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Research Article

Convergence Theorems of Three-Step Iterative Scheme for a Finite Family of Uniformly Quasi-Lipschitzian Mappings in Convex Metric Spaces

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We consider a new Noor-type iterative procedure with errors for approximating the common fixed point of a finite family of uniformly quasi-Lipschitzian mappings in convex metric spaces. Under appropriate conditions, some convergence theorems are proved for such iterative sequences involving a finite family of uniformly quasi-Lipschitzian mappings. The results presented in this paper extend, improve and unify some main results in previous work.

1. Introduction and Preliminaries

Takahashi [1] introduced a notion of convex metric spaces and studied the fixed point theory for nonexpansive mappings in such setting. For the convex metric spaces, Kirk [2] and Goebel and Kirk [3] used the term "hyperbolic type space" when they studied the iteration processes for nonexpansive mappings in the abstract framework. For the Banach space, Petryshyn and Williamson [4] proved a sufficient and necessary condition for Picard iterative sequences and Mann iterative sequence to converge to fixed points for quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [5] extended the results of [4] and gave the sufficient and necessary condition for Ishikawa iterative sequence to converge to fixed points for quasi-nonexpansive mappings. Liu [6–8] proved some sufficient and necessary conditions for Ishikawa iterative sequence and Ishikawa iterative sequence with errors to converge to fixed point for asymptotically quasi-nonexpansive mappings in Banach space and uniform convex Banach space. Tian [9] gave some sufficient and necessary conditions for an Ishikawa iteration sequence for an asymptotically quasi-nonexpansive mapping to converge to a fixed point in convex metric spaces. Very recently, Wang and Liu [10] gave some iteration sequence with errors to approximate a fixed point of two uniformly quasi-Lipschitzian mappings in convex metric spaces. The purpose of this paper is to give some sufficient and necessary conditions for a new Noor-type iterative sequence with errors to approximate a common fixed point for a finite family of uniformly quasi-Lipschitzian mappings in convex metric spaces. The results presented in this paper generalize, improve, and unify some main results of [1–14].

First of all, let us list some definitions and notations.

Let T be a given self mapping of a nonempty convex subset C of an arbitrary real normed space.The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} &x_0 \in C, \\ &x_{n+1} = \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \quad n \geq 0, \\ &y_n = a_n x_n + b_n T z_n + c_n v_n, \\ &z_n = d_n x_n + e_n T x_n + f_n w_n, \end{aligned} \tag{1.1}$$

is called the Noor iterative procedure with errors [11], where $\alpha_n, \beta_n, \gamma_n, a_n, b_n, c_n, d_n, e_n,$ and f_n are appropriate sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + f_n = 1$, $n \geq 0$ and $\{u_n\}$,$\{v_n\}$, and $\{w_n\}$ are bounded sequences in C. If $d_n = 1$ ($e_n = f_n = 0$), $n \geq 0$ then (1.1) reduces to the Ishikawa iterative procedure with errors [15] defined as follows:

$$\begin{aligned} &x_0 \in C, \\ &x_{n+1} = \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \quad n \geq 0, \\ &y_n = a_n x_n + b_n T x_n + c_n v_n. \end{aligned} \tag{1.2}$$

If $a_n = 1$ ($b_n = c_n = 0$) then (1.2) reduces to the following Mann type iterative procedure with errors [15]:

$$\begin{aligned} &x_0 \in C, \\ &x_{n+1} = \alpha_n x_n + \beta_n T x_n + \gamma_n u_n, \quad n \geq 0. \end{aligned} \tag{1.3}$$

Let (E,d) be a metric space. A mapping $T : E \to E$ is said to be asymptotically nonexpansive, if there exists a sequence $\{K_n\} \in [1,\infty]$, $\lim_{n\to\infty} K_n = 1$, such that

$$d(T^n x, T^n y) \leq K_n d(x,y), \quad \forall x, y \in E, \quad n \geq 0. \tag{1.4}$$

