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THE SHALLOW WATER EQUATIONS: CONSERVATION LAWS AND SYMPLECTIC GEOMETRY

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ABSTRACT. We consider the system of nonlinear differential equations governing shallow water waves over a uniform or sloping bottom. By using the hodograph method we construct solutions, conservation laws, and Bäcklund transformations for these equations. We show that these constructions are canonical relative to a symplectic form introduced by Manin.

KEY WORDS AND PHRASES. Shallow water waves, hodograph transformation, simple-wave, Bäcklund transformation, Hamiltonian formalism, symplectic, completely integrable system.
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1. INTRODUCTION.

In recent years there has been a revival of interest in the subject of shallow water wave equations. Numerous articles have been appearing on the subject. This can be explained by all the recent work done on completely integrable systems, their soliton solutions and conservation laws. In this article we are incorporating our work on shallow water wave equations which was published in [1], [2] and was announced in [3].

We consider the following system of equations governing two-dimensional shallow water waves over a uniform (homogenous case, $\beta = 0$) or sloping (non-homogenous case, $\beta \neq 0$) bottom:

$$u_t + uu_x + h_x = g\beta, \tag{1.1a}$$

$$h_t + hu_x + uh_x = 0, \tag{1.1b}$$

where $0 < X < \infty$ is the horizontal coordinate, t is the time, $u(t, x)$ is the horizontal component of the velocity at the point x at time t, and $h(t, x)$ is the depth of the free surface below the point x at time t. In both cases the nonlinear equations can be reduced to linear ones in the hodograph plane, upon which the derivation of conservation laws and solutions of these equations is based. It was interesting for us to see that from a single equation one could construct explicit solutions of the homogenous and nonhomogenous systems as well as conservation laws and Bäcklund transformations associated with them [1].

Manin [4] cast the homogenous system into a Hamiltonian formalism and the symplectic geometry of the homogenous system ($\beta = 0$) has been studied by Kupershmidt and Manin [5], and Lebedev and Manin [6]. We extend Manin's construction to the sloping bottom case (nonhomogenous system, $\beta \neq 0$) and show that the solution space of the above mentioned wave equation (from which we obtain explicit solutions, conservation laws, and Bäcklund transformations) is isotropic relative to the accompanying symplectic form; thus, we simultaneously prove that the conserved quantities are in involution and that the Bäcklund transformations are symplectic, i.e., canonical.

Whether the isotropic space of the solutions of the above mentioned wave equation is Lagrangian is an important question. This is, in fact, the problem of complete integrability of an infinite dimensional Hamiltonian system. See pp. 7 and 9 of Cavalcante and McKean [7] on this important issue.

2. HOMOGENOUS CASE (UNIFORM BOTTOM).

The system (1.1) with $\beta = 0$ is a pair of quasi-linear partial differential equations with no explicit (t, x) dependence. Hence, for any region where the Jacobian $j = u_x h_t - u_t h_x$ is non-zero, (1.1) can be transformed into an equivalent linear system by interchanging the roles of dependent and independent variables, (u,h) and (t, x), respectively. This is a so-called hodograph transformation. Since the system (1.1) is homogenous for $\beta = 0$, from $u_x = jt_h$, $u_t = -jx_h$, $h_x = -jt_u$, $h_t = jx_u$, we see that the highly nonlinear factor j cancels through in (1.1), and we arrive at the following linear differential equations

$$x_u = ut_u - ht_h, \tag{2.1a}$$

$$\text{or } \nabla x = \begin{bmatrix} u & -h \\ -1 & u \end{bmatrix} \nabla t,$$

$$x_h = -t_u + ut_h, \tag{2.1b}$$

where ∇ is the gradient operator $(\partial/\partial u, \partial/\partial h)$ on the (u, h)-plane. By eliminating x in (2.1) we obtain the linear equation

$$t_{uu} = 2t_h + ht_{hh}, \tag{2.2}$$

whose solutions can easily be found in standard tables.

