Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2009, Article ID 493965, 11 pages
doi:10.1155/2009/493965

Research Article

Some Common Fixed Point Results in Cone Metric Spaces

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Received 5 September 2008; Revised 26 December 2008; Accepted 5 February 2009

Recommended by Lech Górniewicz

We prove a result on points of coincidence and common fixed points for three self-mappings satisfying generalized contractive type conditions in cone metric spaces. We deduce some results on common fixed points for two self-mappings satisfying contractive type conditions in cone metric spaces. These results generalize some well-known recent results.

1. Introduction

Huang and Zhang [1] recently have introduced the concept of cone metric space, where the set of real numbers is replaced by an ordered Banach space, and they have established some fixed point theorems for contractive type mappings in a normal cone metric space. Subsequently, some other authors [2–5] have generalized the results of Huang and Zhang [1] and have studied the existence of common fixed points of a pair of self mappings satisfying a contractive type condition in the framework of normal cone metric spaces.

Vetro [5] extends the results of Abbas and Jungck [2] and obtains common fixed point of two mappings satisfying a more general contractive type condition. Rezapour and Hamlbarani [6] prove that there aren't normal cones with normal constant $c < 1$ and for each $k > 1$ there are cones with normal constant $c > k$. Also, omitting the assumption of normality they obtain generalizations of some results of [1]. In [7] Di Bari and Vetro obtain results on points of coincidence and common fixed points in nonnormal cone metric spaces. In this paper, we obtain points of coincidence and common fixed points for three self-mappings satisfying generalized contractive type conditions in a complete cone metric space. Our results improve and generalize the results in [1, 2, 5, 6, 8].

2. Preliminaries

We recall the definition of cone metric spaces and the notion of convergence [1]. Let E be a real Banach space and P be a subset of E. The subset P is called an *order cone* if it has the following properties:

(i) P is nonempty, closed, and $P \neq \{\mathbf{0}\}$;

(ii) $0 \leqslant a, b \in \mathbb{R}$ and $x, y \in P \Rightarrow ax + by \in P$;

(iii) $P \cap (-P) = \{\mathbf{0}\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leqslant$ on E with respect to P by $x \leqslant y$ if and only if $y - x \in P$. We will write $x < y$ if $x \leqslant y$ and $x \neq y$, while $x \ll y$ will stands for $y - x \in \operatorname{Int} P$, where $\operatorname{Int} P$ denotes the interior of P. The cone P is called *normal* if there is a number $\kappa \geqslant 1$ such that for all $x, y \in E$:

$$\mathbf{0} \leqslant x \leqslant y \Longrightarrow \|x\| \leqslant \kappa \|y\|. \tag{2.1}$$

The least number $\kappa \geqslant 1$ satisfying (2.1) is called the *normal constant* of P.

In the following we always suppose that E is a real Banach space and P is an order cone in E with $\operatorname{Int} P \neq \varnothing$ and $\leqslant$ is the partial ordering with respect to P.

Definition 2.1. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

(i) $\mathbf{0} \leqslant d(x, y)$, for all $x, y \in X$, and $d(x, y) = \mathbf{0}$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leqslant d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a *cone metric* on X, and (X, d) is called a *cone metric space*.

Let $\{x_n\}$ be a sequence in X, and $x \in X$. If for every $c \in E$, with $\mathbf{0} \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0, d(x_n, x) \ll c$, then $\{x_n\}$ is said to be *convergent*, $\{x_n\}$ converges to x and x is the *limit* of $\{x_n\}$. We denote this by $\lim_n x_n = x$, or $x_n \to x$, as $n \to \infty$. If for every $c \in E$ with $\mathbf{0} \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a *Cauchy sequence* in X. If every Cauchy sequence is convergent in X, then X is called a *complete cone metric space*.

3. Main Results

First, we establish the result on points of coincidence and common fixed points for three self-mappings and then show that this result generalizes some of recent results of fixed point.