Let $F(T)$ be the set of fixed points of T in E and $F(T) \neq \emptyset$, a mapping T is said to be asymptotically quasi-nonexpansive, if there exists $\{K_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} K_n = 1$ such that

$$d(T^n x, p) \leq K_n d(x,p), \quad \forall x \in E, \ p \in F(T), \ n \geq 0. \tag{1.5}$$

Moreover, T is said to be uniformly quasi-Lipschitzian, if there exists $L > 0$ such that

$$d(T^n x, p) \leq L d(x, p), \quad \forall x \in E, \ p \in F(T), \ n \geq 0. \tag{1.6}$$

Remark 1.1. If $F(T)$ is nonempty, then it follows from the above definitions that an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive, and an asymptotically quasi-nonexpansive mapping must be a uniformly quasi-Lipschitzian with $L = \sup_{n\geq 0}\{K_n\} < \infty$. However, the inverse is not true in general.

Definition 1.2 (see [9]). Let (E, d) be a metric space, and let $I = [0,1], \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$. A mapping $W : E^3 \times I^3 \to X$ is said to be a convex structure on E if, for any $(x, y, z, \alpha_n, \beta_n, \gamma_n) \in E^3 \times I^3$ and $u \in E$,

$$d(W(x, y, z, \alpha_n, \beta_n, \gamma_n)u) \leq \alpha_n d(x, u) + \beta_n d(y, u) + \gamma_n d(z, u). \tag{1.7}$$

If (E, d) is a metric space with a convex structure W, then (E, d) is called a convex metric space. Let (E, d) be a convex metric space, a nonempty subset C of E is said to be convex if

$$W(x, y, z, \lambda_1, \lambda_2, \lambda_3) \in C, \quad \forall (x, y, z, \lambda_1, \lambda_2, \lambda_3) \in C^3 \times I^3. \tag{1.8}$$

Definition 1.3. Let (E, d) be a convex metric space with a convex structure $W : E^3 \times I^3$ and $T_i : E \to E$ be a finite family of uniformly quasi-Lipschitzian mappings with $i = 1, 2, \ldots, N$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}$, and $\{f_n\}$ be nine sequences in $[0,1]$ with

$$\alpha_n + \beta_n + \eta_n = a_n + b_n + c_n = d_n + e_n + f_n = 1, \quad n = 0, 1, 2, \ldots. \tag{1.9}$$

For a given $x_0 \in E$, define a sequence $\{x_n\}$ as follows:

$$\begin{aligned} x_{n+1} &= W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \geq 0, \\ y_n &= W(f(x_n), T_n^n z_n, v_n; a_n, b_n, c_n), \\ z_n &= W(f(x_n), T_n^n x_n, w_n; d_n, e_n, f_n), \end{aligned} \tag{1.10}$$

where $T_n^n = T_{n(\bmod N)}^n$, $f : E \to E$ is a Lipschitz continuous mapping with a Lipschitz constant $\xi > 0$ and $\{u_n\}, \{v_n\}, \{w_n\}$ are any given three sequences in E. Then $\{x_n\}$ is called the Noor-type iterative sequence with errors for a finite family of uniformly quasi-Lipschitzian mappings $\{T_i\}_{i=1}^N$. If $f = I$ (the identity mapping on E) in (1.10), then the sequence $\{x_n\}$ defined by (1.10) can be written as follows:

$$\begin{aligned} x_{n+1} &= W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \geq 0; \\ y_n &= W(x_n, T_n^n z_n, v_n; a_n, b_n, c_n), \\ z_n &= W(x_n, T_n^n x_n, w_n; d_n, e_n, f_n). \end{aligned} \tag{1.11}$$

If $d_n = 1$ for all $n \geq 0$ in (1.10), then $z_n = x_n$ for all $n \geq 0$ and the sequence $\{x_n\}$ defined by (1.10) can be written as follows:

$$\begin{aligned} x_{n+1} &= W\left(f(x_n), T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n\right), \quad n \geq 0, \\ y_n &= W\left(f(x_n), T_n^n x_n, v_n; a_n, b_n, c_n\right). \end{aligned} \tag{1.12}$$

