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Research Article

A Hybrid Extragradient Viscosity Approximation Method for Solving Equilibrium Problems and Fixed Point Problems of Infinitely Many Nonexpansive Mappings

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We introduce a new hybrid extragradient viscosity approximation method for finding the common element of the set of equilibrium problems, the set of solutions of fixed points of an infinitely many nonexpansive mappings, and the set of solutions of the variational inequality problems for β-inverse-strongly monotone mapping in Hilbert spaces. Then, we prove the strong convergence of the proposed iterative scheme to the unique solution of variational inequality, which is the optimality condition for a minimization problem. Results obtained in this paper improve the previously known results in this area.

1. Introduction

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Recall that a mapping T of H into itself is called nonexpansive (see [1]) if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in H$. We denote by $F(T) = \{x \in C : Tx = x\}$ the set of fixed points of T. Recall also that a self-mapping $f : H \to H$ is a *contraction* if there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \le \alpha\|x - y\|$, for all $x, y \in H$. In addition, let $B : C \to H$ be a nonlinear mapping. Let P_C be the projection of H onto C. The classical variational inequality which is denoted by $VI(C, B)$ is to find $u \in C$ such that

$$\langle Bu, v - u\rangle \ge 0, \quad \forall v \in C. \tag{1.1}$$

For a given $z \in H$, $u \in C$ satisfies the inequality

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in C, \tag{1.2}$$

if and only if $u = P_C z$. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{1.3}$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and for all $x \in H, y \in C$,

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \tag{1.4}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2. \tag{1.5}$$

It is easy to see that the following is true:

$$u \in VI(C, B) \Longleftrightarrow u = P_C(u - \lambda Bu), \quad \lambda > 0. \tag{1.6}$$

One can see that the variational inequality (1.1) is equivalent to a fixed point problem. The variational inequality has been extensively studied in literature; see, for instance, [2–6]. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems. Recall the following.

(1) A mapping B of C into H is called monotone if

$$\langle Bx - By, x - y \rangle \geq 0, \quad \forall x, y \in C. \tag{1.7}$$

(2) A mapping B is called β-strongly monotone (see [7, 8]) if there exists a constant $\beta > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in C. \tag{1.8}$$

(3) A mapping B is called k-Lipschitz continuous if there exists a positive real number k such that

$$\|Bx - By\| \leq k \|x - y\|, \quad \forall x, y \in C. \tag{1.9}$$

(4) A mapping B is called β-inverse-strongly monotone (see [7, 8]) if there exists a constant $\beta > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \beta \|Bx - By\|^2, \quad \forall x, y \in C. \tag{1.10}$$

Remark 1.1. It is obvious that any β-inverse-strongly monotone mapping B is monotone and $1/\beta$-Lipschitz continuous.

(5) An operator A is strongly positive on H if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.11}$$

(6) A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$, and $g \in Ty$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H\times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let B be a monotone map of C into H, and let $N_C v$ be the normal cone to C at $v \in C$, that is, $N_C v = \{w \in H : \langle u - v, w\rangle \geq 0, \text{ for all } u \in C\}$, .

$$Tv = \begin{cases} Bv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{1.12}$$

Then T is the maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, B)$; see [9].

(7) Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.13}$$

The set of solutions of (1.13) is denoted by $EP(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $z \in EP(F)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$. Numerous problems in physics, saddle point problem, fixed point problem, variational inequality problems, optimization, and economics are reduced to find a solution of (1.13). Some methods have been proposed to solve the equilibrium problem; see, for instance, [10–16]. Recently, Combettes and Hirstoaga [17] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

In 1976, Korpelevich [18] introduced the following so-called extragradient method:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= P_C(x_n - \lambda B x_n), \\ x_{n+1} &= P_C(x_n - \lambda B y_n) \end{aligned} \tag{1.14}$$

for all $n \geq 0$, where $\lambda \in (0, 1/k)$, C is a closed convex subset of $\mathbb{R}^n$, and B is a monotone and k-Lipschitz continuous mapping of C into $\mathbb{R}^n$. He proved that if $VI(C, B)$ is nonempty, then the sequences $\{x_n\}$ and $\{y_n\}$, generated by (1.14), converge to the same point $z \in VI(C, B)$. For finding a common element of the set of fixed points of a nonexpansive mapping and the set of solution of variational inequalities for β-inverse-strongly monotone, Takahashi and Toyoda [19] introduced the following iterative scheme:

$$\begin{aligned} &x_0 \in C \quad \text{chosen arbitrary,} \\ &x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C (x_n - \lambda_n B x_n), \quad \forall n \geq 0, \end{aligned} \tag{1.15}$$

where B is β-inverse-strongly monotone, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\beta)$. They showed that if $F(S) \cap VI(C, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.15) converges weakly to some $z \in F(S) \cap VI(C, B)$. Recently, Iiduka and Takahashi [20] proposed a new iterative scheme as follows:

$$\begin{aligned} &x_0 = x \in C \quad \text{chosen arbitrary,} \\ &x_{n+1} = \alpha_n x + (1-\alpha_n) S P_C (x_n - \lambda_n B x_n), \quad \forall n \geq 0, \end{aligned} \tag{1.16}$$

where B is β-inverse-strongly monotone, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\beta)$. They showed that if $F(S) \cap VI(C, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.16) converges strongly to some $z \in F(S) \cap VI(C, B)$.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [21–24] and the references therein. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \tag{1.17}$$

where A is a linear bounded operator, C is the fixed point set of a nonexpansive mapping S on H, and b is a given point in H. Moreover, it is shown in [25] that the sequence $\{x_n\}$ defined by the scheme

$$x_{n+1} = \epsilon_n \gamma f(x_n) + (1 - \epsilon_n A) S x_n \tag{1.18}$$

converges strongly to $z = P_{F(S)}(I - A + \gamma f)(z)$. Recently, Plubtieng and Punpaeng [26] proposed the following iterative algorithm:

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n A) S u_n. \end{gathered} \tag{1.19}$$

They prove that if the sequences $\{\epsilon_n\}$ and $\{r_n\}$ of parameters satisfy appropriate condition, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge to the unique solution z of the variational inequality

$$\langle (A - \gamma f) q, q - p \rangle \geq 0, \quad p \in F(S) \cap EP(F), \tag{1.20}$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(S)\cap EP(F)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.21}$$

where h is a potential function for γf (i.e., $h\prime(x) = \gamma f(x)$ for $x \in H$).

