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ON THE AFFINE WEYL GROUP OF TYPE $\tilde{A}_{n-1}$

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ABSTRACT. We study in this paper the affine Weyl group of type $\tilde{A}_{n-1}$, [1]. Coxeter [1] showed that this group is infinite. We see in Bourbaki [2] that $\tilde{A}_{n-1}$ is a split extension of S_n, the symmetric group of degree n, by a group of translations and of a lattice of weights. $\tilde{A}_{n-1}$ is one of the crystallographic Coxeter groups considered by Maxwell [3], [4].

We prove the following:

THEOREM 1. $\tilde{A}_{n-1}$, $n \geq 3$ is a split extension of S_n by the direct product of $(n-1)$ copies of Z.

THEOREM 2. The group $\tilde{A}_2$ is soluble of derived length 3, $\tilde{A}_3$ is soluble of derived .ength 4. For $n > 4$, the second derived group $\tilde{A}''_{n-1}$ coincides with the first $\tilde{A}'_{n-1}$ and so $\tilde{A}_{n-1}$ is not soluble for $n > 4$.

THEOREM 3. The center of $\tilde{A}_{n-1}$ is trivial for $n \geq 3$.

KEY WORDS AND PHRASES. Presentation, Reidemeister-Schreier method, Coxeter group.
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1. INTRODUCTION.

Consider the presentation

$$\begin{aligned} \tilde{A}_{n-1} = \langle y_1, y_2, \ldots, y_n \mid\ & y_i^2 = e \quad \text{if } 1 \leq i \leq n, \\ & y_i y_{i+1} y_i = y_{i+1} y_i y_{i+1} \quad \text{if } 1 \leq i \leq n-1, \\ & y_i y_j = y_j y_i \quad \text{if } 1 \leq i < j-1 < n \text{ and } (i,j) \neq (1,n), \\ & y_1 y_n y_1 = y_n y_1 y_n \rangle \end{aligned}$$

where $n \geq 3$.

This is an irreducible Coxeter group whose graph is a polygon with n vertices. Using some geometrical methods Coxeter showed that $\tilde{A}_{n-1}$ is infinite [4]. This group is also a Weyl group [1]. It is the affine Weyl group of type $\tilde{A}_{n-1}$. We see in Bourbaki [2] that $\tilde{A}_{n-1}$ is a split extension of S_n, the symmetric group of degree n, by a group of translations and of a lattice of weights. This group was also considered by Maxwell [3], [4].

The purpose of this paper is to prove that $\tilde{A}_{n-1}$ is a split extension of S_n by a direct product of $(n-1)$ copies of Z. The method depends on presentations of group extension [5]. We also find that $\tilde{A}_3$ is soluble of derived length 3, $\tilde{A}_4$ is soluble of derived length 4 and that the second derived group $\tilde{A}''_{n-1}$ coincides with the first $\tilde{A}'_{n-1}$ if $n > 4$ and hence $\tilde{A}_{n-1}$ is not soluble in this case. We finally show that the center of $\tilde{A}_{n-1}$ is trivial.

2. THE STRUCTURE OF $\tilde{A}_{n-1}$.

We show in this section that $\tilde{A}_{n-1}$ is a split extension of S_n by the direct product of $(n-1)$ copies of Z. We achieve this by using the method in [5] as follows. We find an epimorphism $\theta: \tilde{A}_{n-1} \to S_n$ such that the extension

$$1 \longrightarrow \ker\theta \longrightarrow \tilde{A}_{n-1} \longrightarrow S_n \longrightarrow 1 \tag{2.1}$$

splits. It will be required to find a presentation for $\ker\theta$. We guess that it will be isomorphic to $A = Z^{\times(n-1)}$ (given by generators and relations). We then construct a new short exact sequence (2.3), where A is embedded as normal subgroup of a group E in such a way that A is the kernel of an epimorphism $\theta': E \longrightarrow G$.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \ker\theta & \longrightarrow & \tilde{G} & \xrightarrow{\theta} & G & \longrightarrow & 1 \qquad (2.2) \\ & & \uparrow\vdots & & \uparrow\vdots & & \uparrow & & \\ 1 & \longrightarrow & A & \longrightarrow & E & \xrightarrow{\theta'} & G & \longrightarrow & 1 \qquad (2.3) \end{array}$$