If $f = I$ and $d_n = 1$ for all $n \geq 0$, then the sequence $\{x_n\}$ defined by (1.10) can be written as follows:

$$\begin{aligned} x_{n+1} &= W\left(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n\right), \quad n \geq 0, \\ y_n &= W(x_n, T_n^n x_n, v_n; a_n, b_n, c_n), \end{aligned} \tag{1.13}$$

which is the Ishikawa type iterative sequence with errors considered in [9]. Further, if $f = I$ and $d_n = a_n = 1$ for all $n \geq 0$, then $z_n = y_n = x_n$ for all $n \geq 0$ and (1.10) reduces to the following Mann type iterative sequence with errors [9]:

$$x_{n+1} \equiv W\left(x_n, T_n^n x_n, u_n; \alpha_n, \beta_n, \gamma_n\right), \quad n \geq 0. \tag{1.14}$$

In order to prove our main results, the following lemmas will be needed.

Lemma 1.4. *Let (E, d) be a convex metric space, $T_i : E \to E$ be a uniformly quasi-Lipschitzian mapping for $i = 1, 2, \ldots, N$ such that $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Then there exists a constant $L \geq 1$ such that, for all $i = 1, 2, \ldots, N$,*

$$d\left(T_i^n x, p\right) \leq L d\left(x, p\right), \quad \forall x \in X, \ p \in F, \ n \geq 0. \tag{1.15}$$

Proof. In fact, for each $i = 1, 2, \ldots, N$, since $T_i : E \to E$ is a uniformly quasi-Lipschitzian mapping, we have

$$d\left(T_i^n x, p\right) \leq L_i d\left(x, p\right) \leq L d\left(x, p\right), \quad \forall x \in E, \ p \in F, \ n \geq 0, \tag{1.16}$$

where

$$L = \max_{i=1,2,\ldots,N} \{\max\{L_i, 1\}\}. \tag{1.17}$$

This completes the proof. □

Lemma 1.5 (see [7]). *Let $\{p_n\}, \{q_n\}, \{r_n\}$ be three nonexpansive squences satisfying the following conditions:*

$$p_{n+1} \leq (1 + q_n)p_n + r_n, \quad \forall n \geq 0, \quad \sum_{n=0}^{\infty} q_n < \infty, \quad \sum_{n=0}^{\infty} r_n < \infty. \tag{1.18}$$

Then

(1) $\lim_{n\to\infty} p_n$ exists;

(2) In addition, if $\liminf_{n\to\infty} p_n = 0$, then $\lim_{n\to\infty} p_n = 0$.

Lemma 1.6. *Let (E, d) be a complete convex metric space and C be a nonempty closed convex subset of E. Let $T_i : C \to C$ be a finite family of uniformly quasi-Lipschitzian mapping for $i = 1, 2, \ldots, N$ such that $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $f : C \to C$ be a contractive mapping with a contractive constant $\xi \in (0, 1)$.Let $\{x_n\}$ be the iterative sequence with errors defined by* (1.10) *and $\{u_n\}, \{v_n\}, \{w_n\}$ be three bounded sequences in C. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}, \{f_n\}$ be sequences in [0,1] satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + f_n = 1, \quad \forall n \geq 0$;

(ii) $\sum_{n=0}^{\infty} (\beta_n + \gamma_n) < \infty$;

(iii) $M_0 = \text{Sup}_{p\in F, n\geq 0} \{d(u_n, p) + d(v_n, p) + d(w_n, p) + d(f(p), p)\} < \infty$.

Then the following conclusions hold:

(1) *for all* $p \in F$ and $n \geq 0$,

$$d(x_{n+1}, p) \leq \left[1 + \beta_n L\left(1 + L + L^2\right)\right] d(x_n, p) + M\eta_n, \tag{1.19}$$

where $L = \max_{i=1,2,\ldots,N}\{L_i\}$, $\eta_n = \beta_n + \gamma_n$ *for all* $n \geq 0$ *and*

$$M = L(1 + L)\left[d(u_n, p) + d(v_n, p) + d(w_n, p) + d(f(p), p)\right]. \tag{1.20}$$

(2) *there exists a constant* $M_i > 0$ *such that*

$$d(x_{n+m}, p) \leq M_1 d(x_n, p) + MM_1 \sum_{k=n}^{n+m-1} \eta_k, \quad \forall p \in F, \tag{1.21}$$

for all $n, m \geq 0$.

