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Research Article

Assad-Kirk-Type Fixed Point Theorems for a Pair of Nonself Mappings on Cone Metric Spaces

S. Janković,[1] Z. Kadelburg,[2] S. Radenović,[3] and B. E. Rhoades[4]

[1] *Mathematical Institute SANU, Knez Mihailova 36, 11001 Beograd, Serbia*
[2] *Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000 Beograd, Serbia*
[3] *Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia*
[4] *Department of Mathematics, Indiana University, Bloomington, IN 47405-7106, USA*

Correspondence should be addressed to S. Radenović, radens@beotel.yu

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New fixed point results for a pair of non-self mappings defined on a closed subset of a metrically convex cone metric space (which is not necessarily normal) are obtained. By adapting Assad-Kirk's method the existence of a unique common fixed point for a pair of non-self mappings is proved, using only the assumption that the cone interior is nonempty. Examples show that the obtained results are proper extensions of the existing ones.

1. Introduction and Preliminaries

Cone metric spaces were introduced by Huang and Zhang in [1], where they investigated the convergence in cone metric spaces, introduced the notion of their completeness, and proved some fixed point theorems for contractive mappings on these spaces. Recently, in [2–4], some common fixed point theorems have been proved for maps on cone metric spaces. However, in [1–3], the authors usually obtain their results for normal cones. In this paper we do not impose the normality condition for the cones.

We need the following definitions and results, consistent with [1], in the sequel.

Let E be a real Banach space. A subset P of E is a *cone* if

(i) P is closed, nonempty and $P \neq \{0\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;

(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define the partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in \operatorname{int} P$ (the interior of P).

There exist two kinds of cones: normal and nonnormal cones. A cone $P \subset E$ is a *normal cone* if

$$\inf\{\|x+y\| : x, y \in P, \|x\| = \|y\| = 1\} > 0 \tag{1.1}$$

or, equivalently, if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \le x \le y \quad \text{implies } \|x\| \le K\|y\|. \tag{1.2}$$

The least positive number satisfying (1.2) is called the normal constant of P. It is clear that $K \ge 1$.

It follows from (1.1) that P is *nonnormal* if and only if there exist sequences $x_n, y_n \in P$ such that

$$0 \le x_n \le x_n + y_n, \quad x_n + y_n \longrightarrow 0 \quad \text{but } x_n \nrightarrow 0. \tag{1.3}$$

So, in this case, the Sandwich theorem does not hold.

Example 1.1 (see [5]). Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \ge 0 \text{ on } [0,1]\}$. This cone is not normal. Consider, for example,

$$x_n(t) = \frac{1 - \sin nt}{n+2}, \qquad y_n(t) = \frac{1 + \sin nt}{n+2}. \tag{1.4}$$

Then $\|x_n\| = \|y_n\| = 1$ and $\|x_n + y_n\| = 2/(n+2) \to 0$.

Definition 1.2 (see [1]). Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

(d1) $0 \le d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(d2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(d3) $d(x,y) \le d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a *cone metric* on X, and (X, d) is called a *cone metric space*.

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space with $E = \mathbb{R}$ and $P = [0, +\infty[$ (see [1, Example 1] and [4, Examples 1.2 and 2.2]).

Let $\{x_n\}$ be a sequence in X, and let $x \in X$. If, for every c in E with $0 \ll c$, there is an $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \ll c$, then it is said that x_n converges to x, and this is denoted by $\lim_{n\to\infty} x_n = x$, or $x_n \to x$, $n \to \infty$. If for every c in E with $0 \ll c$, there is an $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a *Cauchy sequence* in X. If every Cauchy sequence is convergent in X, then X is called a *complete cone metric space*.

Huang and Zhang [1] proved that if P is a normal cone then $x_n \in X$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$, $n \to \infty$, and that $x_n \in X$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$, $n, m \to \infty$.

Let (X, d) be a cone metric space. Then the following properties are often useful (particulary when dealing with cone metric spaces in which the cone needs not to be normal):

(p_1) if $u \leq v$ and $v \ll w$, then $u \ll w$,

(p_2) if $0 \leq u \ll c$ for each $c \in \operatorname{int} P$ then $u = 0$,

(p_3) if $a \leq b + c$ for each $c \in \operatorname{int} P$ then $a \leq b$,

(p_4) if $0 \leq x \leq y$, and $a \geq 0$, then $0 \leq ax \leq ay$,

(p_5) if $0 \leq x_n \leq y_n$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, then $0 \leq x \leq y$,

(p_6) if $0 \leq d(x_n, x) \leq b_n$ and $b_n \to 0$, then $d(x_n, x) \ll c$ where x_n and x are, respectively, a sequence and a given point in X,

(p_7) if E is a real Banach space with a cone P and if $a \leq \lambda a$ where $a \in P$ and $0 < \lambda < 1$, then $a = 0$,

(p_8) if $c \in \operatorname{int} P$, $0 \leq a_n$ and $a_n \to 0$, then there exists n_0 such that for all $n > n_0$ we have $a_n \ll c$.