Furthermore, for finding approximate common fixed points of an infinite countable family of nonexpansive mappings $\{T_n\}$ under very mild conditions on the parameters. Wangkeeree [27] introduced an iterative scheme for finding a common element of the set of solutions of the equilibrium problem (1.13) and the set of common fixed points of a countable family of nonexpansive mappings on C. Starting with an arbitrary initial $x_1 \in C$, define a sequence $\{x_n\}$ recursively by

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n P_C(u_n - \lambda_n B y_n), \quad \forall n \geq 1, \end{gathered} \tag{1.22}$$

where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0, 1)$. It is proved that under certain appropriate conditions imposed on $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{r_n\}$, the sequence $\{x_n\}$ generated by (1.22) strongly converges to the unique solution $q \in \cap_{n=1}^{\infty} F(S_n) \cap VI(C, B) \cap EP(F)$, where $p = P_{\cap_{n=1}^{\infty} F(S_n)\cap VI(C,B)\cap EP(F)} f(q)$ which extend and improve the result of Kumam [14].

Definition 1.2 (see [21]). Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself, and let $\{\mu_n\}$ be a sequence of nonnegative numbers in [0,1]. For each $n \geq 1$, define a mapping W_n of C into itself as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \mu_n T_n U_{n,n+1} + (1 - \mu_n) I, \\ U_{n,n-1} &= \mu_{n-1} T_{n-1} U_{n,n} + (1 - \mu_{n-1}) I, \\ &\vdots \\ U_{n,k} &= \mu_k T_k U_{n,k+1} + (1 - \mu_k) I, \\ U_{n,k-1} &= \mu_{k-1} T_{k-1} U_{n,k} + (1 - \mu_{k-1}) I, \\ &\vdots \\ U_{n,2} &= \mu_2 T_2 U_{n,3} + (1 - \mu_2) I, \\ W_n = U_{n,1} &= \mu_1 T_1 U_{n,2} + (1 - \mu_1) I. \end{aligned} \tag{1.23}$$

Such a mapping W_n is nonexpansive from C to C, and it is called the W-mapping generated by $T_1, T_2, \ldots, T_n$ and $\mu_1, \mu_2, \ldots, \mu_n$.

On the other hand, Colao et al. [28] introduced and considered an iterative scheme for finding a common element of the set of solutions of the equilibrium problem (1.13) and the set of common fixed points of infinitely many nonexpansive mappings on C. Starting with an arbitrary initial $x_0 \in C$, define a sequence $\{x_n\}$ recursively by

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + \big((1-\beta)I - \epsilon_n A\big)W_n u_n, \end{gathered} \tag{1.24}$$

where $\{\epsilon_n\}$ is a sequence in $(0,1)$. It is proved [28] that under certain appropriate conditions imposed on $\{\epsilon_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (1.24) strongly converges to $z \in \cap_{n=1}^{\infty} F(T_n) \cap EP(F)$, where z is an equilibrium point for F and is the unique solution of the variational inequality (1.20), that is, $z = P_{\cap_{n=1}^{\infty} F(T_n) \cap EP(F)}(I - (A - \gamma f))z$.

In this paper, motivated by Wangkeeree [27], Plubtieng and Punpaeng [26], Marino and Xu [25], and Colao, et al. [28], we introduce a new iterative scheme in a Hilbert space H which is mixed by the iterative schemes of (1.18), (1.19), (1.22), and (1.24) as follows.

Let f be a contraction of H into itself, A a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$, and B a β-inverse-strongly monotone mapping of C into H; define sequences $\{x_n\}$, $\{y_n\}$, $\{k_n\}$, and $\{u_n\}$ recursively by

$$\begin{gathered} x_1 = x \in C \quad \text{chosen arbitrary}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ k_n = \alpha_n u_n + (1-\alpha_n) P_C\big(u_n - \lambda_n B y_n\big), \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \beta_n x_n + \big((1-\beta_n)I - \epsilon_n A\big)W_n k_n, \quad \forall n \geq 1, \end{gathered} \tag{1.25}$$

where $\{W_n\}$ is the sequence generated by (1.23), $\{\epsilon_n\}, \{\alpha_n\}$, and $\{\beta_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfying appropriate conditions. We prove that the sequences $\{x_n\}$, $\{y_n\}$, $\{k_n\}$ and $\{u_n\}$ generated by the above iterative scheme (1.25) converge strongly to a common element of the set of solutions of the equilibrium problem (1.13), the set of common fixed points of infinitely family nonexpansive mappings, and the set of solutions of variational inequality (1.1) for a β-inverse-strongly monotone mapping in Hilbert spaces. The results obtained in this paper improve and extend the recent ones announced by Wangkeeree [27], Plubtieng and Punpaeng [26], Marino and Xu [25], Colao, et al. [28], and many others.

2. Preliminaries

We now recall some well-known concepts and results.

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\rightarrow$, respectively.

A space H is said to satisfy Opial's condition [29] if for each sequence $\{x_n\}$ in H which converges weakly to point $x \in H$, we have

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|, \quad \forall y \in H, \quad y \neq x. \tag{2.1}$$

Lemma 2.1 (see [25]). *Let C be a nonempty closed convex subset of H, let f be a contraction of H into itself with $\alpha \in (0,1)$, and let A be a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$. Then , for $0 < \gamma < \overline{\gamma}/\alpha$,*

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y\rangle \geq (\overline{\gamma} - \alpha\gamma)\|x - y\|^2, \quad x, y \in H. \tag{2.2}$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\overline{\gamma} - \gamma\alpha$.

Lemma 2.2 (see [25]). *Assume that A is a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

For solving the equilibrium problem for a bifunction $F : C \times C \to R$, let us assume that F satisfies the following conditions:

$(A1)$ $F(x,x) = 0$ for all $x \in C$;

$(A2)$ F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

$(A3)$ for each $x, y, z \in C, \lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;

$(A4)$ for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [30].

Lemma 2.3 (see [30]). *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \forall y \in C. \tag{2.3}$$

The following lemma was also given in [17].

Lemma 2.4 (see [17]). *Assume that $F : C \times C \to R$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\right\} \tag{2.4}$$

for all $z \in H$. Then, the following holds:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle; \tag{2.5}$$

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ *is closed and convex.*

For each $n, k \in \mathbb{N}$, let the mapping $U_{n,k}$ be defined by (1.23). Then we can have the following crucial conclusions concerning W_n. You can find them in [31]. Now we only need the following similar version in Hilbert spaces.

Lemma 2.5 (see [31]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\mu_1, \mu_2, \ldots$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then, for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.5, one can define a mapping W of C into itself as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x \tag{2.6}$$

for every $x \in C$. Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\mu_1, \mu_2, \ldots$. Throughout this paper, we will assume that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then, we have the following results.