Proof. (1) It follows from (1.7),(1.10), and Lemma 1.4 that

$$\begin{aligned} d(x_{n+1},p) &= d(W(x_n,T_n^n y_n,u_n;\alpha_n,\beta_n,\gamma_n),p) \\ &\le \alpha_n d(x_n,p)+\beta_n d(T_n^n y_n,p)+\gamma_n d(u_n,p) \\ &\le \alpha_n d(x_n,p)+\beta_n L d(y_n,p)+\gamma_n d(u_n,p), \end{aligned} \tag{1.22}$$

$$\begin{aligned} d(y_n,p) &= d(W(f(x_n),T_n^n z_n,v_n;a_n,b_n,c_n),p) \\ &\le a_n d(f(x_n),p)+b_n d(T_n^n z_n,p)+c_n d(v_n,p) \\ &\le a_n d(f(x_n),f(p))+a_n d(f(p),p)+b_n L d(z_n,p)+c_n d(v_n,p) \\ &\le a_n \xi d(x_n,p)+a_n d(f(p),p)+b_n L d(z_n,p)+c_n d(v_n,p), \end{aligned} \tag{1.23}$$

$$\begin{aligned} d(z_n,p) &= d(W(f(x_n),T_n^n x_n,w_n;d_n,e_n,f_n),p) \\ &\le d_n d(f(x_n),p)+e_n d(T_n^n x_n,p)+f_n d(w_n,p) \\ &\le d_n d(f(x_n),f(p))+d_n d(f(p),p)+e_n L d(x_n,p)+f_n d(w_n,p) \\ &\le d_n \xi d(x_n,p)+d_n d(f(p),p)+e_n L d(x_n,p)+f_n d(w_n,p) \\ &\le (d_n\xi+e_nL) d(x_n,p)+d_n d(f(p),p)+f_n d(w_n,p). \end{aligned} \tag{1.24}$$

Substituting (1.23) into (1.22) and simplifying it, we have

$$\begin{aligned} d(x_{n+1},p) \le{}& \alpha_n d(x_n,p) \\ &+\beta_n L\left[a_n\xi d(x_n,p)+a_n d(f(p),p)+b_n L d(z_n,p)+c_n d(v_n,p)\right]+\gamma_n d(u_n,p) \\ \le{}& (\alpha_n+\beta_n L\xi a_n) d(x_n,p)+\beta_n L a_n d(f(p),p) \\ &+\beta_n L^2 b_n d(z_n,p)+\beta_n L c_n d(v_n,p)+\gamma_n d(u_n,p). \end{aligned} \tag{1.25}$$

Substituting (1.24) into (1.25) and simplifying it, we get

$$\begin{aligned} d(x_{n+1},p) \le{}& (\alpha_n+\beta_n L a_n\xi) d(x_n,p) \\ &+\beta_n L^2 b_n\left[(d_n\xi+e_nL) d_n(x_n,p)+d_n d(f(p),p)+f_n d(w_n,p)\right] \\ &+\beta_n L a_n d(f(p),p)+\beta_n L c_n d(v_n,p)+\gamma_n d(u_n,p) \\ ={}& \{\alpha_n+\beta_n L[a_n\xi+L b_n(d_n\xi+e_nL)]\} d(x_n,p)+\beta_n L^2 b_n d_n d(f(p),p) \\ &+\beta_n L a_n d(f(p),p)+\beta_n L^2 b_n f_n d(w_n,p)+\beta_n L c_n d(v_n,p)+\gamma_n d(u_n,p) \end{aligned}$$