Then we use Tietze transformations to identify E with $\tilde{G}$, i.e., to find an isomorphism $\phi: E \longrightarrow \tilde{G}$, which makes the right-hand square commute. It then follows that $A = \ker\theta$. A presentation for the symmetric group of degree $n \geq 2$ is

$$S_n = \langle x_1, \ldots, x_{n-1} \mid x_i^2 = e \text{ if } 1 \leq i \leq n-1,$$
$$x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \text{ if } 1 \leq i \leq n-2,$$
$$x_i x_j = x_j x_i \text{ if } 1 \leq i < j-1 \leq n-1\rangle.$$

We define the mapping $\theta: \tilde{A}_{n-1} \longrightarrow S_n$ by

$$\theta: y_i \longrightarrow x_i \text{ if } 1 \leq i \leq n-1$$
$$y_n \longrightarrow x_1 x_2 \ldots x_{n-2} x_{n-1} x_{n-2} \ldots x_2 x_1.$$

Then θ is an epimorphism. If α is the mapping from S_n to $\tilde{A}_{n-1}$ defined by

$$\alpha: x_i \longrightarrow y_i \text{ if } 1 \leq i \leq n-1,$$

then α is a homomorphism and $\alpha\theta = 1_{S_n}$.

Thus the extension

$$1 \longrightarrow \ker\theta \rightarrow \tilde{A}_{n-1} \underset{\alpha}{\overset{\longrightarrow}{\longleftarrow}} S_n \xrightarrow{\theta} 1 \quad \text{splits.}$$

We construct the short exact sequence

$$1 \longrightarrow A \longrightarrow E \longrightarrow S_n \longrightarrow 1.$$

A presentation of E will be

$$E = \langle \text{generators of } A, \text{ generators of } S_n \mid \text{relations of } A, \text{ relations of } S_n, \text{ action of } S_n \text{ on } A\rangle \text{ [6].}$$

Let $A = \langle a_1, \ldots, a_{n-1} \mid a_i a_k = a_k a_i \quad \text{if} \quad 1 \leq i \leq k < n-1\rangle$ (2.4)

We define the action of S_n on A as follows:

$$a_1^{x_1} = a_1^{-1} \qquad (2.5)$$

$$a_i^{x_1} = a_1^{-1} a_i \quad \text{if} \quad 2 \leq i \leq n-1 \qquad (2.6)$$

$$a_k^{x_i} = \begin{cases} a_{k+1} & \text{if} \quad i = k+1, \quad 1 \leq k \leq n-1 \qquad (2.7) \\ a_{k-1} & \text{if} \quad i = k, \quad 2 \leq i \leq n-1 \qquad (2.8) \\ a_k & \text{otherwise} \qquad (2.9) \end{cases}$$

NOTATION. We let $\Delta_i = x_2 x_3 \ldots x_i$. We also denote the relations $xyx = yxy$ and $ab = ba$ by (x,y) and $[a,b]$ respectively.

To reduce the relations of E to a manageable form we consider the following lemma and proposition.

LEMMA 1. In the group S_n the following identities hold:

(i) $\Delta_k x_i = x_{i+1}\Delta_k$ if $2 \leq i < k$

(ii) $\Delta_k x_i = \Delta_{k-1}$ if $i = k$

(iii) $\Delta_k x_i = \Delta_{k+1}$ if $i = k+1$

(iv) $\Delta_k x_i = x_i \Delta_k$ if $i > k+1$

(v) $\Delta_k \Delta_i = x_3 \ldots x_{i+1}\Delta_k$ if $2 \leq i < k$

(vi) $\Delta_i^2 = x_3 \ldots x_i \Delta_{i-1}$.

PROOF. (i)

$$\begin{aligned} \Delta_k x_i &= x_2 x_3 \ldots x_{i-1} x_i x_{i+1} \cdots x_k x_i \\ &= x_2 \ldots x_{i-1} x_i x_{i+1} x_i \cdots x_k \\ &= x_2 \ldots x_{i-1} x_{i+1} x_i x_{i+1} \cdots x_k \\ &= x_{i+1}\Delta_k . \end{aligned}$$

(ii) to (iv) obvious.

(v) and (vi) application of (i).

PROPOSITION 1. In the group E, relations (2.4) to (2.9) become the following:

(i) Relation (2.5) is equivalent to $(a_1 x_1)^2 = e$.