Lemma 2.6 (see [31]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\mu_1, \mu_2, \ldots$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then, $F(W) = \cap_{n=1}^{\infty} F(T_n)$.*

Lemma 2.7 (see [32]). *If $\{x_n\}$ is a bounded sequence in C, then $\lim_{n\to\infty}\|Wx_n - W_n x_n\| = 0$.*

Lemma 2.8 (see [33]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|z_n - x_n\| = 0$.*

Lemma 2.9 (see [34]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - l_n)a_n + \sigma_n, \quad n \geq 0, \tag{2.7}$$

where $\{l_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} l_n = \infty$;

(2) $\limsup_{n\to\infty}\sigma_n/l_n \leq 0$ *or* $\sum_{n=1}^{\infty}|\sigma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10. *Let H be a real Hilbert space. Then for all $x, y \in H$,*

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;

(2) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$.

3. Main Results

In this section, we prove the strong convergence theorem for infinitely many nonexpansive mappings in a real Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), let $\{T_n\}$ be an infinitely many nonexpansive of C into itself, and let B be an β-inverse-strongly monotone mapping of C into H such that $\Theta := \cap_{n=1}^{\infty} F(T_n) \cap EP(F) \cap VI(C,B) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0,1)$, and let A be a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Let $\{x_n\}$, $\{y_n\}$, $\{k_n\}$, and $\{u_n\}$ be sequences generated by (1.25), where $\{W_n\}$ is the sequence generated by (1.23), $\{\epsilon_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are three sequences in $(0,1)$, and $\{r_n\}$ is a real sequence in $(0,\infty)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty}\epsilon_n = 0$, $\sum_{n=1}^{\infty}\epsilon_n = \infty$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$;

(v) $\{\lambda_n/\beta\} \subset (\tau, 1-\delta)$ *for some* $\tau, \delta \in (0,1)$ *and* $\lim_{n\to\infty}\lambda_n = 0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Theta$ which is the unique solution of the variational inequality

$$\langle (A - \gamma f)z, z - x \rangle \geq 0, \quad \forall x \in \Theta. \tag{3.1}$$

Equivalently, one has $z = P_{\Theta}(I - A + \gamma f)(z)$.

Proof. Note that from the condition (i), we may assume, without loss of generality, that $\epsilon_n \leq (1-\beta_n)\|A\|^{-1}$ for all $n \in \mathbb{N}$. From Lemma 2.2, we know that if $0 \leq \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$. We will assume that $\|I - A\| \leq 1 - \overline{\gamma}$. First, we show that $I - \lambda_n B$ is nonexpansive. Indeed, from the β-inverse-strongly monotone mapping definition on B and condition (v), we have

$$\begin{aligned}
\|(I - \lambda_n B)x - (I - \lambda_n B)y\|^2 &= \|(x - y) - \lambda_n(Bx - By)\|^2 \\
&= \|x - y\|^2 - 2\lambda_n\langle x - y, Bx - By\rangle + \lambda_n^2\|Bx - By\|^2 \\
&\leq \|x - y\|^2 - 2\lambda_n\beta\|Bx - By\|^2 + \lambda_n^2\|Bx - By\|^2 \\
&= \|x - y\|^2 + \lambda_n(\lambda_n - 2\beta)\|Bx - By\|^2 \\
&\leq \|x - y\|^2,
\end{aligned} \tag{3.2}$$

which implies that the mapping $I - \lambda_n B$ is nonexpansive. On the other hand, since A is a strongly positive bounded linear operator on H, we have

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}. \tag{3.3}$$

Observe that

$$\begin{aligned}\langle((1-\beta_n)I-\epsilon_n A)x,x\rangle &= 1-\beta_n-\epsilon_n\langle Ax,x\rangle\\ &\geq 1-\beta_n-\epsilon_n\|A\|\\ &\geq 0,\end{aligned} \tag{3.4}$$

and this show that $(1-\beta_n)I-\epsilon_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I-\epsilon_n A\| &= \sup\{|\langle((1-\beta_n)I-\epsilon_n A)x,x\rangle| : x\in H, \|x\|=1\}\\ &= \sup\{1-\beta_n-\epsilon_n\langle Ax,x\rangle : x\in H, \|x\|=1\}\\ &\leq 1-\beta_n-\epsilon_n\overline{\gamma}.\end{aligned} \tag{3.5}$$

Let $Q=P_\Theta$, where $\Theta := \cap_{n=1}^{\infty}F(T_n)\cap EP(F)\cap VI(C,B)$. Note that f is a contraction of H into itself with $\alpha\in(0,1)$. Then, we have

$$\begin{aligned}\|Q(I-A+\gamma f)(x)-Q(I-A+\gamma f)(y)\| &= \|P_\Theta(I-A+\gamma f)(x)-P_\Theta(I-A+\gamma f)(y)\|\\ &\leq \|(I-A+\gamma f)(x)-(I-A+\gamma f)(y)\|\\ &\leq \|I-A\|\|x-y\|+\gamma\|f(x)-f(y)\|\\ &\leq (1-\overline{\gamma})\|x-y\|+\gamma\alpha\|x-y\|\\ &= (1-\overline{\gamma}+\gamma\alpha)\|x-y\|\\ &= (1-(\overline{\gamma}-\gamma\alpha))\|x-y\|, \quad \forall x,y\in H.\end{aligned} \tag{3.6}$$

Since $0<1-(\overline{\gamma}-\gamma\alpha)<1$, it follows that $Q(I-A+\gamma f)$ is a contraction of H into itself. Therefore by the Banach Contraction Mapping Principle, which implies that there exists a unique element $z\in H$ such that $z=Q(I-A+\gamma f)(z)=P_\Theta(I-A+\gamma f)(z)$.

We will divide the proof into five steps.