$$
\begin{aligned}
&\leq \left[1+\beta_n L\left(1+L+L^2\right)\right] d(x_n,p) + \left[\beta_n L^2 b_n d_n + \beta_n L a_n\right] d(f(p),p) + \gamma_n d(u_n,p) \\
&\quad + \beta_n L c_n d(v_n,p) + \beta_n L^2 b_n f_n d(w_n,p) \\
&\leq \left[1+\beta_n L\left(1+L+L^2\right)\right] d(x_n,p) + \beta_n L(1+L) d(f(p),p) + \gamma_n L(1+L) d(f(p),p) \\
&\quad + L(1+L)(\beta_n+\gamma_n) d(u_n,p) + L(1+L)(\beta_n+\gamma_n) d(v_n,p) \\
&\quad + L(1+L)(\beta_n+\gamma_n) d(w_n,p) \\
&= \left[1+\beta_n L\left(1+L+L^2\right)\right] d(x_n,p) \\
&\quad + L(1+L)(\beta_n+\gamma_n)\left[d(u_n,p)+d(v_n,p)+d(w_n,p)+d(f(p),p)\right] \\
&= \left[1+\beta_n L\left(1+L+L^2\right)\right] d(x_n,p) + M\eta_n, \quad \forall n \geq 0, \ \ p \in F,
\end{aligned}
\tag{1.26}
$$

where

$$
M = L(1+L)\left[d(u_n,p)+d(v_n,p)+d(w_n,p)+d(f(p),p)\right], \quad \eta_n = \beta_n + \gamma_n. \tag{1.27}
$$

(2) Since $1 + x \leq e^x$ for all $x \geq 0$, it follows from (1.26) that, for $n, m \geq 0$ and $p \in F$,

$$
\begin{aligned}
d(x_{n+m},p) &\leq \left[1+\beta_{n+m-1} L\left(1+L+L^2\right)\right] d(x_{n+m-1},p) + M\eta_{n+m-1} \\
&\leq e^{\beta_{n+m-1} L\left(1+L+L^2\right)} d(x_{n+m-1},p) + M\eta_{n+m-1} \\
&\leq e^{\beta_{n+m-1} L\left(1+L+L^2\right)} \left\{\left[1+\beta_{n+m-2} L\left(1+L+L^2\right)\right] d(x_{n+m-2},p) + M\eta_{n+m-2}\right\} + M\eta_{n+m-1} \\
&\leq e^{L\left(1+L+L^2\right)(\beta_{n+m-1}+\beta_{n+m-2})} d(x_{n+m-1},p) + M\left[e^{\beta_{n+m-1} L\left(1+L+L^2\right)}\eta_{n+m-2} + \eta_{n+m-1}\right] \\
&\leq \cdots \\
&\leq M_1 d(x_n,p) + M_1 M \sum_{k=n}^{n+m-1} \eta_k,
\end{aligned}
\tag{1.28}
$$

where

$$
M_1 = e^{L\left(1+L+L^2\right)\sum_{k=0}^{\infty}\beta_k}. \tag{1.29}
$$

This completes the proof. □

2. Main Results

Theorem 2.1. *Let (E, d) be a complete convex metric space and C be a nonempty closed convex subset of E. Let $T_i : C \to C$ be a finite family of uniformly quasi-Lipschitzian mapping for $i = 1, 2, \ldots, N$ such that $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $f : C \to C$ be a contractive mapping with a contractive constant $\xi \in (0, 1)$. Let $\{x_n\}$ be the iterative sequence with errors defined by* (1.10) *and $\{u_n\}$,$\{v_n\}$,$\{w_n\}$ be three bounded sequence in C and $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\}$,$\{a_n\}$,$\{b_n\}$,$\{c_n\}$,$\{d_n\}$,$\{e_n\}$ and $\{f_n\}$ be nine sequences in [0,1] satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + f_n = 1, \forall n \geq 0$,

(ii) $\sum_{n=0}^{\infty}(\beta_n + \gamma_n) < \infty$,

(iii) $M_0 = \mathrm{Sup}_{p\in F, n\geq 0}\{d(u_n, p) + d(v_n, p) + d(w_n, p) + d(f(p), p)\} < \infty$.

Then the sequence $\{x_n\}$ converges to a common fixed point $p \in F$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf\{d(x, F), p \in F\}$.