A pair (f, T) of self-mappings on X is said to be weakly compatible if they commute at their coincidence point (i.e., $fTx = Tfx$ whenever $fx = Tx$). A point $y \in X$ is called point of coincidence of a family T_j, $j \in J$, of self-mappings on X if there exists a point $x \in X$ such that $y = T_j x$ for all $j \in J$.

Lemma 3.1. *Let X be a nonempty set and the mappings $S, T, f : X \to X$ have a unique point of coincidence v in X. If (S, f) and (T, f) are weakly compatibles, then S, T, and f have a unique common fixed point.*

Proof. Since v is a point of coincidence of S, T, and f. Therefore, $v = fu = Su = Tu$ for some $u \in X$. By weakly compatibility of (S, f) and (T, f) we have

$$Sv = Sfu = fSu = fv, \qquad Tv = Tfu = fTu = fv. \tag{3.1}$$

It implies that $Sv = Tv = fv = w$ (say). Then w is a point of coincidence of S, T, and f. Therefore, $v = w$ by uniqueness. Thus v is a unique common fixed point of S, T, and f. □

Let (X, d) be a cone metric space, S, T, f be self-mappings on X such that $S(X) \cup T(X) \subseteq f(X)$ and $x_0 \in X$. Choose a point x_1 in X such that $fx_1 = Sx_0$. This can be done since $S(X) \subseteq f(X)$. Successively, choose a point x_2 in X such that $fx_2 = Tx_1$. Continuing this process having chosen $x_1, \ldots, x_{2k}$, we choose x_{2k+1} and x_{2k+2} in X such that

$$\begin{aligned} fx_{2k+1} &= Sx_{2k}, \\ fx_{2k+2} &= Tx_{2k+1}, \quad k = 0, 1, 2, \ldots. \end{aligned} \tag{3.2}$$

The sequence $\{fx_n\}$ is called an *S-T-sequence* with initial point x_0.

Proposition 3.2. *Let (X, d) be a cone metric space and P be an order cone. Let $S, T, f : X \to X$ be such that $S(X) \cup T(X) \subseteq f(X)$. Assume that the following conditions hold:*

(i) *$d(Sx, Ty) \leqslant \alpha d(fx, Sx) + \beta d(fy, Ty) + \gamma d(fx, fy)$, for all $x, y \in X$, with $x \neq y$, where α, β, γ are nonnegative real numbers with $\alpha + \beta + \gamma < 1$;*

(ii) *$d(Sx, Tx) < d(fx, Sx) + d(fx, Tx)$, for all $x \in X$, whenever $Sx \neq Tx$.*

Then every S-T-sequence with initial point $x_0 \in X$ is a Cauchy sequence.

Proof. Let x_0 be an arbitrary point in X and $\{fx_n\}$ be an S-T-sequence with initial point x_0. First, we assume that $fx_n \neq fx_{n+1}$ for all $n \in \mathbb{N}$. It implies that $x_n \neq x_{n+1}$ for all n. Then,

$$\begin{aligned} d(fx_{2k+1}, fx_{2k+2}) &= d(Sx_{2k}, Tx_{2k+1}) \\ &\leqslant \alpha d(fx_{2k}, Sx_{2k}) + \beta d(fx_{2k+1}, Tx_{2k+1}) + \gamma d(fx_{2k}, fx_{2k+1}) \\ &\leqslant [\alpha + \gamma] d(fx_{2k}, fx_{2k+1}) + \beta d(fx_{2k+1}, fx_{2k+2}). \end{aligned} \tag{3.3}$$

It implies that

$$[1 - \beta] d(fx_{2k+1}, fx_{2k+2}) \leqslant [\alpha + \gamma] d(fx_{2k}, fx_{2k+1}), \tag{3.4}$$

so

$$d(fx_{2k+1}, fx_{2k+2}) \leqslant \left[\frac{\alpha + \gamma}{1 - \beta}\right] d(fx_{2k}, fx_{2k+1}). \tag{3.5}$$