Step 1. We claim that $\{x_n\}$ is bounded. Indeed, pick any $p\in\Theta$. From the definition of T_r, we note that $u_n=T_{r_n}x_n$. If follows that

$$\|u_n-p\| = \|T_{r_n}x_n-T_{r_n}p\| \leq \|x_n-p\|. \tag{3.7}$$

Since $I-\lambda_n B$ is nonexpansive and $p=P_C(p-\lambda_n Bp)$ from (1.6), we have

$$\begin{aligned}\|y_n-p\| &= \|P_C(u_n-\lambda_n Bu_n)-P_C(p-\lambda_n Bp)\|\\ &\leq \|(u_n-\lambda_n Au_n)-(p-\lambda_n Bp)\|\\ &= \|(I-\lambda_n A)u_n-(I-\lambda_n B)p\|\\ &\leq \|u_n-p\|\leq\|x_n-p\|.\end{aligned} \tag{3.8}$$

Put $v_n = P_C(u_n - \lambda_n By_n)$. Since $p \in VI(C,B)$, we have $p = P_C(p - \lambda_n Bp)$. Substituting $x = u_n - \lambda_n Ay_n$ and $y = p$ in (1.5), we can write

$$\begin{aligned}
\|v_n - p\|^2 &\le \|u_n - \lambda_n By_n - p\|^2 - \|u_n - \lambda_n By_n - v_n\|^2 \\
&= \|u_n - p\|^2 - 2\lambda_n\langle By_n, u_n - p\rangle + \lambda_n^2\|By_n\|^2 \\
&\quad - \|u_n - v_n\|^2 + 2\lambda_n\langle By_n, u_n - v_n\rangle - \lambda_n^2\|By_n\|^2 \\
&= \|u_n - p\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle By_n, p - v_n\rangle \\
&= \|u_n - p\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle By_n - Bp, p - y_n\rangle \\
&\quad + 2\lambda_n\langle Bp, p - y_n\rangle + 2\lambda_n\langle By_n, y_n - v_n\rangle.
\end{aligned} \tag{3.9}$$

Using the fact that B is β-inverse-strongly monotone mapping, and p is a solution of the variational inequality problem $VI(C,B)$, we also have

$$\langle By_n - Bp, p - y_n\rangle \le 0, \qquad \langle Bp, p - y_n\rangle \le 0. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\begin{aligned}
\|v_n - p\|^2 &\le \|u_n - p\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle By_n, y_n - v_n\rangle \\
&= \|u_n - p\|^2 - \|(u_n - y_n) + (y_n - v_n)\|^2 + 2\lambda_n\langle By_n, y_n - v_n\rangle \\
&\le \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\
&\quad - 2\langle u_n - y_n, y_n - v_n\rangle + 2\lambda_n\langle By_n, y_n - v_n\rangle \\
&= \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\langle u_n - \lambda_n By_n - y_n, v_n - y_n\rangle.
\end{aligned} \tag{3.11}$$

Substituting x by $u_n - \lambda_n Bu_n$ and $y = v_n$ in (1.4), we obtain

$$\langle u_n - \lambda_n Bu_n - y_n, v_n - y_n\rangle \le 0. \tag{3.12}$$

It follows that

$$\begin{aligned}
\langle u_n - \lambda_n By_n - y_n, v_n - y_n\rangle &= \langle u_n - \lambda_n Bu_n - y_n, v_n - y_n\rangle \\
&\quad + \langle \lambda_n Bu_n - \lambda_n By_n, v_n - y_n\rangle \\
&\le \langle \lambda_n Bu_n - \lambda_n By_n, v_n - y_n\rangle \\
&\le \lambda_n\|Bu_n - By_n\|\|v_n - y_n\| \\
&\le \frac{\lambda_n}{\beta}\|u_n - y_n\|\|v_n - y_n\|.
\end{aligned} \tag{3.13}$$

Substituting (3.13) into (3.11), we have

$$\begin{aligned}\|v_n-p\|^2 &\le \|u_n-p\|^2-\|u_n-y_n\|^2-\|y_n-v_n\|^2+2\langle u_n-\lambda_n By_n-y_n,v_n-y_n\rangle\\ &\le \|u_n-p\|^2-\|u_n-y_n\|^2-\|y_n-v_n\|^2+2\frac{\lambda_n}{\beta}\|u_n-y_n\|\|v_n-y_n\|\\ &\le \|u_n-p\|^2-\|u_n-y_n\|^2-\|y_n-v_n\|^2+\frac{\lambda_n^2}{\beta^2}\|u_n-y_n\|^2+\|v_n-y_n\|^2\\ &= \|u_n-p\|^2-\|u_n-y_n\|^2+\frac{\lambda_n^2}{\beta^2}\|u_n-y_n\|^2\\ &= \|u_n-p\|^2+\left(\frac{\lambda_n^2}{\beta^2}-1\right)\|u_n-y_n\|^2\\ &\le \|u_n-p\|^2\le\|x_n-p\|^2.\end{aligned}\tag{3.14}$$

Setting $k_n=\alpha_n u_n+(1-\alpha_n)v_n$, we can calculate

$$\begin{aligned}\|x_{n+1}-p\| &= \|\epsilon_n(\gamma f(x_n)-Ap)+\beta_n(x_n-p)+((1-\beta_n)I-\epsilon_nA)(W_nk_n-p)\|\\ &\le (1-\beta_n-\epsilon_n\overline{\gamma})\|k_n-p\|+\beta_n\|x_n-p\|+\epsilon_n\|\gamma f(x_n)-Ap\|\\ &\le (1-\beta_n-\epsilon_n\overline{\gamma})\{\alpha_n\|u_n-p\|+(1-\alpha_n)\|v_n-p\|\}\\ &\quad+\beta_n\|x_n-p\|+\epsilon_n\|\gamma f(x_n)-Ap\|\\ &\le (1-\beta_n-\epsilon_n\overline{\gamma})\{\alpha_n\|x_n-p\|+(1-\alpha_n)\|x_n-p\|\}\\ &\quad+\beta_n\|x_n-p\|+\epsilon_n\|\gamma f(x_n)-Ap\|\\ &= (1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-p\|+\beta_n\|x_n-p\|+\epsilon_n\|\gamma f(x_n)-Ap\|\\ &= (1-\epsilon_n\overline{\gamma})\|x_n-p\|+\epsilon_n\gamma\|f(x_n)-f(p)\|+\epsilon_n\|\gamma f(p)-Ap\|\\ &\le (1-\epsilon_n\overline{\gamma})\|x_n-p\|+\epsilon_n\gamma\alpha\|x_n-p\|+\epsilon_n\|\gamma f(p)-Ap\|\\ &= (1-(\overline{\gamma}-\gamma\alpha)\epsilon_n)\|x_n-p\|+(\overline{\gamma}-\gamma\alpha)\epsilon_n\frac{\|\gamma f(p)-Ap\|}{\overline{\gamma}-\gamma\alpha}.\end{aligned}\tag{3.15}$$

By induction,

$$\|x_n-p\|\le\max\left\{\|x_1-p\|,\frac{\|\gamma f(p)-Ap\|}{\overline{\gamma}-\gamma\alpha}\right\},\quad n\in\mathbb{N}.\tag{3.16}$$

Hence, $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{v_n\}$, $\{W_nk_n\}$, $\{f(x_n)\}$, $\{Bu_n\}$, $\{y_n\}$, and $\{By_n\}$.

