Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2009, Article ID 962303, 7 pages
doi:10.1155/2009/962303

Research Article

Strong Convergence Theorems for Infinitely Nonexpansive Mappings in Hilbert Space

Yi-An Chen

College of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, China

Correspondence should be addressed to Yi-An Chen, chenyian1969@sohu.com

Received 23 June 2009; Accepted 12 October 2009

Recommended by Anthony To Ming Lau

We introduce a modified Ishikawa iterative process for approximating a fixed point of two infinitely nonexpansive self-mappings by using the hybrid method in a Hilbert space and prove that the modified Ishikawa iterative sequence converges strongly to a common fixed point of two infinitely nonexpansive self-mappings.

Copyright © 2009 Yi-An Chen. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Let C be a nonempty closed convex subset of a Hilbert space H, T a self-mapping of C. Recall that T is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in C$.

Construction of fixed points of nonexpansive mappings via Mann's iteration [1] has extensively been investigated in literature (see, e.g., [2–5] and reference therein). But the convergence about Mann's iteration and Ishikawa's iteration is in general not strong (see the counterexample in [6]). In order to get strong convergence, one must modify them. In 2003, Nakajo and Takahashi [7] proposed such a modification for a nonexpansive mapping T.

Consider the algorithm,

$$
\begin{gathered}
x_0 \in C \text{ chosen arbitrarity,} \\
y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\
C_n = \{v \in C : \|y_n - v\| \le \|x_n - v\|\}, \\
Q_n = \{v \in C : \langle x_n - v, x_n - x_0 \rangle \le 0\}, \\
x_{n+1} = P_{C_n \cap Q_n}(x_0),
\end{gathered}
\tag{1.1}
$$

where P_C denotes the metric projection from H onto a closed convex subset C of H. They prove the sequence $\{x_n\}$ generated by that algorithm (1.1) converges strongly to a fixed point of T provided that the control sequence $\{\alpha_n\}$ is chosen so that $\sup_{n\geq 0}\alpha_n < 1$.

Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings of C, $\{\lambda_n\}_{n=1}^{\infty}$ a sequence of nonnegative numbers in $[0,1]$. For each $n \geq 1$, defined a mapping W_n of C into itself as follows:

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I,\\
&\vdots\\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I,\\
&\vdots\\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
W_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned} \tag{1.2}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \dots, T_1$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$; see [8].

In this paper, motivated by [9], for any given $x_i \in C$ ($i = 0, 1, \dots, q, q \in \mathbb{N}$ is a fixed number), we will propose the following iterative progress for two infinitely nonexpansive mappings $\{T_n^{(1)}\}$ and $\{T_n^{(2)}\}$ in a Hilbert space H:

$$\begin{gathered}
x_0, x_1, \dots, x_q \in C \text{ chosen arbitrarity},\\
y_n = \alpha_n x_n + (1-\alpha_n) W_n^{(1)} z_{n-q},\\
z_n = \overline{\alpha}_n x_n + (1-\overline{\alpha}_n) W_n^{(2)} x_n,\\
C_n = \left\{ v \in K : \left\| y_n - v \right\|^2 \leq \|x_n - v\|^2 + (1-\alpha_n)\left(\left\| x_{n-q} - x^* \right\|^2 - \|x_n - x^*\|^2 \right) \right\},\\
Q_n = \{ v \in K : \langle x_n - v, x_n - x_q \rangle \leq 0 \},\\
x_{n+1} = P_{C_n \cap Q_n}(x_q), n \geq q
\end{gathered} \tag{1.3}$$

and prove, $\{x_n\}$ converges strongly to a fixed point of $\{T_n^{(1)}\}$ and $\{T_n^{(2)}\}$.

We will use the notation:

$\rightharpoonup$ for weak convergence and $\to$ for strong convergence.

$\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of x_n.

2. Preliminaries

In this paper, we need some facts and tools which are listed as lemmas below.