It follows from (p_8) that the sequence x_n converges to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$ and x_n is a Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$. In the case when the cone is not necessarily normal, we have only one half of the statements of Lemmas 1 and 4 from [1]. Also, in this case, the fact that $d(x_n, y_n) \to d(x, y)$ if $x_n \to x$ and $y_n \to y$ is not applicable.

There exist a lot of fixed-point theorems for self-mappings defined on closed subsets of Banach spaces. However, for applications (numerical analysis, optimization, etc.) it is important to consider functions that are not self-mappings, and it is natural to search for sufficient conditions which would guarantee the existence of fixed points for such mappings.

In what follows we suppose only that E is a Banach space, that P is a cone in E with $\operatorname{int} P \neq \emptyset$ and that $\leq$ is the partial ordering with respect to P.

Rhoades [6] proved the following result, generalizing theorems of Assad [7] and Assad and Kirk [8].

Theorem 1.3. *Let X be a Banach space, C a nonempty closed subset of X, and let $T : C \to X$ be a mapping from C into X satisfying the condition*

$$d(Tx, Ty) \leq h \max\left\{ \frac{d(x, y)}{2}, d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{q} \right\}, \tag{1.5}$$

for some $0 < h < 1$, $q \geq 1 + 2h$, and for all x, y in C. Let T have the additional property that for each $x \in \partial C$, (the boundary of C), $Tx \in C$. Then T has the unique fixed point.

Recently Imdad and Kumar [9] extended this result of Rhoades by considering a pair of maps in the following way.

Theorem 1.4. *Let X be a Banach space, let C be a nonempty closed subset of X, and let $F, T : C \to X$ be two mappings satisfying the condition*

$$d(Fx, Fy) \leq h \max\left\{ \frac{d(Tx, Ty)}{2}, d(Tx, Fx), d(Ty, Fy), \frac{d(Tx, Fy) + d(Ty, Fx)}{q} \right\}, \tag{1.6}$$

for some $0 < h < 1$, $q \geq 1 + 2h$, *and for all* $x, y \in C$ *and suppose*

(i) $\partial C \subseteq TC$, $FC \cap C \subset TC$,

(ii) $Tx \in \partial C \Rightarrow Fx \in C$,

(iii) TC *is closed in* X.

Then there exists a coincidence point z *of* F, T *in* X*. Moreover, if* F *and* T *are coincidentally commuting, then* z *is the unique common fixed point of* F *and* T*.*

Recall that a pair (f, g) of mappings is *coincidentally commuting* (see, e.g., [2]) if they commute at their coincidence point, that is, if $fx = gx$ for some $x \in X$, implies $fgx = gfx$.

In [10, 11] these results were extended using complete metric spaces of hyperbolic type, instead of Banach spaces.

2. Results

2.1. Main Result

In [12], assuming only that int $P \neq \emptyset$, Theorems 1.3 and 1.4 are extended to the setting of cone metric spaces. Thus, proper generalizations of the results of Rhoades [6] (for one map) and of Imdad and Kumar [9] (for two maps) were obtained. Example 1.1 of a nonnormal cone shows that the method of proof used in [6, 8, 9] cannot be fully applied in the new setting.

The purpose of this paper is to extend the previous results to the cone metric spaces, but with new contractive conditions. This is worthwhile, since from [2, 13] we know that self-mappings that satisfy the new conditions (given below) do have a unique common fixed point. Let us note that the questions concerning common fixed points for self-mappings in metric spaces, under similar conditions, were considered in [14]. It seems that these questions were not considered for nonself mappings. This is an additional motivation for studying these problems.

We begin with the following definition.

Definition 2.1. Let (X, d) be a cone metric space, let C be a nonempty closed subset of X, and let $f, g : C \to X$. Denote, for $x, y \in C$,

$$M_1^{f,g}(C; x, y) = \left\{ d(gx, gy), d(fx, gx), d(fy, gy), \frac{d(fx, gy) + d(fy, gx)}{2} \right\}. \tag{2.1}$$

Then f is called a *generalized* g_{M_1}*-contractive mapping* of C into X if for some $\lambda \in (0, \sqrt{2} - 1)$ there exists

$$u(x, y) \in M_1^{f,g}(C; x, y), \tag{2.2}$$

such that for all x, y in C

$$d(fx, fy) \leq \lambda \cdot u(x, y). \tag{2.3}$$

Our main result is the following.

Theorem 2.2. *Let (X, d) be a complete cone metric space, let C be a nonempty closed subset of X such that, for each $x \in C$ and each $y \notin C$, there exists a point $z \in \partial C$ such that*

$$d(x,z) + d(z,y) = d(x,y). \tag{2.4}$$

Suppose that f is a generalized g_{M_1}-contractive mapping of C into X and

(i) $\partial C \subseteq gC$, $fC \cap C \subset gC$,

(ii) $gx \in \partial C \Rightarrow fx \in C$,

(iii) *gC is closed in X.*

Then there exists a coincidence point z of f, g in C. Moreover, if the pair (f, g) is coincidentally commuting, then z is the unique common fixed point of f and g.

Proof. We prove the theorem under the hypothesis that neither of the mappings f and g is necessarily a self-mapping. We proceed in several steps.

Step 1 (construction of three sequences). The following construction is the same as the construction used in [10] in the case of hyperbolic metric spaces. It differs slightly from the constructions in [6, 9].