Step 2. We claim that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Observing that $u_n = T_{r_n} x_n$ and $u_{n+1} = T_{r_{n+1}} x_{n+1}$, we get

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 \quad \forall y \in H \tag{3.17}$$

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0 \quad \forall y \in H. \tag{3.18}$$

Putting $y = u_{n+1}$ in (3.17) and $y = u_n$ in (3.18), we have

$$\begin{aligned} F(u_n, u_{n+1}) + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0 \\ F(u_{n+1}, u_n) + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0. \end{aligned} \tag{3.19}$$

So, from (A2) we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0, \tag{3.20}$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}} (u_{n+1} - x_{n+1}) \right\rangle \geq 0. \tag{3.21}$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\}, \end{aligned} \tag{3.22}$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{M_1}{c} |r_{n+1} - r_n|, \end{aligned} \tag{3.23}$$

where $M_1 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. Note that

$$\begin{aligned}
\|v_{n+1} - v_n\| &\leq \left\|P_C\left(u_{n+1} - \lambda_{n+1}By_{n+1}\right) - P_C\left(u_n - \lambda_n By_n\right)\right\| \\
&\leq \left\|u_{n+1} - \lambda_{n+1}By_{n+1} - \left(u_n - \lambda_n By_n\right)\right\| \\
&= \|(u_{n+1} - \lambda_{n+1}Bu_{n+1}) - (u_n - \lambda_{n+1}Bu_n) \\
&\quad + \lambda_{n+1}\left(Bu_{n+1} - By_{n+1} - Bu_n\right) + \lambda_n By_n\| \\
&\leq \|(u_{n+1} - \lambda_{n+1}Bu_{n+1}) - (u_n - \lambda_{n+1}Bu_n)\| \\
&\quad + \lambda_{n+1}\left(\|Bu_{n+1}\| + \left\|By_{n+1}\right\| + \|Bu_n\|\right) + \lambda_n\|By_n\| \\
&\leq \|u_{n+1} - u_n\| + \lambda_{n+1}\left(\|Bu_{n+1}\| + \left\|By_{n+1}\right\| + \|Bu_n\|\right) + \lambda_n\|By_n\|, \\
\|k_{n+1} - k_n\| &= \|\alpha_{n+1}u_{n+1} + (1 - \alpha_{n+1})v_{n+1} - \alpha_n u_n - (1 - \alpha_n)v_n\| \\
&= \|\alpha_{n+1}(u_{n+1} - u_n) + (\alpha_{n+1} - \alpha_n)u_n \\
&\quad + (1 - \alpha_{n+1})(v_{n+1} - v_n) + (\alpha_n - \alpha_{n+1})v_n\| \\
&\leq \alpha_{n+1}\|u_{n+1} - u_n\| + (1 - \alpha_{n+1})\|v_{n+1} - v_n\| + |\alpha_n - \alpha_{n+1}|\|u_n + v_n\| \\
&= \alpha_{n+1}\|u_{n+1} - u_n\| + (1 - \alpha_{n+1}) \\
&\quad \times \left\{\|u_{n+1} - u_n\| + \lambda_{n+1}\left(\|Bu_{n+1}\| + \left\|By_{n+1}\right\| + \|Bu_n\|\right)\right. \\
&\qquad \left. + \lambda_n\|By_n\|\right\} + |\alpha_n - \alpha_{n+1}|\|u_n + v_n\| \\
&= \|u_{n+1} - u_n\| + (1 - \alpha_{n+1})\lambda_{n+1}\left(\|Bu_{n+1}\| + \left\|By_{n+1}\right\| + \|Bu_n\|\right) \\
&\quad + (1 - \alpha_{n+1})\lambda_n\left\|By_n\right\| + |\alpha_n - \alpha_{n+1}|\|u_n + v_n\| \\
&\leq \|x_{n+1} - x_n\| + \frac{M_1}{c}|r_{n+1} - r_n| + (1 - \alpha_{n+1})\lambda_{n+1} \\
&\quad \times \left(\|Bu_{n+1}\| + \left\|By_{n+1}\right\| + \|Bu_n\|\right) \\
&\quad + (1 - \alpha_{n+1})\lambda_n\left\|By_n\right\| + |\alpha_n - \alpha_{n+1}|\|u_n + v_n\|.
\end{aligned} \tag{3.24}$$

Setting

$$z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\epsilon_n \gamma f(x_n) + \left((1 - \beta_n)I - \epsilon_n A\right)W_n k_n}{1 - \beta_n}, \tag{3.25}$$

we have $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$, $n \geq 1$. It follows that

$$\begin{aligned}
z_{n+1} - z_n &= \frac{\epsilon_{n+1}\gamma f(x_{n+1}) + ((1-\beta_{n+1})I - \epsilon_{n+1}A)W_{n+1}k_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\epsilon_n\gamma f(x_n) + ((1-\beta_n)I - \epsilon_n A)W_n k_n}{1-\beta_n} \\
&= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1}) - \frac{\epsilon_n}{1-\beta_n}\gamma f(x_n) + W_{n+1}k_{n+1} - W_n k_n \\
&\quad + \frac{\epsilon_n}{1-\beta_n}AW_n k_n - \frac{\epsilon_{n+1}}{1-\beta_{n+1}}AW_{n+1}k_{n+1} \\
&= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - AW_{n+1}k_{n+1}) + \frac{\epsilon_n}{1-\beta_n}(AW_n k_n - \gamma f(x_n)) \\
&\quad + W_{n+1}k_{n+1} - W_{n+1}k_n + W_{n+1}k_n - W_n k_n.
\end{aligned} \tag{3.26}$$