Since the application of the hodograph transformation depends on the assumption that $j \neq 0$, solutions for which $j = 0$ cannot be obtained by this method. Such solutions are called simple waves, and they are important tools for the solution of flow problems; for instance, wave-breaking occurs when $j = 0$ due to the multivaluedness, i.e., shocks. As an example, the solution

$$u = 2x/3t, \quad h = (x/3t)^2, \tag{2.3}$$

which is found in Nutku [8] by a scale-invariance argument, represents a simple wave. So, we could not possibly obtain it by the hodograph method.

The system of equations (2.1) can also be written in the following equivalent form

$$(x - ut)_u = -(ht)_h, \tag{2.4a}$$

$$(x - ut)_h = -t_u \,. \tag{2.4b}$$

These, in return, suggest the existence of potentials $\Psi(u, h)$ and $\Phi(u, h)$ satisfying

$$\Psi_u = -ht, \quad \Psi_h = x - ut, \tag{2.5a}$$

$$\Phi_u = x - ut, \quad \Phi_h = -t. \tag{2.5b}$$

Solving (2.5) for x and t we obtain

$$x = \Psi_h - \frac{u}{h}\Psi_u, \quad t = -\frac{1}{h}\Psi_u, \tag{2.6a}$$

and

$$x = \Phi_u - u\Phi_h, \quad t = -\Phi_h. \tag{2.6b}$$

Combining (2.6a) with (2.4b) gives the following wave equation

$$\Psi_{uu} = h\Psi_{hh}. \tag{2.7}$$

Similarly, (2.6b) together with (2.4a) gives

$$\Phi_{uu} = (h\Phi_h)_h. \tag{2.8}$$

Thus, the potentials Ψ and Φ satisfy linear equations whose solutions can be obtained by standard methods. From any one of these potentials, via (2.6), we can easily construct hodograph solutions of our original system (1.1). As an example, we give the following solution

$$h = x + t^2/2, \quad u = -t, \tag{2.9}$$

obtained from $\Psi = \frac{1}{2}(u^2h + h^2)$. (2.10)

3. NONHOMOGENOUS CASE (SLOPING BOTTOM).

In 1958, Carrier and Greenspan [9] applied Riemann's characteristic forms together with a hodograph transformation to the system (1.1) with $\beta \neq 0$ and obtained solutions for this system in terms of the solutions of the wave equation (2.7) above. Explicitly, their formulas read

$$g\beta x = -\Psi_h + \Psi_v^2/2h^2 + h, \tag{3.1a}$$

$$g\beta t = \Psi_v/h - v, \tag{3.1b}$$

where $v = u - g\beta t$ and Ψ satisfies the wave equation

$$\Psi_{vv} = h\Psi_{hh}. \tag{3.2}$$

Again, we miss the so-called simple-wave solutions which correspond to the case in which v and h are functionally dependent.

It is interesting to note that by letting $\beta = 0$ in (3.1) we obtain

$$\Psi = \frac{1}{2}(u^2h + h^2)$$

with the corresponding solution (2.9) in Section 2 of this article. That is, we do not get all the solutions of the homogenous system by setting the nonhomogenous term to zero in the solutions of the nonhomogenous problem. This is quite contrary to what happens in the linear case.

We note that the equations (2.7) and (3.2) have the same form. Hence, after the necessary relabelling of the variables, a solution of any of them can be used to generate a solution to either of the problems: homogenous and nonhomogenous. In this way, we find a correspondence between the non-simple wave solutions of the two systems. This correspondence can be thought of as a Bäcklund transformation between the homogenous and nonhomogenous problems. By using the linear nature of the same wave equation, we can also consruct auto-Bäcklund transformations for each system. These are, of course, nothing but superposition principles for the nonlinear systems under consideration: Given two solutions of (1.1), add the corresponding solutions of (3.2), and then construct the solution of (1.1) corresponding to this sum.