Similarly, we obtain

$$d(fx_{2k+2}, fx_{2k+3}) \leqslant \left[\frac{\beta+\gamma}{1-\alpha}\right] d(fx_{2k+1}, fx_{2k+2}). \tag{3.6}$$

Now, by induction, for each $k = 0, 1, 2, \ldots$, we deduce

$$\begin{aligned} d(fx_{2k+1}, fx_{2k+2}) &\leqslant \left[\frac{\alpha+\gamma}{1-\beta}\right] d(fx_{2k}, fx_{2k+1}) \\ &\leqslant \left[\frac{\alpha+\gamma}{1-\beta}\right]\left[\frac{\beta+\gamma}{1-\alpha}\right] d(fx_{2k-1}, fx_{2k}) \\ &\leqslant \cdots \leqslant \left[\frac{\alpha+\gamma}{1-\beta}\right]\left(\left[\frac{\beta+\gamma}{1-\alpha}\right]\left[\frac{\alpha+\gamma}{1-\beta}\right]\right)^{k} d(fx_0, fx_1), \\ d(fx_{2k+2}, fx_{2k+3}) &\leqslant \left[\frac{\beta+\gamma}{1-\alpha}\right] d(fx_{2k+1}, fx_{2k+2}) \\ &\leqslant \cdots \leqslant \left(\left[\frac{\beta+\gamma}{1-\alpha}\right]\left[\frac{\alpha+\gamma}{1-\beta}\right]\right)^{k+1} d(fx_0, fx_1). \end{aligned} \tag{3.7}$$

Let

$$\lambda = \left[\frac{\alpha+\gamma}{1-\beta}\right], \qquad \mu = \left[\frac{\beta+\gamma}{1-\alpha}\right]. \tag{3.8}$$

Then $\lambda\mu < 1$. Now, for $p < q$, we have

$$\begin{aligned} d(fx_{2p+1}, fx_{2q+1}) &\leqslant d(fx_{2p+1}, fx_{2p+2}) + d(fx_{2p+2}, fx_{2p+3}) + d(fx_{2p+3}, fx_{2p+4}) \\ &\quad + \cdots + d(fx_{2q}, fx_{2q+1}) \\ &\leqslant \left[\lambda\sum_{i=p}^{q-1}(\lambda\mu)^i + \sum_{i=p+1}^{q}(\lambda\mu)^i\right] d(fx_0, fx_1) \\ &\leqslant \left[\frac{\lambda(\lambda\mu)^p}{1-\lambda\mu} + \frac{(\lambda\mu)^{p+1}}{1-\lambda\mu}\right] d(fx_0, fx_1) \\ &\leqslant (1+\mu)\lambda\frac{(\lambda\mu)^p}{1-\lambda\mu} d(fx_0, fx_1) \\ &\leqslant \frac{2(\lambda\mu)^p}{1-\lambda\mu} d(fx_0, fx_1). \end{aligned} \tag{3.9}$$

In analogous way, we deduce

$$d(fx_{2p}, fx_{2q+1}) \leqslant (1+\lambda)\frac{(\lambda\mu)^p}{1-\lambda\mu}d(fx_0, fx_1) \leq \frac{2(\lambda\mu)^p}{1-\lambda\mu}d(fx_0, fx_1),$$
$$d(fx_{2p}, fx_{2q}) \leqslant (1+\lambda)\frac{(\lambda\mu)^p}{1-\lambda\mu}d(fx_0, fx_1) \leq \frac{2(\lambda\mu)^p}{1-\lambda\mu}d(fx_0, fx_1), \tag{3.10}$$
$$d(fx_{2p+1}, fx_{2q}) \leqslant (1+\mu)\lambda\frac{(\lambda\mu)^p}{1-\lambda\mu}d(fx_0, fx_1) \leq \frac{2(\lambda\mu)^p}{1-\lambda\mu}d(fx_0, fx_1).$$

Hence, for $0 < n < m$

$$d(fx_n, fx_m) \leqslant \frac{2(\lambda\mu)^p}{1-\lambda\mu}, \tag{3.11}$$

where p is the integer part of $n/2$.