It follows from (3.24) and (3.26) that

$$\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(x_{n+1})\right\| + \|AW_{n+1}k_{n+1}\|\right) \\
&\quad + \frac{\epsilon_n}{1-\beta_n}\left(\|AW_n k_n\| + \left\|\gamma f(x_n)\right\|\right) + \|W_{n+1}k_{n+1} - W_{n+1}k_n\| \\
&\quad + \|W_{n+1}k_n - W_n k_n\| - \|x_{n+1} - x_n\| \\
&\leq \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(x_{n+1})\right\| + \|AW_{n+1}k_{n+1}\|\right) \\
&\quad + \frac{\epsilon_n}{1-\beta_n}\left(\|AW_n k_n\| + \left\|\gamma f(x_n)\right\|\right) + \|k_{n+1} - k_n\| \\
&\quad + \|W_{n+1}k_n - W_n k_n\| - \|x_{n+1} - x_n\| \\
&\leq \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(x_{n+1})\right\| + \|AW_{n+1}k_{n+1}\|\right) \\
&\quad + \frac{\epsilon_n}{1-\beta_n}\left(\|AW_n k_n\| + \left\|\gamma f(x_n)\right\|\right) + \frac{M_1}{c}|r_{n+1} - r_n| \\
&\quad + (1-\alpha_{n+1})\lambda_{n+1}\left(\|Bu_{n+1}\| + \left\|By_{n+1}\right\| + \|Bu_n\|\right) \\
&\quad + (1-\alpha_{n+1})\lambda_n\left\|By_n\right\| + |\alpha_n - \alpha_{n+1}|\|u_n + v_n\| \\
&\quad + \|W_{n+1}k_n - W_n k_n\|.
\end{aligned} \tag{3.27}$$

Since T_i and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned}
\|W_{n+1}k_n - W_nk_n\| &= \left\|\mu_1 T_1 U_{n+1,2}k_n - \mu_1 T_1 U_{n,2}k_n\right\| \\
&\le \mu_1\|U_{n+1,2}k_n - U_{n,2}k_n\| \\
&= \mu_1\left\|\mu_2 T_2 U_{n+1,3}k_n - \mu_2 T_2 U_{n,3}k_n\right\| \\
&\le \mu_1\mu_2\|U_{n+1,3}k_n - U_{n,3}k_n\| \\
&\ \ \vdots \\
&\le \mu_1\mu_2\cdots\mu_n\|U_{n+1,n+1}k_n - U_{n,n+1}k_n\| \\
&\le M_2\prod_{i=1}^{n}\mu_i,
\end{aligned} \tag{3.28}$$

where $M_2 \ge 0$ is a constant such that $\|U_{n+1,n+1}k_n - U_{n,n+1}k_n\| \le M_2$ for all $n \ge 0$.

Combining (3.27) and (3.28), we have

$$\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le{}& \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(x_{n+1})\right\| + \|AW_{n+1}k_{n+1}\|\right) \\
&+ \frac{\epsilon_n}{1-\beta_n}\left(\|AW_nk_n\| + \left\|\gamma f(x_n)\right\|\right) + \frac{M_1}{c}|r_{n+1} - r_n| \\
&+ (1-\alpha_{n+1})\lambda_{n+1}\left(\|Bu_{n+1}\| + \left\|By_{n+1}\right\| + \|Bu_n\|\right) \\
&+ (1-\alpha_{n+1})\lambda_n\left\|By_n\right\| + |\alpha_n - \alpha_{n+1}|\|u_n + v_n\| \\
&+ M_2\prod_{i=1}^{n}\mu_i,
\end{aligned} \tag{3.29}$$

which implies that (noting that (i), (ii), (iii), (iv), (v), and $0 < \mu_i \le b < 1$, for all $i \ge 1$)

$$\limsup_{n\to\infty}\left(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\right) \le 0. \tag{3.30}$$

Hence, by Lemma 2.8, we obtain

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.31}$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{3.32}$$

Applying (3.32) and (ii), (iv), and (v) to (3.23) and (3.24), we obtain that

$$\lim_{n\to\infty}\|u_{n+1}-u_n\| = \lim_{n\to\infty}\|k_{n+1}-k_n\| = 0. \tag{3.33}$$

Since $x_{n+1} = \epsilon_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A)W_n k_n$, we have

$$\begin{aligned}\|x_n - W_n k_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - W_n k_n\| \\ &= \|x_n - x_{n+1}\| + \left\|\epsilon_n\gamma f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \epsilon_n A\right)W_n k_n - W_n k_n\right\| \\ &= \|x_n - x_{n+1}\| + \left\|\epsilon_n\left(\gamma f(x_n) - AW_n k_n\right) + \beta_n(x_n - W_n k_n)\right\| \\ &\le \|x_n - x_{n+1}\| + \epsilon_n\left(\left\|\gamma f(x_n)\right\| + \|AW_n k_n\|\right) + \beta_n\|x_n - W_n k_n\|,\end{aligned} \tag{3.34}$$

that is

$$\|x_n - W_n k_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\epsilon_n}{1-\beta_n}\left(\left\|\gamma f(x_n)\right\| + \|AW_n k_n\|\right). \tag{3.35}$$

By (i), (iii), and (3.32) it follows that

$$\lim_{n\to\infty}\|W_n k_n - x_n\| = 0. \tag{3.36}$$

Step 3. We claim that the following statements hold:

(i) $\lim_{n\to\infty}\|u_n - k_n\| = 0$;

(ii) $\lim_{n\to\infty}\|x_n - u_n\| = 0$.

For any $p \in \Theta := \cap_{n=1}^{\infty} F(T_n) \cap EP(F) \cap VI(C,B)$ and (3.14), we have

$$\begin{aligned}\left\|k_n - p\right\|^2 &= \left\|\alpha_n(u_n - p) + (1-\alpha_n)(v_n - p)\right\|^2 \\ &\le \alpha_n\left\|u_n - p\right\|^2 + (1-\alpha_n)\left\|v_n - p\right\|^2 \\ &\le \alpha_n\left\|u_n - p\right\|^2 + (1-\alpha_n)\left\{\left\|u_n - p\right\|^2 + \left(\frac{\lambda_n^2}{\beta^2} - 1\right)\left\|u_n - y_n\right\|^2\right\} \\ &= \left\|u_n - p\right\|^2 + (1-\alpha_n)\left(\frac{\lambda_n^2}{\beta^2} - 1\right)\left\|u_n - y_n\right\|^2 \\ &\le \left\|x_n - p\right\|^2 + (1-\alpha_n)\left(\frac{\lambda_n^2}{\beta^2} - 1\right)\left\|u_n - y_n\right\|^2.\end{aligned} \tag{3.37}$$