Lemma 2.1 (see [10]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, and T a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$.*

Lemma 2.2 (see [11]). *Let C be a nonempty bounded closed convex subset of a Hilbert space H. Given also a real number $a \in \mathbb{R}$ and $x, y, z \in H$. Then the set $D := \{v \in C : \|y-v\|^2 \leq \|x-v\|^2 + \langle z, v\rangle + a\}$ is closed and convex.*

Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C, where C is a nonempty closed convex subset of a strictly convex Banach space E. Given a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $[0,1]$, one defines a sequence $\{W_n\}_{n=1}^{\infty}$ of self-mappings on C by (1.2). Then one has the following results.

Lemma 2.3 (see [8]). *Let C be a nonempty closed convex subset of a strictly convex Banach space E, $\{T_n\}_{n=1}^{\infty}$ a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and let $\{\lambda_n\}$ be a sequence in $(0,b]$ for some $b \in (0,1)$. Then, for every $x \in C$ and $k \geq 1$ the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Remark 2.4. It can be known from Lemma 2.3 that if D is a nonempty bounded subset of C, then for $\varepsilon > 0$ there exists $n_0 \geq k$ such that $\sup_{x\in D}\|U_{n,k}x - U_k x\| \leq \varepsilon$ for all $n > n_0$.

Remark 2.5. Using Lemma 2.3, we can define a mapping $W : C \to C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x \tag{2.1}$$

for all $x \in C$. Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Since W_n is nonexpansive mapping, $W : C \to C$ is also nonexpansive. Indeed, observe that for each $x, y \in C$,

$$\|Wx - Wy\| = \lim_{n\to\infty} \|W_n x - W_n y\| \leq \|x - y\|. \tag{2.2}$$

If $\{x_n\}$ is a bounded sequence in C, then we put $D = \{x_n : n \geq 0\}$. Hence, it is clear from Remark 2.4 that for $\varepsilon > 0$ there exists $N_0 \geq 1$ such that for all $n > N_0$, $\|W_n x_n - W x_n\| = \|U_{n,1}x_n - U_1 x_n\| \leq \sup_{x\in D}\|U_{n,1}x - U_1 x\| \leq \varepsilon$. This implies that

$$\lim_{n\to\infty} \|W_n x_n - W x_n\| = 0. \tag{2.3}$$

Lemma 2.6 (see [8]). *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and let $\{\lambda_n\}$ be a sequence in $(0,b]$ for some $b \in (0,1)$. Then, $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

3. Strong Convergence Theorem

Theorem 3.1. *Let C be a closed convex subset of a Hilbert space H and let $\{W_n^{(1)}\}$ and $\{W_n^{(2)}\}$ be defined as* (1.2). *Assume that $\alpha_n \le a$ for all n and for some $0 < a < 1$, and $\{\overline{\alpha}_n\} \in [b, c]$ for all n and $0 < b < c < 1$. If $F = \bigcap_{n=1}^{\infty}[F(T_n^{(1)}) \bigcap F(T_n^{(2)})] \neq \emptyset$, then $\{x_n\}$ generated by* (1.3) *converges strongly to $P_F(x_q)$.*

Proof. Firstly, we observe that C_n is convex by Lemma 2.2. Next, we show that $F \subset C_n$ for all n.