Fix $\mathbf{0} \ll c$ and choose $I(\mathbf{0}, \delta) = \{x \in E : \|x\| < \delta\}$ such that $c + I(\mathbf{0}, \delta) \subset \operatorname{Int} P$. Since

$$\lim_{p\to\infty} \frac{2(\lambda\mu)^p}{1-\lambda\mu}d(fx_0, fx_1) = \mathbf{0}, \tag{3.12}$$

there exists $n_0 \in \mathbb{N}$ be such that

$$\frac{2(\lambda\mu)^p}{1-\lambda\mu}d(fx_0, fx_1) \in I(\mathbf{0}, \delta) \tag{3.13}$$

for all $p \geq n_0$. The choice of $I(\mathbf{0}, \delta)$ assures

$$c - \frac{2(\lambda\mu)^p}{1-\lambda\mu}d(fx_0, fx_1) \in \operatorname{Int} P, \tag{3.14}$$

so

$$\frac{2(\lambda\mu)^p}{1-\lambda\mu}d(fx_0, fx_1) \ll c. \tag{3.15}$$

Consequently, for all $n, m \in \mathbb{N}$, with $2n_0 < n < m$, we have

$$d(fx_n, fx_m) \ll c, \tag{3.16}$$

and hence $\{fx_n\}$ is a Cauchy sequence.

Now, we suppose that $fx_m = fx_{m+1}$ for some $m \in \mathbb{N}$. If $x_m = x_{m+1}$ and $m = 2k$, by (ii) we have

$$\begin{aligned} d(fx_{2k+1}, fx_{2k+2}) &= d(Sx_{2k}, Tx_{2k+1}) \\ &< d(fx_{2k}, Sx_{2k}) + d(fx_{2k+1}, Tx_{2k+1}) \\ &= d(fx_{2k+1}, fx_{2k+2}), \end{aligned} \tag{3.17}$$

which implies $fx_{2k+1} = fx_{2k+2}$. If $x_m \neq x_{m+1}$ we use (i) to obtain $fx_{2k+1} = fx_{2k+2}$. Similarly, we deduce that $fx_{2k+2} = fx_{2k+3}$ and so $fx_n = fx_m$ for every $n \geq m$. Hence $\{fx_n\}$ is a Cauchy sequence. □

Theorem 3.3. *Let (X, d) be a cone metric space and P be an order cone. Let $S, T, f : X \to X$ be such that $S(X) \cup T(X) \subseteq f(X)$. Assume that the following conditions hold:*

(i) *$d(Sx, Ty) \leqslant \alpha d(fx, Sx) + \beta d(fy, Ty) + \gamma d(fx, fy)$, for all $x, y \in X$, with $x \neq y$, where α, β, γ are nonnegative real numbers with $\alpha + \beta + \gamma < 1$;*

(ii) *$d(Sx, Tx) < d(fx, Sx) + d(fx, Tx)$, for all $x \in X$, whenever $Sx \neq Tx$.*

If $f(X)$ or $S(X) \cup T(X)$ is a complete subspace of X, then S, T, and f have a unique point of coincidence. Moreover, if (S, f) and (T, f) are weakly compatibles, then S, T, and f have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. By Proposition 3.2 every S-T-sequence $\{fx_n\}$ with initial point x_0 is a Cauchy sequence. If $f(X)$ is a complete subspace of X, there exist $u, v \in X$ such that $fx_n \to v = fu$ (this holds also if $S(X) \cup T(X)$ is complete with $v \in S(X) \cup T(X)$). From

$$\begin{aligned} d(fu, Su) &\leqslant d(fu, fx_{2n}) + d(fx_{2n}, Su) \\ &\leqslant d(v, fx_{2n}) + d(Tx_{2n-1}, Su) \\ &\leqslant d(v, fx_{2n}) + \alpha d(fu, Su) + \beta d(fx_{2n-1}, Tx_{2n-1}) + \gamma d(fu, fx_{2n-1}), \end{aligned} \tag{3.18}$$

we obtain

$$d(fu, Su) \leqslant \frac{1}{1-\alpha}\left[d(v, fx_{2n}) + \beta d(fx_{2n-1}, fx_{2n}) + \gamma d(v, fx_{2n-1})\right]. \tag{3.19}$$