Proof. The necessity is obvious. Now prove the sufficiency. In fact, from Lemma 1.6, we have

$$d(x_{n+1}, F) \leq \left[1 + \beta_n L\left(1 + L + L^2\right)\right] d(x_n, F) + M\eta_n, \quad \forall n \geq 0, \tag{2.1}$$

where $\eta_n = \beta_n + \gamma_n$. By conditions (i) and (ii), we know that

$$\sum_{n=0}^{\infty} \eta_n < \infty, \qquad \sum_{n=0}^{\infty} \beta_n < \infty. \tag{2.2}$$

It follows from Lemma 1.5 that $\lim_{n\to\infty} d(x_n, F)$ exists. Since $\liminf_{n\to\infty} d(x_n, F) = 0$, we have

$$\lim_{n\to\infty} d(x_n, F) = 0. \tag{2.3}$$

Next prove that $\{x_n\}$ is a Cauchy sequence in C. In fact, for any given $\varepsilon > 0$, there exists a positive integer N_0 such that

$$d(x_n, F) \leq \frac{\varepsilon}{8M_1}, \qquad \sum_{n=N_0}^{\infty} \eta_n \leq \frac{\varepsilon}{4M_1 M}, \quad \forall n \geq 0. \tag{2.4}$$

From (2.4), there exist $p_1 \in F$ and positive integer $N_1 > N_0$ such that

$$d(x_{N_1}, p_1) < \frac{\varepsilon}{4M_1}. \tag{2.5}$$

Thus Lemma 1.6 implies that, for any positive integers n, m with $n > N_1$,

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p_1) + d(p_1, x_n) \\ &\leq M_1 d(x_{N_1}, P_1) + M_1 M \sum_{k=N_1}^{n+m-1} \eta_k + M_1 d(x_{N_1}, p_1) + M_1 M \sum_{k=N_1}^{n-1} \eta_k \\ &\leq 2M_1 \frac{\varepsilon}{4M_1} + 2M_1 M \frac{\varepsilon}{4M_1 M} \\ &= \varepsilon. \end{aligned} \tag{2.6}$$

This shows that $\{x_n\}$ is a Cauchy sequence in a nonempty closed convex subset C of a complete convex metric space E. Without loss of generality, we can assume that $\lim_{n\to\infty} x_n = p^* \in E$. Next prove that $p^* \in F$. In fact, for any given $\varepsilon' > 0$, there exists a positive integer N_2 such that for all $n \geq N_2$,

$$d(x_n, p^*) \leq \frac{\varepsilon'}{4L}, \quad d(x_n, F) \leq \frac{\varepsilon'}{8L}. \tag{2.7}$$

Again from (2.7), there exist $p_2 \in F$ and positive integer $N_3 > N$ such that

$$d(x_{N_3}, P_2) \leq \frac{\varepsilon'}{4L}. \tag{2.8}$$

Thus, for any $i = 1, 2, \ldots, N$, from (2.7) and (2.8), we have

$$\begin{aligned} d(T_i P^*, P^*) &\leq d(T_i P^*, P_2) + d(P_2, T_i x_{N_3}) + d(T_i x_{N_3}, P^*) \\ &\leq L d(P^*, p_2) + L d(p_2, x_{N_3}) + L d(x_{N_3}, P^*) \\ &\leq L\{d(P^*, x_{N_3}) + d(x_{N_3}, p_2)\} + L d(p_2, x_{N_3}) + L d(x_{N_3}, P^*) \\ &= 2L d(P^*, x_{N_3}) + 2L d(x_{N_3}, p_2) \\ &< \frac{\varepsilon'}{2} + \frac{\varepsilon'}{2} = \varepsilon'. \end{aligned} \tag{2.9}$$

By the arbitrariness of $\varepsilon' > 0$, we know that $T_i P^* = P^*$ for all $i = 1, 2, \ldots, N$, that is, $p^* \in F$. This completes the proof of Theorem 2.1. □

Taking $f = I$ in Theorem 2.1, then we have the following theorem.

