Scattering by Obstacles, (Dordrecht: Reidel).