Fix $\mathbf{0} \ll c$ and choose $n_0 \in \mathbb{N}$ be such that

$$d(v, fx_{2n}) \ll kc, \qquad d(fx_{2n-1}, fx_{2n}) \ll kc, \qquad d(v, fx_{2n-1}) \ll kc \tag{3.20}$$

for all $n \geq n_0$, where $k = (1-\alpha)/(1+\beta+\gamma)$. Consequently $d(fu, Su) \ll c$ and hence $d(fu, Su) \ll c/m$ for every $m \in \mathbb{N}$. From

$$\frac{c}{m} - d(fu, Su) \in \operatorname{Int} P, \tag{3.21}$$

being P closed, as $m \to \infty$, we deduce $-d(fu, Su) \in P$ and so $d(fu, Su) = \mathbf{0}$. This implies that $fu = Su$.

Similarly, by using the inequality,

$$d(fu, Tu) \leqslant d(fu, fx_{2n+1}) + d(fx_{2n+1}, Tu), \tag{3.22}$$

we can show that $fu = Tu$. It implies that v is a point of coincidence of S, T, and f, that is

$$v = fu = Su = Tu. \tag{3.23}$$

Now, we show that S, T, and f have a unique point of coincidence. For this, assume that there exists another point v^* in X such that $v^* = fu^* = Su^* = Tu^*$, for some u^* in X. From

$$\begin{aligned} d(v, v^*) &= d(Su, Tu^*) \\ &\leqslant \alpha d(fu, Su) + \beta d(fu^*, Tu^*) + \gamma d(fu, fu^*) \\ &\leqslant \alpha d(v, v) + \beta d(v^*, v^*) + \gamma d(v, v^*) \\ &\leqslant \gamma d(v, v^*) \end{aligned} \tag{3.24}$$

we deduce $v = v^*$. Moreover, if (S, f) and (T, f) are weakly compatibles, then

$$Sv = Sfu = fSu = fv, \qquad Tv = Tfu = fTu = fv, \tag{3.25}$$

which implies $Sv = Tv = fv = w$ (say). Then w is a point of coincidence of S, T, and f therefore, $v = w$, by uniqueness. Thus v is a unique common fixed point of S, T, and f. □

From Theorem 3.3, if we choose $S = T$, we deduce the following theorem.

Theorem 3.4. *Let (X, d) be a cone metric space, P be an order cone and $T, f : X \to X$ be such that $T(X) \subseteq f(X)$. Assume that the following condition holds:*

$$d(Tx, Ty) \leqslant \alpha d(fx, Tx) + \beta d(fy, Ty) + \gamma d(fx, fy) \tag{3.26}$$

for all $x, y \in X$ where $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$.

If $f(X)$ or $T(X)$ is a complete subspace of X, then T and f have a unique point of coincidence. Moreover, if the pair (T, f) is weakly compatible, then T and f have a unique common fixed point.

Theorem 3.4 generalizes Theorem 1 of [5].

Remark 3.5. In Theorem 3.4 the condition (3.26) can be replaced by

$$d(Tx, Ty) \leqslant \alpha[d(fx, Tx) + d(fy, Ty)] + \gamma d(fx, fy) \tag{3.27}$$

for all $x, y \in X$, where $\alpha, \gamma \in [0, 1)$ with $2\alpha + \gamma < 1$.

(3.27)⇒(3.26) is obivious. (3.26)⇒(3.27). If in (3.26) interchanging the roles of x and y and adding the resultant inequality to (3.26), we obtain

$$d(Tx,Ty) \leqslant \frac{\alpha+\beta}{2}[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy). \tag{3.28}$$

From Theorem 3.4, we deduce the followings corollaries.