Indeed, for all $x^* \in F$,

$$\begin{aligned}
\left\|y_n - x^*\right\|^2 &\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\left\|z_{n-q} - x^*\right\|^2 \\
&= \|x_n - x^*\|^2 + (1-\alpha_n)\Big(\left\|z_{n-q} - x^*\right\|^2 - \|x_n - x^*\|^2\Big), \\
\left\|z_{n-q} - x^*\right\|^2 &= \left\|\overline{\alpha}_{n-q}x_{n-q} + (1-\overline{\alpha}_{n-q})W_{n-q}^{(2)}x_{n-q} - x^*\right\| \\
&= \overline{\alpha}_{n-q}\left\|x_{n-q} - x^*\right\|^2 + (1-\overline{\alpha}_{n-q})\left\|W_{n-q}^{(2)}x_{n-q} - x^*\right\|^2 \\
&\quad - \overline{\alpha}_{n-q}(1-\overline{\alpha}_{n-q})\left\|W_{n-q}^{(2)}x_{n-q} - x_{n-q}\right\|^2 \\
&\le \overline{\alpha}_{n-q}\left\|x_{n-q} - x^*\right\|^2 + (1-\overline{\alpha}_{n-q})\left\|x_{n-q} - x^*\right\|^2 \\
&\quad - \overline{\alpha}_{n-q}(1-\overline{\alpha}_{n-q})\left\|W_{n-q}^{(2)}x_{n-q} - x_{n-q}\right\|^2 \\
&= \left\|x_{n-q} - x^*\right\|^2 - \overline{\alpha}_{n-q}(1-\overline{\alpha}_{n-q})\left\|W_{n-q}^{(2)}x_{n-q} - x_{n-q}\right\|^2 \\
&\le \left\|x_{n-q} - x^*\right\|^2.
\end{aligned} \tag{3.1}$$

Therefore,

$$\left\|y_n - x^*\right\|^2 \le \|x_n - x^*\|^2 + (1-\alpha_n)\Big(\left\|x_{n-q} - x^*\right\|^2 - \|x_n - x^*\|^2\Big). \tag{3.2}$$

That is $x^* \in C_n$ for all $n \ge q$. Next we show that $F \subset Q_n$ for all $n \ge q$.

We prove this by induction. For $n = q$, we have $F \subset C = Q_q$. Assume that $F \subset Q_n$ for all $n \ge q+1$, since x_{n+1} is the projection of x_q onto $C_n \bigcap Q_n$, so

$$\langle x_{n+1} - z, x_q - x_{n+1}\rangle \ge 0, \quad \forall z \in C_n \bigcap Q_n. \tag{3.3}$$

As $F \subset C_n \bigcap Q_n$ by the induction assumption, the last inequality holds, in particular, for all $x^* \in F$. This together with definition of Q_{n+1} implies that $F \subset Q_{n+1}$. Hence $F \subset C_n \bigcap Q_n$ for all $n \ge q$.

Notice that the definition of Q_n implies $x_n = P_{Q_n}x_q$. This together with the fact $F \subset Q_n$ further implies $\|x_n - x_q\| \le \|x^* - x_q\|$ for all $x^* \in F$.

The fact $x_{n+1} \in Q_n$ asserts that $\langle x_{n+1} - x_n, x_n - x_q \rangle \geq 0$ implies

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \left\|(x_{n+1} - x_q) - (x_n - x_q)\right\|^2 \\ &= \left\|x_{n+1} - x_q\right\|^2 - \left\|x_n - x_q\right\|^2 - 2\langle x_{n+1} - x_n, x_n - x_q \rangle \\ &\leq \left\|x_{n+1} - x_q\right\|^2 - \left\|x_n - x_q\right\|^2 \longrightarrow 0 \ (n \longrightarrow \infty). \end{aligned} \tag{3.4}$$

We now claim that $\|W^{(1)}x_n - x_n\| \to 0$ and $\|W^{(2)}x_n - x_n\| \to 0$. Indeed,

$$\begin{aligned} \left\|x_n - W_n^{(1)} z_{n-q}\right\| &= \frac{\|x_n - y_n\|}{1 - \alpha_n} \\ &\leq \frac{\|x_n - x_{n+1}\| + \left\|x_{n+1} - y_n\right\|}{1 - \alpha_n}, \end{aligned} \tag{3.5}$$

since $x_{n+1} \in C_n$, we have

$$\left\|y_n - x_{n+1}\right\|^2 \leq \|x_n - x_{n+1}\|^2 + (1 - \alpha_n)\left(\left\|x_{n-q} - x^*\right\|^2 - \|x_n - x^*\|^2\right) \longrightarrow 0. \tag{3.6}$$