Observe that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|((1-\beta_n)I-\epsilon_n A)(W_nk_n-p)+\beta_n(x_n-p)+\epsilon_n(\gamma f(x_n)-Ap)\|^2\\
&= \|((1-\beta_n)I-\epsilon_n A)(W_nk_n-p)+\beta_n(x_n-p)\|^2+\epsilon_n^2\|\gamma f(x_n)-Ap\|^2\\
&\quad+\ 2\beta_n\epsilon_n\langle x_n-p,\gamma f(x_n)-Ap\rangle\\
&\quad+2\epsilon_n\langle((1-\beta_n)I-\epsilon_n A)(W_nk_n-p),\gamma f(x_n)-Ap\rangle\\
&\le [(1-\beta_n-\epsilon_n\overline{\gamma})\|W_nk_n-p\|+\beta_n\|x_n-p\|]^2\\
&\quad+\epsilon_n^2\|\gamma f(x_n)-Ap\|^2\\
&\quad+\ 2\beta_n\epsilon_n\langle x_n-p,\gamma f(x_n)-Ap\rangle\\
&\quad+2\epsilon_n\langle((1-\beta_n)I-\epsilon_n A)(W_nk_n-p),\gamma f(x_n)-Ap\rangle\\
&\le [(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n-p\|+\beta_n\|x_n-p\|]^2+c_n\\
&\le (1-\beta_n-\epsilon_n\overline{\gamma})^2\|k_n-p\|^2+\beta_n^2\|x_n-p\|^2\\
&\quad+2(1-\beta_n-\epsilon_n\overline{\gamma})\beta_n\|k_n-p\|\|x_n-p\|+c_n\\
&\le (1-\beta_n-\epsilon_n\overline{\gamma})^2\|k_n-p\|^2+\beta_n^2\|x_n-p\|^2\\
&\quad+(1-\beta_n-\epsilon_n\overline{\gamma})\beta_n\left(\|k_n-p\|^2+\|x_n-p\|^2\right)+c_n\\
&= \left[(1-\epsilon_n\overline{\gamma})^2-2(1-\epsilon_n\overline{\gamma})\beta_n+\beta_n^2\right]\|k_n-p\|^2+\beta_n^2\|x_n-p\|^2\\
&\quad+\ \left((1-\epsilon_n\overline{\gamma})\beta_n-\beta_n^2\right)\left(\|k_n-p\|^2+\|x_n-p\|^2\right)+c_n\\
&= (1-\epsilon_n\overline{\gamma})^2\|k_n-p\|^2-(1-\epsilon_n\overline{\gamma})\beta_n\|k_n-p\|^2\\
&\quad+(1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n-p\|^2+(1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2+c_n,
\end{aligned}\tag{3.38}$$

where

$$\begin{aligned}
c_n &= \epsilon_n^2\|\gamma f(x_n)-Ap\|^2+2\beta_n\epsilon_n\langle x_n-p,\gamma f(x_n)-Ap\rangle\\
&\quad+\ 2\epsilon_n\langle((1-\beta_n)I-\epsilon_n A)(W_nk_n-p),\gamma f(x_n)-Ap\rangle.
\end{aligned}\tag{3.39}$$

It follows from condition (i) that

$$\lim_{n\to\infty} c_n = 0.\tag{3.40}$$

Substituting (3.37) into (3.38), and using (v), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\{\|x_n-p\|^2+(1-\alpha_n)\left(\frac{\lambda_n^2}{\beta^2}-1\right)\|u_n-y_n\|^2\right\}\\ &\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2+c_n\\ &= (1-\epsilon_n\overline{\gamma})^2\|x_n-p\|^2\\ &\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\left(\frac{\lambda_n^2}{\beta^2}-1\right)\|u_n-y_n\|^2+c_n\\ &\le \|x_n-p\|^2+(1-\alpha_n)\left(\frac{\lambda_n^2}{\beta^2}-1\right)\|u_n-y_n\|^2+c_n.\end{aligned} \tag{3.41}$$

It follows that

$$\begin{aligned}(1-\alpha_n)\delta\|u_n-y_n\|^2 &\le (1-\alpha_n)\left(1-\frac{\lambda_n^2}{\beta^2}\right)\|u_n-y_n\|^2\\ &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+c_n\\ &= (\|x_n-p\|-\|x_{n+1}-p\|)(\|x_n-p\|+\|x_{n+1}-p\|)+c_n\\ &\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+c_n.\end{aligned} \tag{3.42}$$

Since $\lim_{n\to\infty}c_n=0$ and from (3.32), we obtain

$$\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{3.43}$$

Note that

$$k_n-v_n=\alpha_n(u_n-v_n). \tag{3.44}$$

Since $\lim_{n\to\infty}\alpha_n=0$, we have

$$\lim_{n\to\infty}\|k_n-v_n\|=0. \tag{3.45}$$

As B is $1/\beta$-Lipschitz continuous, we obtain

$$\begin{aligned}\|v_n - y_n\| &= \|P_C(u_n - \lambda_n B y_n) - P_C(u_n - \lambda_n B u_n)\| \\ &\leq \|(u_n - \lambda_n B y_n) - (u_n - \lambda_n B u_n)\| \\ &= \lambda_n \|B u_n - B y_n\| \\ &\leq \frac{\lambda_n}{\beta}\|u_n - y_n\|,\end{aligned} \tag{3.46}$$

then, we get

$$\lim_{n\to\infty}\|v_n - y_n\| = 0. \tag{3.47}$$

from

$$\|u_n - k_n\| \leq \|u_n - y_n\| + \|y_n - v_n\| + \|v_n - k_n\|. \tag{3.48}$$

Applying (3.43), (3.45), and (3.47), we have

$$\lim_{n\to\infty}\|u_n - k_n\| = 0. \tag{3.49}$$

For any $p \in \Theta$, note that T_r is firmly nonexpansive (Lemma 2.4), then we have

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n} x_n - T_{r_n} p\|^2 \\ &\leq \langle T_{r_n} x_n - T_{r_n} p, x_n - p\rangle \\ &= \langle u_n - p, x_n - p\rangle \\ &= \frac{1}{2}\left(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2\right),\end{aligned} \tag{3.50}$$

and hence

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2, \tag{3.51}$$

which together with (3.38) gives

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n-p\|^2+(1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&= (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\\
&\quad\times\left\{\|k_n-u_n\|^2+\|u_n-p\|^2+2\langle k_n-u_n,u_n-p\rangle\right\}\\
&\quad+\ (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&\le (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|^2\\
&\quad+(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|u_n-p\|^2\\
&\quad+\ 2(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|\|u_n-p\|\\
&\quad+(1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&\le (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|^2\\
&\quad+\ (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\left\{\|x_n-p\|^2-\|x_n-u_n\|^2\right\}\\
&\quad+\ 2(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|\|u_n-p\|\\
&\quad+(1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&\le (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|^2\\
&\quad+\ (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|x_n-p\|^2\\
&\quad-(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|x_n-u_n\|^2\\
&\quad+\ 2(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|\|u_n-p\|\\
&\quad+(1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&= (1-\epsilon_n\overline{\gamma})^2\|x_n-p\|^2-(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|x_n-u_n\|^2\\
&\quad+\ (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|^2\\
&\quad+\ 2(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|\|u_n-p\|+c_n\\
&= \left[1-2\epsilon_n\overline{\gamma}+(\epsilon_n\overline{\gamma})^2\right]\|x_n-p\|^2\\
&\quad-(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|x_n-u_n\|^2\\
&\quad+\ (1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|^2\\
&\quad+2(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|\|u_n-p\|+c_n\\
&\le \|x_n-p\|^2+(\epsilon_n\overline{\gamma})^2\|x_n-p\|^2\\
&\quad+(1-\epsilon_n\overline{\gamma})(1-\epsilon_n\overline{\gamma}-\beta_n)\|k_n-u_n\|^2
\end{aligned}
$$