Corollary 3.6. *Let (X,d) be a cone metric space, P be an order cone and the mappings $T,f : X \to X$ satisfy*

$$d(Tx,Ty) \leqslant \gamma d(fx,fy) \tag{3.29}$$

for all $x,y \in X$ where, $0 \leqslant \gamma < 1$. If $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, then T and f have a unique point of coincidence. Moreover, if the pair (T,f) is weakly compatible, then T and f have a unique common fixed point.

Corollary 3.6 generalizes Theorem 2.1 of [2], Theorem 1 of [1], and Theorem 2.3 of [6].

Corollary 3.7. *Let (X,d) be a cone metric space, P be an order cone and the mappings $T,f : X \to X$ satisfy*

$$d(Tx,Ty) \leqslant \alpha[d(fx,Tx)+d(fy,Ty)] \tag{3.30}$$

for all $x,y \in X$, where $0 \leqslant \alpha < 1/2$. If $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, then T and f have a unique point of coincidence. Moreover, if the pair (T,f) is weakly compatible, then T and f have a unique common fixed point.

Corollary 3.7 generalizes Theorem 2.3 of [2], Theorem 3 of [1], and Theorem 2.6 of [6].

Example 3.8. Let $X = \{a,b,c\}$, $E = \mathbb{R}^2$ and $P = \{(x,y) \in E \mid x,y \geqslant 0\}$. Define $d : X \times X \to E$ as follows:

$$d(x,y) = \begin{cases} (0,0) & \text{if } x = y, \\ \left(\frac{5}{7},5\right) & \text{if } x \neq y,\ x,y \in X-\{b\}, \\ (1,7) & \text{if } x \neq y,\ x,y \in X-\{c\}, \\ \left(\frac{4}{7},4\right) & \text{if } x \neq y,\ x,y \in X-\{a\}. \end{cases} \tag{3.31}$$

Define mappings $f,T : X \to X$ as follow:

$$f(x) = x, \qquad T(x) = \begin{cases} c, & \text{if } x \neq b, \\ a, & \text{if } x = b. \end{cases} \tag{3.32}$$

Then, if $2\alpha + \gamma < 1$

$$\begin{aligned}\left(\frac{7\alpha+4\gamma}{7}, 7\alpha+4\gamma\right) &\leqslant \left(\frac{8\alpha+4\gamma}{7}, 8\alpha+4\gamma\right)\\ &\leqslant \left(\frac{4(2\alpha+\gamma)}{7}, 4(2\alpha+\gamma)\right)\\ &< \left(\frac{4}{7}, 4\right) < \left(\frac{5}{7}, 5\right),\end{aligned} \tag{3.33}$$

which implies

$$\alpha[d(fb, Tb) + d(fc, Tc)] + \gamma d(fb, fc) < d(Tb, Tc), \tag{3.34}$$

for all $\alpha, \gamma \in [0, 1)$ with $2\alpha + \gamma < 1$.

Therefore, Theorem 3.4 is not applicable to obtain fixed point of T or common fixed points of f and T.

Now define a constant mapping $S : X \to X$ by $Sx = c$, then for $\alpha = 0 = \gamma, \beta = 5/7$.

$$\begin{aligned}d(Sx, Ty) &= \begin{cases}(0,0), & \text{if } y \neq b,\\ \left(\frac{5}{7}, 5\right), & \text{if } y = b,\end{cases}\\ \alpha d(fx, Sx) + \beta d(fy, Ty) + \gamma d(fx, fy) &= \left(\frac{5}{7}, 5\right) \quad \text{if } y = b.\end{aligned} \tag{3.35}$$

It follows that all conditions of Theorem 3.3 are satisfied for $\alpha = 0 = \gamma, \beta = 5/7$ and so S, T, and f have a unique point of coincidence and a unique common fixed point c.