Let $x \in \partial C$ be arbitrary. We construct three sequences: $\{x_n\}$ and $\{z_n\}$ in C and $\{y_n\}$ in $fC \subseteq X$ in the following way. Set $z_0 = x$. Since $z_0 \in \partial C$, by (i) there exists a point $x_0 \in C$ such that $z_0 = gx_0$. Since $gx_0 \in \partial C$, from (ii) we conclude that $fx_0 \in C \cap fC$. Then from (i), $fx_0 \in gC$. Thus, there exists $x_1 \in C$ such that $gx_1 = fx_0 \in C$. Set $z_1 = y_1 = fx_0 = gx_1$ and $y_2 = fx_1$.

If $y_2 \in fC \cap C$, then from (i), $y_2 \in gC$ and so there is a point $x_2 \in C$ such that $gx_2 = y_2 = z_2 = fx_1$.

If $y_2 = fx_1 \notin C$, then z_2 is a point in ∂C, $(z_2 \neq y_2)$ such that $d(y_1, z_2) + d(z_2, y_2) = d(y_1, y_2) = d(fx_0, fx_1)$. By (i), there is $x_2 \in C$ such that $gx_2 = z_2$. Thus $z_2 \in \partial C$ and $d(y_1, z_2) + d(z_2, y_2) = d(y_1, y_2) = d(fx_0, fx_1)$.

Now we set $y_3 = fx_2 = z_3$. Since $fx_2 \in fC \cap C \subseteq gC$, from (ii) there is a point $x_3 \in C$ such that $gx_3 = y_3$.

Note that in the case $z_2 \neq y_2 = fx_1$, we have $z_1 = y_1 = fx_0$ and $z_3 = y_3 = fx_2$.

Continuing the foregoing procedure we construct three sequences: $\{x_n\} \subseteq C$, $\{z_n\} \subset C$ and $\{y_n\} \subseteq fC \subset X$ such that:

(a) $y_n = fx_{n-1}$;

(b) $z_n = gx_n$;

(c) $z_n = y_n$ if and only if $y_n \in C$;

(d) $z_n \neq y_n$ whenever $y_n \notin C$ and then $z_n \in \partial C$ and $d(y_{n-1}, z_n) + d(z_n, y_n) = d(y_{n-1}, y_n)$.

Step 2 ($\{z_n\}$ is a Cauchy sequence). First, note that if $z_n \neq y_n$, then $z_n \in \partial C$, which then implies, by (b), (ii), and (a), that $z_{n+1} = y_{n+1} \in C$. Also, $z_n \neq y_n$ implies that $z_{n-1} = y_{n-1} \in C$, since otherwise $z_{n-1} \in \partial C$, which then implies $z_n = y_n \in C$.

Proof of Step 2

Now we have to estimate $d(z_n, z_{n+1})$. If $d(z_n, z_{n+1}) = 0$ for some n, then it is easy to show that $d(z_n, z_{n+k}) = 0$ for all $k \geq 1$.

Suppose that $d(z_n, z_{n+1}) > 0$ for all n. There are three possibilities:

(1) $z_n = y_n \in C$ and $z_{n+1} = y_{n+1} \in C$;

(2) $z_n = y_n \in C$, but $z_{n+1} \neq y_{n+1}$; and

(3) $z_n \neq y_n$, in which case $z_n \in \partial C$ and $d(y_{n-1}, z_n) + d(z_n, y_n) = d(y_{n-1}, y_n) = d(fx_{n-2}, fx_{n-1})$.

Note that the estimate of $d(z_n, z_{n+1})$ in this cone version differs from those from [6, 8–11]. In the case of convex metric spaces it can be used that, for each $x, y, u \in X$ and each $\lambda \in (0,1)$, it is $\lambda d(u,x) + (1-\lambda)d(u,y) \leq \max\{d(u,x), d(u,y)\}$. In cone spaces the maximum of the set $\{d(u,x), d(u,y)\}$needs not to exist. Therefore, besides (2.4), we have to use here the relation "$\in$", and to consider several cases. In cone metric spaces as well as in metric spaces the key step is Assad-Kirk's induction.

Case 1. Let $z_n = y_n \in C$, and let $z_{n+1} = y_{n+1} \in C$. Then $z_n = y_n = fx_{n-1}$, $z_{n+1} = y_{n+1} = fx_n$ and $z_{n-1} = gx_{n-1}$ (observe that it is not necessarily $z_{n-1} = y_{n-1}$). Then from (2.3),

$$d(z_n, z_{n+1}) = d(y_n, y_{n+1}) = d(fx_{n-1}, fx_n) \leq \lambda \cdot u_n, \tag{2.5}$$

where

$$\begin{aligned} u_n &\in \left\{ d(gx_{n-1}, gx_n), d(fx_{n-1}, gx_{n-1}), d(fx_n, gx_n), \frac{d(fx_{n-1}, gx_n) + d(fx_n, gx_{n-1})}{2} \right\} \\ &= \left\{ d(z_{n-1}, z_n), d(y_n, z_{n-1}), d(y_{n+1}, z_n), \frac{d(y_n, z_n) + d(y_{n+1}, z_{n-1})}{2} \right\} \\ &= \left\{ d(z_{n-1}, z_n), d(z_n, z_{n+1}), \frac{d(z_{n-1}, z_{n+1})}{2} \right\}. \end{aligned} \tag{2.6}$$