(ii) Relation (2.7) is equivalent to $a_i = a_1^{\Delta_i}$ $2 \leq i \leq n-1$.

(iii) Relation (2.6) is equivalent to $(a_1 x_1, x_2)$.

(iv) Relation (2.8) follows from (ii).

(v) Relation (2.9) is equivalent to $[a, x_i]$ for $3 \leq i \leq n-1$.

(vi) Relation (2.4) is equivalent to $(x_2 a_1)^2 = (a_1 x_2)^2$.

PROOF. (i) Obvious

(ii) Easy by induction on i.

(iii) Using part (ii) relation (2.6) becomes

$$x_1\Delta_i^{-1}a_1\Delta_i x_1 = a_1^{-1}\Delta_i^{-1}a_1\Delta_i .$$

Using relation (2.9) it reduces to (a_1x_1, x_2).

(iv) Obvious by using part (ii).

(v) Using part (ii) relation (2.9) becomes

$$\Delta_k x_i \Delta_k^{-1} a_1 = a_1\Delta_k x_i \Delta_k^{-1}, \quad i \neq k, \quad i \neq k+1.$$

If $i > k+1$, then by Lemma 1 (iv) we get

$[x_i, a_1]$ for $3 \leq i \leq n-1$.

If $i < k$ then by Lemma 1 (i), we get

$[x_{i+1}, a_1]$ for $2 \leq i \leq n-1$.

Therefore relation (2.9) is equivalent to $[a_1, x_i]$ for $3 \leq i \leq n-1$.

(vi) Using part (ii) relation (2.4) becomes

$$\Delta_k\Delta_i^{-1}a_1\Delta_i\Delta_k^{-1}a_1 = a_1\Delta_k\Delta_i^{-1}a_1\Delta_i\Delta_k^{-1}, \quad 1 \leq i < k \leq n-1.$$

Using Lemma 1 (v) and relation (2.9), we get

$$(x_2a_1)^2 = (a_1x_2)^2.$$

THEOREM 1. The group E is isomorphic to $\tilde{A}_{n-1}$ and so $\tilde{A}_{n-1}$ is a split extension of S_n by A where $n \geq 3$.

PROOF. In Proposition 1, we let $a_1x_1 = b$. Then E has the following generators: $x_1, x_2, \ldots, x_{n-1}, b$. Relations of E are:

Relations of S_n,

$$b^2 = e , \tag{2.10}$$

$$(b, x_2) \tag{2.11}$$

$$[bx_1, x_i] \text{ for } 3 \leq i \leq n-1 \tag{2.12}$$

$$(x_2bx_1)^2 = (bx_1x_2)^2. \tag{2.13}$$

We change relation (2.13) to the form

$$(b, x_1x_2x_1). \tag{2.14}$$

We change relation (2.12) to $[b, x_i]$ for $3 \leq i \leq n-1$ (2.15)

We let $c = \Delta_{n-1}^{-1}b\Delta_{n-1}$. Then $c^2 = e$.

Using relation (2.11) and Lemma 1 (i), we get (c, x_1).

$$x_{n-1}\, cx_{n-1} = \Delta_{n-1}^{-1}b\Delta_{n-1} .$$

Using Lemma 1 (ii) and (v) and (2.15)

$$cx_{n-1}c = \Delta_{n-1}^{-2}b\Delta_{n-1}$$

Using Lemma 1 (vi)

$$cx_{n-1}c = \Delta_{n-2}^{-1}b\Delta_{n-2}^{2} = x_{n-1}cx_{n-1}.$$

Therefore (c, x_{n-1}).

Using Lemma 1 (i) and (2.15) we get

$$[c, x_i] \text{ for } 2 \leq i < n-1.$$

Thus E has the following presentation

$$\begin{aligned} E = \langle x_1, \ldots, x_{n-1}, c | x_i^2 = e \text{ for } 1 \leq i \leq n-1, \\ c^2 = e, \\ (x_i, x_{i+1}) \text{ for } 1 \leq i \leq n-2 \\ [x_i, x_k] \text{ for } 1 \leq i < k-1 < n-1, \\ (x_{n-1}, c), (x_1, c), \\ [x_i, c] \text{ for } 2 \leq i \leq n-1\rangle. \end{aligned}$$

Let $c = x_n$. Then it is clear that E is the same as $\tilde{A}_{n-1}$ and the theorem is proved.

REMARK 1. We notice the special cases $\tilde{A}_0 = S_2 = Z_2$.