Thus

$$\left\|x_n - W_n^{(1)} z_{n-q}\right\| \longrightarrow 0. \tag{3.7}$$

We now show $\lim_{n\to\infty}\|W_n^{(2)}x_n - x_n\| = 0$. Let $\{\|W_{n_k}^{(2)}x_{n_k} - x_{n_k}\|\}$ be any subsequence of $\{\|W_n^{(2)}x_n - x_n\|\}$. Since C is a bounded subset of H, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that

$$\lim_{j\to\infty}\left\|x_{n_{k_j}} - x^*\right\| = \limsup_{k\to\infty}\|x_{n_k} - x^*\| := r. \tag{3.8}$$

Since

$$\begin{aligned} \left\|x_{n_{k_j}} - x^*\right\| &\leq \left\|x_{n_{k_j}} - W_{n_{k_j}}^{(1)} z_{n_{k_j}-q}\right\| + \left\|W_{n_{k_j}}^{(1)} z_{n_{k_j}-q} - x^*\right\| \\ &\leq \left\|x_{n_{k_j}} - W_{n_{k_j}}^{(1)} z_{n_{k_j}-q}\right\| + \left\|z_{n_{k_j}-q} - x^*\right\|, \end{aligned} \tag{3.9}$$

it follows that $r = \lim_{j\to\infty}\|x_{n_{k_j}} - x^*\| \leq \liminf_{j\to\infty}\|z_{n_{k_j}} - x^*\|$. By (3.1), we have

$$\left\|z_{n_{k_j}} - x^*\right\| \leq \left\|x_{n_{k_j}} - x^*\right\|^2. \tag{3.10}$$

Hence

$$\limsup_{j\to\infty}\left\|z_{n_{k_j}} - x^*\right\| \leq \lim_{j\to\infty}\left\|x_{n_{k_j}} - x^*\right\| = r. \tag{3.11}$$

Thus,

$$\lim_{j\to\infty}\left\|z_{n_{k_j}}-x^*\right\|=r=\lim_{j\to\infty}\left\|x_{n_{k_j}}-x^*\right\|. \quad (3.12)$$

Using (3.1) again, we obtain that

$$\overline{\alpha}_{n_{k_j}-q}\left(1-\overline{\alpha}_{n_{k_j}-q}\right)\left\|W^{(2)}_{n_{k_j}-q}x_{n_{k_j}-q}-x_{n_{k_j}-q}\right\|^2\le\left\|x_{n_{k_j}-q}-x^*\right\|^2-\left\|z_{n_{k_j}-q}-x^*\right\|^2\longrightarrow 0. \quad (3.13)$$

This imply that $\lim_{j\to\infty}\|W^{(2)}_{n_{k_j}}x_{n_{k_j}}-x_{n_{k_j}}\|=0$. For the arbitrariness of $\{x_{n_k}\}\subset\{x_n\}$, we have $\lim_{n\to\infty}\|W^{(2)}_n x_n-x_n\|=0$ and

$$\|z_n-x_n\|=(1-\overline{\alpha}_n)\left\|W^{(2)}_n x_n-x_n\right\|\longrightarrow 0. \quad (3.14)$$

Thus, by (3.4), (3.7) and (3.14), we have

$$\begin{aligned}\left\|W^{(1)}_n x_n-x_n\right\| &\le \left\|W^{(1)}_n x_n-W^{(1)}_n z_{n-q}\right\|+\left\|W^{(1)}_n z_{n-q}-x_n\right\|\\ &\le \left\|z_{n-q}-x_n\right\|+\left\|W^{(1)}_n z_{n-q}-x_n\right\|\\ &\le \left\|W^{(1)}_n z_{n-q}-x_n\right\|+\left\|z_{n-q}-x_{n-q}\right\|+\left\|x_{n-q}-x_{n-q+1}\right\|\\ &\quad+\left\|x_{n-q+1}-x_{n-q+2}\right\|+\cdots+\|x_{n-1}-x_n\|\\ &\longrightarrow 0.\end{aligned} \quad (3.15)$$