Clearly, there are infinitely many n's such that at least one of the following cases holds:

(I) $d(z_n, z_{n+1}) \leq \lambda \cdot d(z_{n-1}, z_n)$,

(II) $d(z_n, z_{n+1}) \leq \lambda \cdot d(z_n, z_{n+1}) \Rightarrow d(z_n, z_{n+1}) = 0$, contradicting the assumption that $d(z_n, z_{n+1}) > 0$ for each n. Hence, (I) holds,

(III) $d(z_n, z_{n+1}) \leq \lambda \cdot (d(z_{n-1}, z_{n+1})/2) \leq (\lambda/2) d(z_{n-1}, z_n) + (1/2) d(z_n, z_{n+1}) \Rightarrow d(z_n, z_{n+1}) \leq \lambda d(z_{n-1}, z_n)$, that is, (I) holds.

From (I), (II), and (III) it follows that in Case 1

$$d(z_n, z_{n+1}) \leq \lambda \cdot d(z_{n-1}, z_n). \tag{2.7}$$

Case 2. Let $z_n = y_n \in C$ but $z_{n+1} \neq y_{n+1}$. Then $z_{n+1} \in \partial C$ and $d(y_n, z_{n+1}) + d(z_{n+1}, y_{n+1}) = d(y_n, y_{n+1})$. It follows that

$$d(z_n, z_{n+1}) = d(y_n, z_{n+1}) = d(y_n, y_{n+1}) - d(z_{n+1}, y_{n+1}) < d(y_n, y_{n+1}), \tag{2.8}$$

that is, according to (2.3), $d(y_n, y_{n+1}) = d(fx_{n-1}, fx_n) \leq h \cdot u_n$, where

$$\begin{aligned}
u_n &\in \left\{ d(gx_{n-1}, gx_n), \mathrm{d}(fx_{n-1}, gx_{n-1}), d(fx_n, gx_n), \frac{d(fx_{n-1}, gx_n) + d(fx_n, gx_{n-1})}{2} \right\} \\
&= \left\{ d(z_{n-1}, z_n), d(y_n, z_{n-1}), d(y_{n+1}, y_n), \frac{d(y_{n+1}, z_{n-1})}{2} \right\} \\
&= \left\{ d(z_{n-1}, z_n), d(z_n, z_{n-1}), d(y_{n+1}, y_n), \frac{d(y_{n+1}, z_{n-1})}{2} \right\} \\
&= \left\{ d(z_{n-1}, z_n), d(y_{n+1}, y_n), \frac{d(y_{n+1}, z_{n-1})}{2} \right\}.
\end{aligned} \tag{2.9}$$

Again, we obtain the following three cases

(I) $d(y_n, y_{n+1}) \leq \lambda \cdot d(z_{n-1}, z_n)$.

(II) $d(y_n, y_{n+1}) \leq \lambda \cdot d(y_n, y_{n+1}) \Rightarrow d(y_n, y_{n+1}) = 0$, contradicting the assumption that $d(z_n, z_{n+1}) > 0$ for each n. It follows that (I) holds.

(III) $d(y_n, y_{n+1}) \leq \lambda \cdot (d(y_{n+1}, z_{n-1})/2) \leq (\lambda/2)d(y_{n+1}, y_n) + (\lambda/2)d(y_n, z_{n-1}) \leq (1/2)d(y_{n+1}, y_n) + (\lambda/2)d(z_n, z_{n-1})$, that is $d(y_n, y_{n+1}) \leq \lambda \cdot d(z_{n-1}, z_n)$.

From (2.8), (I), (II), and (III), we have

$$d(z_n, z_{n+1}) \leq \lambda \cdot d(z_{n-1}, z_n), \tag{2.10}$$

in Case 2.

Case 3. Let $z_n \neq y_n$. Then $z_n \in \partial C, d(y_{n-1}, z_n) + d(z_n, y_n) = d(y_{n-1}, y_n)$ and we have $z_{n+1} = y_{n+1}$ and $z_{n-1} = y_{n-1}$. From this and using (2.4) we get

$$\begin{aligned}
d(z_n, z_{n+1}) = d(z_n, y_{n+1}) &\leq d(z_n, y_n) + d(y_n, y_{n+1}) \\
&= d(y_{n-1}, y_n) - d(z_{n-1}, z_n) + \mathrm{d}(y_n, y_{n+1}).
\end{aligned} \tag{2.11}$$

We have to estimate $d(y_{n-1}, y_n)$ and $d(y_n, y_{n+1})$. Since $y_{n-1} = z_{n-1}$, one can conclude that

$$d(y_{n-1}, y_n) \leq \lambda \cdot d(z_{n-2}, z_{n-1}), \tag{2.12}$$

in view of Case 2. Further,

$$d(y_n, y_{n+1}) = d(fx_{n-1}, fx_n) \leq \lambda \cdot u_n, \tag{2.13}$$

where

$$u_n \in \left\{ d(gx_{n-1}, gx_n), d(fx_{n-1}, gx_{n-1}), d(fx_n, gx_n), \frac{d(fx_{n-1}, gx_n) + d(fx_n, gx_{n-1})}{2} \right\}$$
$$= \left\{ d(z_{n-1}, z_n), d(y_n, y_{n-1}), d(z_n, z_{n+1}), \frac{d(y_n, z_n) + d(y_{n+1}, z_{n-1})}{2} \right\}. \tag{2.14}$$