4. Applications

In this section, we prove an existence theorem for the common solutions for two Urysohn integral equations. Throughout this section let $X = C([a, b], \mathbb{R}^n)$, $P = \{(u, v) \in \mathbb{R}^2 : u, v \geq 0\}$, and $d(x, y) = (\|x - y\|_\infty, p\|x - y\|_\infty)$ for every $x, y \in X$, where $p \geq 0$ is a constant. It is easily seen that (X, d) is a complete cone metric space.

Theorem 4.1. *Consider the Urysohn integral equations*

$$\begin{aligned}x(t) &= \int_a^b K_1(t, s, x(s))ds + g(t),\\ x(t) &= \int_a^b K_2(t, s, x(s))ds + h(t),\end{aligned} \tag{4.1}$$

where $t \in [a, b] \subset \mathbb{R}$, $x, g, h \in X$. Assume that $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ are such that

(i) $F_x, G_x \in X$ *for each* $x \in X$, *where*

$$F_x(t) = \int_a^b K_1(t, s, x(s))ds, \quad G_x(t) = \int_a^b K_2(t, s, x(s))ds \quad \forall t \in [a, b], \tag{4.2}$$

(ii) *there exist* $\alpha, \beta, \gamma \geq 0$ *such that*

$$\begin{aligned}
&\left(\left|F_x(t) - G_y(t) + g(t) - h(t)\right|,\ p\left|F_x(t) - G_y(t) + g(t) - h(t)\right|\right) \\
&\quad \leq \alpha\left(\left|F_x(t) + g(t) - x(t)\right|,\ p\left|F_x(t) + g(t) - x(t)\right|\right) \\
&\qquad + \beta\left(\left|G_y(t) + h(t) - y(t)\right|,\ p\left|G_y(t) + h(t) - y(t)\right|\right) \\
&\qquad + \gamma(|x(t) - y(t)|,\ p|x(t) - y(t)|),
\end{aligned} \tag{4.3}$$

where $\alpha + \beta + \gamma < 1$, *for every* $x, y \in X$ *with* $x \neq y$ *and* $t \in [a, b]$.

(iii) *whenever* $F_x + g \neq G_x + h$

$$\begin{aligned}
&\sup_{t\in[a,b]} \left(\left|F_x(t) - G_x(t) + g(t) - h(t)\right|,\ p\left|F_x(t) - G_x(t) + g(t) - h(t)\right|\right) \\
&\quad < \sup_{t\in[a,b]} \left(\left|F_x(t) + g(t) - x(t)\right|,\ p\left|F_x(t) + g(t) - x(t)\right|\right) \\
&\qquad + \sup_{t\in[a,b]} \left(\left|G_x(t) + h(t) - x(t)\right|,\ p\left|G_x(t) + h(t) - x(t)\right|\right),
\end{aligned} \tag{4.4}$$

for every $x \in X$.

Then the system of integral equations (4.1) *have a unique common solution.*

Proof. Define $S, T : X \to X$ by $S(x) = F_x + g$, $T(x) = G_x + h$. It is easily seen that

$$\begin{aligned}
\left(\|S - T\|_\infty,\ p\|S - T\|_\infty\right) \leq\ & \alpha\left(\left\|S(x) - x\right\|_\infty,\ p\left\|S(x) - x\right\|_\infty\right) \\
& + \beta\left(\left\|T(y) - y\right\|_\infty,\ p\left\|T(y) - y\right\|_\infty\right) \\
& + \gamma\left(\|x - y\|_\infty,\ p\|x - y\|_\infty\right),
\end{aligned} \tag{4.5}$$

for every $x, y \in X$, with $x \neq y$ and if $S(x) \neq T(x)$

$$\begin{aligned}
\left(\|S - T\|_\infty,\ p\|S - T\|_\infty\right) <\ & \left(\left\|S(x) - x\right\|_\infty,\ p\left\|S(x) - x\right\|_\infty\right) \\
& + \left(\left\|T(x) - x\right\|_\infty,\ p\left\|T(x) - x\right\|_\infty\right)
\end{aligned} \tag{4.6}$$

for every $x \in X$. By Theorem 3.3, if f is the identity map on X, the Urysohn integral equations (4.1) have a unique common solution. □

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