Since $\lim_{n\to\infty}\|W^{(1)}_n x_n-W^{(1)}x_n\|=0$ and $\lim_{n\to\infty}\|W^{(2)}_n x_n-W^{(2)}x_n\|=0$, we have

$$\begin{aligned}&\lim_{n\to\infty}\left\|W^{(1)}x_n-x_n\right\|=0,\\ &\lim_{n\to\infty}\left\|W^{(2)}x_n-x_n\right\|=0.\end{aligned} \quad (3.16)$$

Thus, using (3.16), Lemma 2.1, and the boundedness of $\{x_n\}$, we get that $\emptyset\ne\omega_w(x_n)\subset F$. Since $x_n=P_{Q_n}(x_q)$ and $F\subset Q_n$, we have $\|x_n-x_q\|\le\|x^*-x_q\|$ where $x^*:=P_F(x_q)$. By the weak lower semicontinuity of the norm, we have $\|w-x_q\|\le\|x^*-x_q\|$ for all $w\in\omega_w(x_n)$. However, since $\omega_w(x_n)\subset F$, we must have $w=x^*$ for all $w\in\omega_w(x_n)$. Hence $x_n\rightharpoonup x^*=P_F(x_q)$ and

$$\begin{aligned}\|x_n-x^*\|^2&=\left\|x_n-x_q\right\|^2+2\langle x_n-x_q,x_q-x^*\rangle+\left\|x_q-x^*\right\|^2\\ &\le 2\left(\left\|x^*-x_q\right\|^2+\langle x_n-x_q,x_q-x^*\rangle\right)\longrightarrow 0.\end{aligned} \quad (3.17)$$

That is, $\{x_n\}$ converges to $P_F(x_q)$.

This completes the proof. □

Acknowledgment

This work is supported by Grant KJ080725 of the Chongqing Municipal Education Commission.

References

[1] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, pp. 506–510, 1953.
[2] S. Ishikawa, "Fixed points by a new iteration method," *Proceedings of the American Mathematical Society*, vol. 44, pp. 147–150, 1974.
[3] L. Deng, "On Chidume's open questions," *Journal of Mathematical Analysis and Applications*, vol. 174, no. 2, pp. 441–449, 1993.
[4] C. E. Chidume, "An iterative process for nonlinear Lipschitzian strongly accretive mappings in L_p spaces," *Journal of Mathematical Analysis and Applications*, vol. 151, no. 2, pp. 453–461, 1990.
[5] B. E. Rhoades, "Comments on two fixed point iteration methods," *Journal of Mathematical Analysis and Applications*, vol. 56, no. 3, pp. 741–750, 1976.
[6] A. Genel and J. Lindenstrauss, "An example concerning fixed points," *Israel Journal of Mathematics*, vol. 22, no. 1, pp. 81–86, 1975.
[7] K. Nakajo and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 2, pp. 372–379, 2003.
[8] K. Shimoji and W. Takahashi, "Strong convergence to common fixed points of infinite nonexpansive mappings and applications," *Taiwanese Journal of Mathematics*, vol. 5, no. 2, pp. 387–404, 2001.
[9] L. Deng and Q. Liu, "Iterative scheme for nonself generalized asymptotically quasi-nonexpansive mappings," *Applied Mathematics and Computation*, vol. 205, no. 1, pp. 317–324, 2008.
[10] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, vol. 28 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1990.
[11] C. Martinez-Yanes and H.-K. Xu, "Strong convergence of the CQ method for fixed point iteration processes," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 11, pp. 2400–2411, 2006.