Since

$$\begin{aligned}\frac{d(y_n, z_n) + d(y_{n+1}, z_{n-1})}{2} &= \frac{d(y_n, z_n) + d(z_{n+1}, z_{n-1})}{2}\\ &= \frac{d(y_n, y_{n-1}) - d(z_{n-1}, z_n) + d(z_{n+1}, z_{n-1})}{2}\\ &\leq \frac{d(y_n, y_{n-1}) - d(z_{n-1}, z_n) + d(z_{n-1}, z_n) + d(z_n, z_{n+1})}{2}\\ &= \frac{d(y_n, y_{n-1}) + d(z_n, z_{n+1})}{2},\end{aligned} \tag{2.15}$$

$y_{n-1} = z_{n-1}$, $y_{n+1} = z_{n+1}$, and $d(y_{n-1}, y_n) \leq \lambda \cdot d(z_{n-2}, z_{n-1})$, we have that

$$d(y_n, y_{n+1}) \leq \lambda \cdot u_n, \tag{2.16}$$

where

$$u_n \in \left\{ d(z_{n-1}, z_n), \lambda \cdot d(z_{n-2}, z_{n-1}), d(z_{n+1}, z_n), \frac{\lambda \cdot d(z_{n-2}, z_{n-1}) + d(z_n, z_{n+1})}{2} \right\}. \tag{2.17}$$

Substituting (2.12) and (2.16) into (2.11) we get

$$d(z_n, z_{n+1}) \leq \lambda \cdot d(z_{n-2}, z_{n-1}) - d(z_{n-1}, z_n) + \lambda \cdot u_n. \tag{2.18}$$

We have now the following four cases:

(I)

$$\begin{aligned}d(z_n, z_{n+1}) &\leq \lambda \cdot d(z_{n-2}, z_{n-1}) - d(z_{n-1}, z_n) + \lambda \cdot d(z_{n-1}, z_n)\\ &= \lambda \cdot d(z_{n-2}, z_{n-1}) - (1-\lambda) d(z_{n-1}, z_n) \leq \lambda \cdot d(z_{n-2}, z_{n-1});\end{aligned} \tag{2.19}$$

(II)

$$\begin{aligned} d(z_n, z_{n+1}) &\le \lambda d(z_{n-2}, z_{n-1}) - d(z_{n-1}, z_n) + \lambda^2 d(z_{n-2}, z_{n-1}) \\ &= \left(\lambda + \lambda^2\right) d(z_{n-2}, z_{n-1}) - d(z_{n-1}, z_n) \le \left(\lambda + \lambda^2\right) d(z_{n-2}, z_{n-1}); \end{aligned} \tag{2.20}$$

(III)

$$\begin{aligned} d(z_n, z_{n+1}) \le \lambda \cdot d(z_{n-2}, z_{n-1}) - d(z_{n-1}, z_n) + \lambda \cdot d(z_n, z_{n+1}) &\Longrightarrow (1-\lambda) d(z_n, z_{n+1}) \\ \le \lambda d(z_{n-2}, z_{n-1}) \Longrightarrow d(z_n, z_{n+1}) \le \frac{\lambda}{1-\lambda} d(z_{n-2}, z_{n-1}); \end{aligned} \tag{2.21}$$

(IV)

$$\begin{aligned} d(z_n, z_{n+1}) &\le \lambda d(z_{n-2}, z_{n-1}) - d(z_{n-1}, z_n) + \frac{\lambda}{2}(\lambda d(z_{n-2}, z_{n-1}) + d(z_n, z_{n+1})) \\ &\le \left(\lambda + \frac{\lambda^2}{2}\right) d(z_{n-2}, z_{n-1}) + \frac{1}{2} d(z_n, z_{n+1}) \Longrightarrow d(z_n, z_{n+1}) \\ &\le \left(2\lambda + \lambda^2\right) d(z_{n-2}, z_{n-1}). \end{aligned} \tag{2.22}$$

It follows from (I), (II), (III), and (IV) that

$$\begin{aligned} &d(z_n, z_{n+1}) \le \mu \cdot d(z_{n-2}, z_{n-1}), \text{ where} \\ &\mu = \max\left\{\lambda, \lambda + \lambda^2, \frac{\lambda}{1-\lambda}, 2\lambda + \lambda^2\right\} = \max\left\{\frac{\lambda}{1-\lambda}, 2\lambda + \lambda^2\right\}. \end{aligned} \tag{2.23}$$