$\tilde{A}_1 = S_3$.

$\tilde{A}_2 = \Delta(3, 3, 3)$ the triangle group $\Delta(3, 3, 3)$ [6].

REMARK 2. We used the Reidemeister-Schreier process to find $A = \ker\theta$ for $n = 3, 4$. From the computations involved we found the action of S_n on A. For $n \geq 5$, we guessed that $A = Z^{\times(n-1)}$ and the action is a generalization for the case when $n = 3,4$. We then proved this guess by the method in [6].

3. THE DERIVED SERIES OF $\tilde{A}_{n-1}$.

We prove in this section the following theorem:

THEOREM 2. The group $\tilde{A}_3$ is soluble of derived length 3, $\tilde{A}_4$ is soluble of derived length 4. For $n > 4$, the second derived group $\tilde{A}''_{n-1}$ coincides with the first $\tilde{A}'_{n-1}$ and so $\tilde{A}_{n-1}$ is not soluble for $n > 4$.

To prove the theorem we consider the derived series of $\tilde{A}_{n-1}$. We notice that $\frac{\tilde{A}_{n-1}}{\tilde{A}'_{n-1}} = \langle y_1 | y_1^2\rangle$. Hence $\{e, y_1\}$ is a transversal for $\tilde{A}'_{n-1}$ in $\tilde{A}_{n-1}$. Using the Reidemeister-Shreier process we find the following presentation for $\tilde{A}'_{n-1}$:

$$\begin{aligned} \tilde{A}'_{n-1} = \langle b_1, b_2, \ldots, b_{n-1} | b_1^3 = b_i^2 = b_{n-1}^3 \text{ if } 1 \leq i \leq n-2, \\ (b_i b_{i+1}^{-1})^3 = e \text{ if } 1 \leq i \leq n-2, \\ (b_i b_j^{-1})^2 = e \text{ if } 1 \leq i \leq j-1 < n-1\rangle. \end{aligned}$$

We now consider the following cases:

i) If $n = 3$, $\frac{\tilde{A}_2'}{\tilde{A}_2''} = \langle b_1, b_2 | b_1^3 = b_2^3 = [b_1, b_2] = e\rangle$.

Using the Reidemeister-Schreier process we find that $A_3'' = z \times z$.

Therefore $\tilde{A}_2$ is soluble of derived length 3.

ii) If $n = 4$, $\frac{\tilde{A}_3'}{\tilde{A}_3''} = \langle b_1 | b_1^3 = e\rangle$. We use the Reidemeister-Schreier process to find the following presentation for $\tilde{A}_4''$

$$\tilde{A}_4'' = \langle x,y,z,t | x^2 = y^2 = z^2 = t^2 = [x,z] = [y,t] = e$$

$$xytz\ xtzy = e\rangle.$$

$$\frac{\tilde{A}_4''}{\tilde{A}_4'''} = \langle x,y,z,t | x^2 = y^2 = z^2 = t^2 = [x,y] = [x,z] = [x,t] =$$

$$[y,x] = [y,t] = [z,t] = e\rangle.$$

We use the Reidemeister-Schreier process to find that $\tilde{A}_4''' = z \times z$. Therefore $\tilde{A}_4$ is soluble at derived length 4.

iii) If $n > 4$, $\frac{\tilde{A}_{n-1}'}{\tilde{A}_{n-1}''}$ is trivial. So the second derived group $\tilde{A}_{n-1}''$ coincides with first derived group $\tilde{A}_{n-1}$. Hence $\tilde{A}_{n-1}$ is not soluble for $n > 4$.

4. THE CENTER OF $\tilde{A}_{n-1}$.

We prove in this section that the center of $\tilde{A}_{n-1}$ is trivial for $n \geq 3$.

LEMMA 2. The identity of A is the only element fixed by S_n.

PROOF. We let w be an element of A. We can write w in the form $a_1^{m_1} a_2^{m_2} \ldots a_{n-1}^{m_{n-1}}$ where $m_j \in Z$ for $1 \leq j \leq n-1$. Let $w^{x_i} = w$ for $1 \leq i \leq n-1$. We therefore get the equation

$$\left[a_1^{m_1} a_2^{m_2} \ldots a_{n-1}^{m_{n-1}}\right]^{x_i} = a_1^{m_1} a_2^{m_2} \ldots a_{n-1}^{m_{n-1}} \tag{4.1}$$

for $1 \leq i \leq n-1$.