Theorem 2.2. *Let (E, d) be a complete convex metric space and C be a nonempty closed convex subset of E. Let $T_i : C \to C$ be a finite family of uniformly quasi-Lipschitzian mapping for $i = 1, 2, \ldots, N$ such that $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the iterative sequence with errors defined by (1.11) and $\{u_n\}, \{v_n\}, \{w_n\}$ be three bounded sequence in C, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}$, and $\{f_n\}$ be nine sequence in [0,1] satisfying the* *conditions (i)–(iii) of Theorem 2.1. Then the sequence* $\{x_n\}$ *converges to a common fixed point* $p \in F$ *if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0, \tag{2.10}$$

where $d(x,F) = \inf\{d(x,F), p \in F\}$.

Taking $d_n = 1$ in Theorem 2.1, then we have the following theorem.

Theorem 2.3. *Let* (E,d) *be a complete convex metric space and* C *be a nonempty closed convex subset of* E*. Let* $T_i : C \to C$ *be a finite family of uniformly quasi-Lipschitzian mapping for* $i = 1,2,\ldots,N$ *such that* $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ *and* $f : C \to C$ *be a contractive mapping with a contractive constant* $\xi \in (0,1)$*. Let* $\{x_n\}$ *be the iterative sequence with errors defined by* (1.12) *and* $\{u_n\},\{v_n\}$ *be two bounded sequences in* C *and* $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{a_n\},\{b_n\},\{c_n\}$ *be nine sequences in* $[0,1]$ *satisfying the conditions (ii) and (iii) of Theorem 2.1 and* $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = 1$ *for all* $n \geq 0$*. Then the sequence* $\{x_n\}$ *converges to a common fixed point* $p \in F$ *if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0, \tag{2.11}$$

where $d(x,F) = \inf\{d(x,p), p \in F\}$.

Remark 2.4. Theorems 2.1–2.3 generalize, improve, and unify some corresponding results in [1–14].

Similarly, we can obtain the following results.

Theorem 2.5. *Let* (E,d) *be a complete convex metric space and* C *be a nonempty closed convex subset of* E*. Let* $T_i : C \to C$ *be a finite family of asymptotically quasi-nonexpansive mapping for* $i = 1,2,\ldots,N$ *such that* $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ *and* $f : C \to C$ *be a contractive mapping with a contractive constant* $\xi \in (0,1)$*. Let* $\{x_n\}$ *be the iterative sequence with errors defined by* (1.10) *and* $\{u_n\},\{v_n\},\{w_n\}$ *be three bounded sequences in* C *and* $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{a_n\},\{b_n\},\{c_n\},\{d_n\},\{e_n\}$*, and* $\{f_n\}$ *be nine sequences in* $[0,1]$ *satisfying the conditions (i)–(iii) of Theorem 2.1. Then the sequence* $\{x_n\}$ *converges to a common fixed point* $p \in F$ *if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0, \tag{2.12}$$

where $d(x,F) = \inf f\{d(x,p), p \in F\}$.

Proof. From Remark 1.1, we know that each asymptotically quasi-nonexpansive mapping $T_i : C \to C, i = 1,2,\ldots,N$ must be a uniformly quasi-Lipschitzian with

$$L_i = \operatorname*{Sup}_{n\geq 0} \left\{k_n^{(i)}\right\} < \infty, \tag{2.13}$$

where $\{k_n^{(i)}\} \subset [1,\infty)$ is the sequence appeared in (1.5). Hence the conclusion of Theorem 2.5 can be obtained from Theorem 2.1 immediately. This completes the proof. □

Theorem 2.6. *Let* (E, d) *be a complete convex metric space and* C *be a nonempty closed convex subset of* E*. Let* $T_i : C \to C$ *be a finite family of asymptotically quasi-nonexpansive mapping for,* $i = 1, 2, \ldots, N$ *such that* $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$*. Let* $\{x_n\}$ *be the iterative sequence with errors defined by* (1.11) *and* $\{u_n\}, \{v_n\}, \{w_n\}$ *be three bounded sequence in* C *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}$*, and* $\{f_n\}$ *be nine sequence in* $[0, 1]$ *satisfying the conditions (i)–(iii) of Theorem 2.1. Then the sequence* $\{x_n\}$ *converges to a common fixed point* $p \in F$ *if and only if*

$$\liminf_{n \to \infty} d(x_n, F) = 0, \tag{2.14}$$

where $d(x, F) = \inf \{d(x, p), p \in F\}$.

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