Thus, in all Cases 1–3,

$$d(z_n, z_{n+1}) \le \mu \cdot w_n, \tag{2.24}$$

where $w_n \in \{d(z_{n-2}, z_{n-1}), d(z_{n-1}, z_n)\}$ and

$$\mu = \max\left\{\lambda, \frac{\lambda}{2-\lambda}, \lambda + \lambda^2, \frac{\lambda}{1-\lambda}, 2\lambda + \lambda^2\right\} = \max\left\{\frac{\lambda}{1-\lambda}, 2\lambda + \lambda^2\right\}. \tag{2.25}$$

It is not hard to conclude that for $0 < \lambda < \sqrt{2} - 1$,

$$\max\left\{ \frac{\lambda}{1-\lambda}, 2\lambda + \lambda^2 \right\} = 2\lambda + \lambda^2 < 1. \tag{2.26}$$

Now, following the procedure of Assad and Kirk [8], it can be shown by induction that, for $n > 1$,

$$d(z_n, z_{n+1}) \leq \mu^{(n-1)/2} \cdot w_2, \tag{2.27}$$

where $w_2 \in \{d(z_0, z_1), d(z_1, z_2)\}$.

From (2.27) and using the triangle inequality, we have for $n > m$

$$\begin{aligned} d(z_n, z_m) &\leq d(z_n, z_{n-1}) + d(z_{n-1}, z_{n-2}) + \cdots + d(z_{m+1}, z_m) \\ &\leq \left(\mu^{(n-1)/2} + \mu^{(n-2)/2} + \cdots + \mu^{(m-1)/2} \right) \cdot w_2 \\ &\leq \frac{\sqrt{\mu}^{m-1}}{1-\sqrt{\mu}} \cdot w_2 \longrightarrow 0, \quad \text{as } m \longrightarrow \infty. \end{aligned} \tag{2.28}$$

According to the property (p_8) from the Introduction, $d(z_n, z_m) \ll c$; that is, $\{z_n\}$ is a Cauchy sequence.

Step 3 (Common fixed point for f and g). In this step we use only the definition of convergence in the terms of the relation "$\ll$". The only assumption is that the interior of the cone P is nonempty; so we use neither continuity of vector metric d, nor the Sandwich theorem.

Since $z_n = gx_n \in C \cap gC$ and $C \cap gC$ is complete, there is some point $z \in C \cap gC$ such that $z_n \to z$. Let $w \in C$ be such that $gw = z$. By the construction of $\{z_n\}$, there is a subsequence $\{z_{n(k)}\}$ such that $z_{n(k)} = y_{n(k)} = fx_{n(k)-1}$ and hence $fx_{n(k)-1} \to z$.

We now prove that $fw = z$. We have

$$d(fw, z) \leq d(fw, fx_{n(k)-1}) + d(fx_{n(k)-1}, z) \leq \lambda \cdot u_{n(k)} + d(fx_{n(k)-1}, z), \tag{2.29}$$

where

$$u_{n(k)} \in \left\{ d(gx_{n(k)-1}, gw), d(fx_{n(k)-1}, gx_{n(k)-1}), d(fw, gw), \frac{d(fx_{n(k)-1}, gw) + d(fw, gx_{n(k)-1})}{2} \right\}. \tag{2.30}$$

From the definition of convergence and the fact that $z_{n(k)} = y_{n(k)} = fx_{n(k)-1} \to z$, as $k \to \infty$, we obtain (for the given $c \in E$ with $0 \ll c$)

$$
\begin{aligned}
&(1)\ d(fw,z) \le \lambda \cdot d(gx_{n(k)-1},z) + d(fx_{n(k)-1},z) \ll \lambda \cdot \frac{c}{2\lambda} + \frac{c}{2} = c; \\
&(2)\ d(fw,z) \le \lambda \cdot d(fx_{n(k)-1}, gx_{n(k)-1}) + d(fx_{n(k)-1},z) \\
&\qquad \le \lambda \cdot \left(d(fx_{n(k)-1},z) + d(z, gx_{n(k)-1})\right) + d(fx_{n(k)-1},z) \\
&\qquad = (\lambda+1) \cdot d(fx_{n(k)-1},z) + \lambda \cdot d(z, gx_{n(k)-1}) \\
&\qquad \ll (\lambda+1) \cdot \frac{c}{2(\lambda+1)} + \lambda \cdot \frac{c}{2\lambda} = c; \\
&(3)\ d(fw,z) \le \lambda \cdot d(fw,z) + d(fx_{n(k)-1},z) \Longrightarrow d(fw,z) \\
&\qquad \le \frac{1}{1-\lambda} \cdot d(fx_{n(k)-1},z) \ll \frac{1}{1-\lambda} \cdot \frac{c}{1/(1-\lambda)} = c; \\
&(4)\ d(fw,z) \le \lambda \cdot \frac{d(fx_{n(k)-1},z) + d(fw, gx_{n(k)-1})}{2} + d(fx_{n(k)-1},z) \\
&\qquad \le \lambda \cdot \frac{d(fx_{n(k)-1},z) + d(z, gx_{n(k)-1})}{2} + \frac{1}{2} d(fw,z) + d(fx_{n(k)-1},z); \text{ i.e.,} \\
&\quad d(fw,z) \le (\lambda+2) \cdot d(fx_{n(k)-1},z) + \lambda \cdot d(z, gx_{n(k)-1}) \\
&\qquad \ll (\lambda+2) \frac{c}{2(\lambda+2)} + \lambda \frac{c}{2\lambda} = c.
\end{aligned}
\tag{2.31}
$$

In all the cases we obtain $d(fw,z) \ll c$ for each $c \in \operatorname{int} P$. According to the property (p_2), it follows that $d(fw,z) = 0$, that is, $fw = z$.