Using the action of S_n on A [in Section 2] equation (4.1) for $i = 1$ implies

$$a_1^{2m_1+m_2+\ldots+m_{n-1}} = e.$$

Since A is free abelian this equation gives

$$2m_1 + m_2 + \ldots + m_{n-1} = 0. \tag{4.2}$$

Using the action of S_n on A, equation (4.1) for $2 \leq i \leq n-1$ implies

$$a_{i-1}^{m_i - m_{i-1}} = a_i^{m_i - m_{i-1}}.$$

Since A is free abelian this gives

$$m_i - m_{i-1} = 0 \quad \text{for} \quad 2 \leq i \leq n-1. \tag{4.3}$$

From (4.2) and (4.3) we get $m_1 = m_2 = \ldots = m_{n-1} = 0$. Therefore $w = e$ as required.

THEOREM 3. The center of $\tilde{A}_{n-1}$ is trivial for $n \geq 3$.

PROOF. We know that

$$1 \longrightarrow A \longrightarrow \tilde{A}_{n-1} \longrightarrow S_n \longrightarrow 1.$$

We let $x \in Z(\tilde{A}_{n-1})$ so $x = as$ where $a \in A$ and $s \in S_n$. We let $x_1 = a_1 s_1$ be a typical element of $\tilde{A}_{n-1}$. Hence $xx_1 = x_1 x$ implies $asa_1s_1 = a_1s_1as$. Applying the epimorphism θ we get $\theta(s)\theta(s_1) = \theta(s_1)\theta(s)$ and so $\theta(s) \in Z(S_n) = \{e\}$. Hence $s \in \ker\theta = A \cap S_n \Rightarrow s = e$. Therefore $x = a$ commutes elementwise with S_n. Using Lemma 3, $a = e$ and so $Z(\tilde{A}_{n-1}) = \{e\}$.

REMARK 3. From Remark 1 we notice that $Z(\tilde{A}_0) = Z$ and $Z(\tilde{A}_1) = Z(S_3) = \{e\}$.

REMARK 4. We notice that $\frac{\tilde{A}_{n-1}}{A} \cong S_n$ from Theorem 1. Since S_3 and S_4 are soluble of length 3 and 4 respectively, we get that $\tilde{A}_2$ and $\tilde{A}_3$ are soluble of length 3 and 4 respectively. S_n is not soluble for $n > 4$ and A is soluble, it follows that $\tilde{A}_{n-1}$ is not soluble for $n > 4$.

REMARK 5. One way to view $\tilde{A}_{n-1}$ is as a subgroup of the wreath product $Z \wr S_n$ defined as follows: Let Z^{xn} be the free abelian group with base $P_0, \ldots, P_{n-1}$ on which S_n acts by permuting the basis, $x_i = (i-1, i)$, exchanges P_{i-1} and P_i and fixes the others. The subgroup $\{P_0^{k_0} \ldots P_{n-1}^{k_{n-1}} \mid \sum_{j=0}^{n-1} k_n = 0\} = H$ is S_n-invariant, and has basis $\{a_i = P_i - P_0 \mid 1 \leq i \leq n-1$, and $\tilde{A}_{n-1}$ is just this split extension of S_n by H. Therefore $\tilde{A}_{n-1}$ is the subgroup of the natural wreath product of $Z \wr S_n$ consisting of those elements in which the component from the base group has exponent sum zero.

REMARK 6. The motivation behind studying this group $\tilde{A}_{n-1}$ was to get some information about the circular braid group $\hat{B}_n$ [7]. We see that $\tilde{A}_{n-1}$ is the Coxeter group corresponding to the Artin group $\hat{B}_n$. Consider the diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & Y & \longrightarrow & F & \longrightarrow & Z^{x(n-1)} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & X & \longrightarrow & \hat{B}_n & \longrightarrow & A_{n-1} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & \longrightarrow & U_n & \longrightarrow & B_n & \longrightarrow & S_n & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Here B_n is Artin's braid group [6], U_n the unpermuted braid group, F a free group of countably infinite rank [7] and $Z^{x(n-1)}$ as described in this paper. Knowing $Z^{x(n-1)}$ did not help us to describe the structure of $\hat{B}_n$ which was described in a different way [7]. We are still unable to find the groups X and Y.

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