In Akyildiz [1], we were also able to construct polynomial conserved quantities for the nonhomogenous system (1.1) in the form $\int_0^\infty \Psi dx$, where Ψ is a solution of the same wave equation (3.2). (The transformation equation between this article and [1] is $h = c^2$). In this way, via the solutions of (3.2), we establish correspondence between conservation laws, nonsimple wave solutions and Bäcklund transformations of both the homogenous and nonhomogenous systems. (Perhaps we ought to call (3.2) the moduli equation for the system (1.1) since the solution space of (3.2) parameterizes the non-simple solutions as well as the conservation laws of the system (1.1)).

4. SYMPLECTIC GEOMETRY.

Finally, we shall cast the system (1.1) into Hamiltonian form and show that the solution space of the wave equation (3.2) is isotropic relative to the accompanying symplectic form (Poisson bracket); hence, proving that all the constructions carried out above are canonical. After absorbing the nonhomogenous term $g\beta t$ in (1.1a) as

$$(u - g\beta t)_t + uu_x + h_x = 0, \tag{4.1}$$

we can introduce the following Hamiltonian formalism for the system (1.1):

$$\begin{bmatrix} u - g\beta t \\ h \end{bmatrix}_t = J\nabla H, \quad J = -\begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix}, \quad H = \frac{1}{2}(u^2h + h^2), \tag{4.2}$$

where J is the Hamiltonian operator, H the Hamiltonian, D differentiation with respect to x, and ∇ the gradient operator in (v, h)-space with $v = u - g\beta t$.

Since $u_x = v_x$, we equivalently have

$$\begin{bmatrix} v \\ h \end{bmatrix}_t = J\nabla H, \text{ with } H = \frac{1}{2}(v^2h + h^2). \tag{4.3}$$

The associated Poisson bracket is defined to be

$$[A, B] = \int_0^\infty \nabla A J \nabla B \, dx \tag{4.4}$$

for two functions A and B of the variables v and h. It is easy to verify that (4.4) satisfies the Jacobi identity. We assume that the boundary conditions $u = 0$, $h = 0$ are satisfied at $x = 0$ and ∞. Since $v = u - g\beta t$, in order to have meaningful space integrals in (4.4), we must not have v's appearing on their own in our expressions (because t may become arbitrarily large). v's must always come multiplied with u's or c's. This was an essential point in constructing conserved quantities for the system (1.1) in our work [1], cf. p. 1727.

Now, we shall show that the Poisson bracket (4.4) vanishes on the solution space of the wave equation (3.2), which parameterizes both the non-simple wave solutions and conservation laws of the system (1.1): Let A and B be two solutions of (3.2). The integrand in (4.4) is

$$(A_v DB_h + A_h DB_v)dx = A_v dB_h + A_h dB_v. \tag{4.5}$$

Since

$$\begin{aligned} d(A_v dB_h + A_h dB_v) &= dA_v \wedge dB_h + dA_h \wedge dB_v \\ &= (A_{vv}dv + A_{vh}dh) \wedge (B_{hv}dv + B_{hh}dh) \\ &\quad + (A_{hv}dv + A_{hh}dh) \wedge (B_{vv}dv + B_{vh}dh) \\ &= (A_{vv}B_{hh} - A_{hh}B_{vv})dv \wedge dh \\ &= 0 \qquad \text{by (3.2)} \end{aligned}$$

the integrand in (4.4) is closed and is, therefore, exact on simply connected regions; that is,

$$[A, B] = \int_0^\infty dC = 0,$$

for a function, C, of the arguments v and h. This finishes the proof that the Poisson bracket (4.4) vanishes on the solution space of the wave equation (3.2), from which we obtain solutions, Bäcklund transformations and conservation laws for the system (1.1). Thus, we have simultaneously proved that the Bäcklund transformations above are symplectic and that conserved quantities found in Akyildiz [1] for the nonhomogenous system (1.1) are in involution relative to the symplectic structure introduced in this article.

Recently, Professor Yu. I. Manin informed us that his paper with B. A. Kuperschmidt [5] also contains explicit solutions to the homogenous system ($\beta = 0$) of equations (1.1). See (also) the book by Stoker [10], p. 337. We would like to thank Professor Manin for his cooperation.

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