Suppose now that f and g are coincidentally commuting. Then

$$z = fw = gw \Longrightarrow fz = fgw = gfw = gz. \tag{2.32}$$

Then from (2.3),

$$d(fz,z) = d(fz,fw) \le \lambda \cdot u, \tag{2.33}$$

where

$$u \in \left\{ d(gz, gw), d(fz, gz), d(fw, gw), \frac{d(fz, gw) + d(fw, gz)}{2} \right\}$$
$$= \left\{ d(fz, z), d(fz, gz), d(z, z), \frac{d(fz, z) + d(z, fz)}{2} \right\} = \{d(fz, z), 0\}. \quad (2.34)$$

Hence, we obtain the following cases:

$$d(fz, z) \le \lambda \cdot d(fz, z) \Longrightarrow d(fz, z) = 0,$$
$$d(fz, z) \le \lambda \cdot 0 = 0 \Longrightarrow d(fz, z) = 0, \quad (2.35)$$

which implies that $fz = z$, that is, z is a common fixed point of f and g.

Uniqueness of the common fixed point follows easily. This completes the proof of the theorem. □

2.2. Examples

We present now two examples showing that Theorem 2.2 is a proper extension of the known results. In both examples, the conditions of Theorem 2.2 are fulfilled, but in the first one (because of nonnormality of the cone) the main theorems from [6, 9] cannot be applied. This shows that Theorem 2.2 is more general, that is, the main theorems from [6, 9] can be obtained as its special cases (for $0 < \lambda < \sqrt{2} - 1$) taking $\|\cdot\| = |\cdot|$, $E = \mathbb{R}$ and $P = [0, +\infty[$.

Example 2.3 (The case of a nonnormal cone). Let $X = \mathbb{R}$, let $C = [0, 1]$, and $E = C^1_{\mathbb{R}}[0, 1]$, and let $P = \{\varphi \in E : \varphi(t) \ge 0,\ t \in [0, 1]\}$. The mapping $d : X \times X \to E$ is defined in the following way: $d(x, y) = |x - y|\varphi$, where $\varphi \in P$ is a fixed function, for example, $\varphi(t) = e^t$. Take functions $f(x) = ax$, $g(x) = bx$, $0 < a < 1 < b$, so that $a/b \le \lambda < \sqrt{2} - 1$, which map the set $C = [0, 1]$ into $\mathbb{R}$. We have that (X, d) is a complete cone metric space with a nonnormal cone having the nonempty interior. The topological and "metric" notions are used in the sense of definitions from [15, 16]. For example, one easily checks the condition (2.4), that is, that for $x \in [0, 1]$, $y \notin [0, 1]$ the following holds

$$d(x, 1) + d(1, y) = d(x, y) \Longleftrightarrow |1 - x|\varphi + |y - 1|\varphi$$
$$= |y - x|\varphi \Longleftrightarrow (1 - x)\varphi + (y - 1)\varphi = (y - x)\varphi. \quad (2.36)$$

The mappings f and g are weakly compatible, that is, they commute in their fixed point $x = 0$. All the conditions of Theorem 2.2 are fulfilled, and so the nonself mappings f and g have a unique common fixed point $x = 0$.

Example 2.4 (The case of a normal cone). Let $X = [0, +\infty[$, let $C = [0, 1] \subset X$, let $E = \mathbb{R}^2$, and let $P = \{(x, y) : x \ge 0, y \ge 0\}$. The mapping $d : X \times X \to E$ is defined as $d(x, y) = (|x - y|, \alpha|x - y|)$, $\alpha \ge 0$. Take the functions $f(x) = ax$, $g(x) = bx$, $0 < a < 1 < b$, so that $a/b < \sqrt{2} - 1$, which map the set $C = [0,1]$ into $\mathbb{R}$. We have that (X,d) is a complete cone metric space with a normal cone having the normal coefficient $K = 1$, whose interior is obviously nonempty. All the conditions of Theorem 2.2 are fulfilled. We check again the condition (2.4), that is, that for $x \in C = [0,1]$, $y \notin C = [0,1]$ the following holds

$$\begin{aligned} d(x,1) + d(1,y) = d(x,y) &\Longleftrightarrow (|1-x|, \alpha|1-x|) + \left(|y-1|, \alpha|y-1|\right) \\ &= \left(|y-x|, \alpha|y-x|\right) \Longleftrightarrow (1-x) + (y-1) \\ &= y - x, \\ \alpha(1-x) + \alpha(y-1) = \alpha(y-x). & \end{aligned} \tag{2.37}$$

The mappings f and g are weakly compatible, that is, they commute in their fixed point $x = 0$. All the conditions of Theorem 2.2 are again fulfilled. The point $x = 0$ is the unique common fixed point for nonself mappings f and g.

2.3. *Further Results*

Remark 2.5. The following definition is a special case of Definition 2.1 when (X,d) is a metric space. But when (X,d) is a cone metric space, which is not a metric space, this is not true. Indeed, there may exist $x, y \in X$ such that the vectors $d(fx,gx), d(fy,gy)$ and $(1/2)(d(fx,gx) + d(fy,gy))$ are incomparable. For the same reason Theorems 2.2 and 2.7 (given below) are incomparable.

Definition 2.6. Let (X,d) be a cone metric space, let C be a nonempty closed subset of X, and let $f, g : C \to X$. Denote

$$M_2^{f,g}(C;x,y) = \left\{ d(gx,gy), \frac{d(fx,gx) + d(fy,gy)}{2}, \frac{d(fx,gy) + d(fy,gx)}{2} \right\}. \tag{2.38}$$

Then f is called a *generalized* g_{M_2}*-contractive mapping* from C into X if for some $\lambda \in (0, \sqrt{2}-1)$ there exists

$$u(x,y) \in M_2^{f,g}(C;x,y), \tag{2.39}$$

such that for all x, y in C

$$d(fx,fy) \le \lambda \cdot u(x,y). \tag{2.40}$$

Our next result is the following.

Theorem 2.7. *Let (X,d) be a complete cone metric space, and let C be a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x,z) + d(z,y) = d(x,y). \tag{2.41}$$

Suppose that f is a generalized g_{M_2}-contractive mapping of C into X and

(i) $\partial C \subseteq gC$, $fC \cap C \subset gC$,

(ii) $gx \in \partial C \Rightarrow fx \in C$,

(iii) gC *is closed in* X.

Then there exists a coincidence point z of f and g in C. Moreover, if the pair (f, g) is coincidentally commuting, then z is the unique common fixed point of f and g.

The proof of this theorem is very similar to the proof of Theorem 2.2 and it is omitted. We now list some corollaries of Theorems 2.2 and 2.7.

Corollary 2.8. *Let (X, d) be a complete cone metric space, and let C be a nonempty closed subset of X such that, for each $x \in C$ and each $y \notin C$, there exists a point $z \in \partial C$ such that*

$$d(x,z) + d(z,y) = d(x,y). \tag{2.42}$$

Let $f, g : C \to X$ be such that

$$d(fx, fy) \leq \lambda \cdot d(gx, gy), \tag{2.43}$$

for some $0 < \lambda < \sqrt{2} - 1$ and for all $x, y \in C$.

Suppose, further, that f, g, and C satisfy the following conditions:

(i) $\partial C \subseteq gC$, $fC \cap C \subset gC$,

(ii) $gx \in \partial C \Rightarrow fx \in C$,

(iii) gC *is closed in* X.

Then there exists a coincidence point z of f and g in C. Moreover, if (f, g) is a coincidentally commuting pair, then z is the unique common fixed point of f and g.

Corollary 2.9. *Let (X, d) be a complete cone metric space, and let C be a nonempty closed subset of X such that, for each $x \in C$ and each $y \notin C$, there exists a point $z \in \partial C$ such that*

$$d(x,z) + d(z,y) = d(x,y). \tag{2.44}$$

Let $f, g : C \to X$ be such that

$$d(fx, fy) \leq \lambda \cdot (d(fx, gx) + d(fy, gy)), \tag{2.45}$$

for some $0 < \lambda < \sqrt{2} - 1$ and for all $x, y \in C$.

Suppose, further, that f, g, and C satisfy the following conditions:

(i) $\partial C \subseteq gC$, $fC \cap C \subset gC$,

(ii) $gx \in \partial C \Rightarrow fx \in C$,

(iii) gC *is closed in* X.

Then there exists a coincidence point z of f and g in C. Moreover, if (f, g) is a coincidentally commuting pair, then z is the unique common fixed point of f and g.

Corollary 2.10. *Let (X, d) be a complete cone metric space, and let C be a nonempty closed subset of X such that, for each $x \in C$ and each $y \notin C$, there exists a point $z \in \partial C$ such that*

$$d(x, z) + d(z, y) = d(x, y). \tag{2.46}$$

Let $f, g : C \to X$ be such that

$$d(fx, fy) \leq \lambda \cdot (d(fx, gy) + d(fy, gx)), \tag{2.47}$$

for some $0 < \lambda < \sqrt{2} - 1$ and for all $x, y \in C$.

Suppose, further, that f, g, and C satisfy the following conditions:

(i) $\partial C \subseteq gC$, $fC \cap C \subset gC$,

(ii) $gx \in \partial C \Rightarrow fx \in C$,

(iii) gC *is closed in* X.

Then there exists a coincidence point z of f and g in C. Moreover, if (f, g) is a coincidentally commuting pair, then z is the unique common fixed point of f and g.

Remark 2.11. Corollaries 2.8–2.10 are the corresponding theorems of Abbas and Jungck from [2] in the case that f, g are nonself mappings.

Remark 2.12. If (X, d) is a metrically convex cone metric space, that is, if for each $x, y \in X$, $x \neq y$ there is $z \in X$, $x \neq z \neq y$ such that $d(x, z) = d(x, y) + d(y, z)$, we do not know whether (2.4) holds for every nonempty closed subset C in X (see [8]).

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