$$\begin{aligned}
&- \left(1-\epsilon_n\overline{\gamma}\right)\left(1-\epsilon_n\overline{\gamma}-\beta_n\right)\|x_n-u_n\|^2 \\
&+ 2\left(1-\epsilon_n\overline{\gamma}\right)\left(1-\epsilon_n\overline{\gamma}-\beta_n\right)\|k_n-u_n\|\left\|u_n-p\right\| + c_n.
\end{aligned} \tag{3.52}$$

So

$$\begin{aligned}
&\left(1-\epsilon_n\overline{\gamma}\right)\left(1-\epsilon_n\overline{\gamma}-\beta_n\right)\|x_n-u_n\|^2 \\
&\quad \le \left\|x_n-p\right\|^2 - \left\|x_{n+1}-p\right\|^2 + \left(\epsilon_n\overline{\gamma}\right)^2\left\|x_n-p\right\|^2 \\
&\qquad + \left(1-\epsilon_n\overline{\gamma}\right)\left(1-\epsilon_n\overline{\gamma}-\beta_n\right)\|k_n-u_n\|^2 \\
&\qquad + 2\left(1-\epsilon_n\overline{\gamma}\right)\left(1-\epsilon_n\overline{\gamma}-\beta_n\right)\|k_n-u_n\|\left\|u_n-p\right\| + c_n \\
&\quad = \left(\left\|x_n-p\right\| - \left\|x_{n+1}-p\right\|\right)\left(\left\|x_n-p\right\| + \left\|x_{n+1}-p\right\|\right) \\
&\qquad + \left(\epsilon_n\overline{\gamma}\right)^2\left\|x_n-p\right\|^2 + \left(1-\epsilon_n\overline{\gamma}\right)\left(1-\epsilon_n\overline{\gamma}-\beta_n\right)\|k_n-u_n\|^2 \\
&\qquad + 2\left(1-\epsilon_n\overline{\gamma}\right)\left(1-\epsilon_n\overline{\gamma}-\beta_n\right)\|k_n-u_n\|\left\|u_n-p\right\| + c_n \\
&\quad \le \|x_n-x_{n+1}\|\left(\left\|x_n-p\right\| + \left\|x_{n+1}-p\right\|\right) + \left(\epsilon_n\overline{\gamma}\right)^2\left\|x_n-p\right\|^2 \\
&\qquad + \left(1-\epsilon_n\overline{\gamma}\right)\left(1-\epsilon_n\overline{\gamma}-\beta_n\right)\|k_n-u_n\|^2 \\
&\qquad + 2\left(1-\epsilon_n\overline{\gamma}\right)\left(1-\epsilon_n\overline{\gamma}-\beta_n\right)\|k_n-u_n\|\left\|u_n-p\right\| + c_n.
\end{aligned} \tag{3.53}$$

Using $\epsilon_n \to 0$, $c_n \to 0$ as $n \to \infty$, (3.32), and (3.49), we obtain

$$\lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{3.54}$$

Since $\liminf_{n\to\infty} r_n > 0$, we obtain

$$\lim_{n\to\infty}\left\|\frac{x_n-u_n}{r_n}\right\| = \lim_{n\to\infty}\frac{1}{r_n}\|x_n-u_n\| = 0. \tag{3.55}$$

Observe that

$$\begin{aligned}
\|W_nu_n-u_n\| &\le \|W_nu_n-W_nk_n\| + \|W_nk_n-x_n\| + \|x_n-u_n\| \\
&\le \|u_n-k_n\| + \|W_nk_n-x_n\| + \|x_n-u_n\|.
\end{aligned} \tag{3.56}$$

Applying (3.36), (3.49), and (3.54) to the last inequality, we obtain

$$\lim_{n\to\infty}\|W_nu_n-u_n\| = 0. \tag{3.57}$$

Let W be the mapping defined by (2.6). Since $\{u_n\}$ is bounded, applying Lemma 2.7 and (3.57), we have

$$\|Wu_n - u_n\| \leq \|Wu_n - W_n u_n\| + \|W_n u_n - u_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.58}$$

Step 4. We claim that $\limsup_{n\to\infty}\langle (A-\gamma f)z, z - x_n\rangle \leq 0$, where z is the unique solution of the variational inequality $\langle (A-\gamma f)z, z-x\rangle \geq 0$, for all $x \in \Theta$.

Since $z = P_\Theta(I - A + \gamma f)(z)$ is a unique solution of the variational inequality (3.1), to show this inequality, we choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\lim_{i\to\infty}\langle (A-\gamma f)z, z - u_{n_i}\rangle = \limsup_{n\to\infty}\langle (A-\gamma f)z, z - u_n\rangle. \tag{3.59}$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to $w \in C$. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup w$. From $\|Wu_n - u_n\| \to 0$, we obtain $Wu_{n_i} \rightharpoonup w$. Next, We show that $w \in \Theta$, where $\Theta := \cap_{n=1}^{\infty} F(T_n) \cap EP(F) \cap VI(C,B)$. First, we show that $w \in EP(F)$. Since $u_n = T_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C. \tag{3.60}$$

If follows from (A2) that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq -F(u_n, y) \geq F(y, u_n), \tag{3.61}$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq F(y, u_{n_i}). \tag{3.62}$$

Since $(u_{n_i} - x_{n_i})/r_{n_i} \to 0$ and $u_{n_i} \rightharpoonup w$, it follows by (A4) that $F(y,w) \leq 0$ for all $y \in H$. For t with $0 < t \leq 1$ and $y \in H$, let $y_t = ty + (1-t)w$. Since $y \in H$ and $w \in H$, we have $y_t \in H$ and hence $F(y_t, w) \leq 0$. So, from (A1) and (A4) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, w) \leq tF(y_t, y), \tag{3.63}$$

and hence $F(y_t, y) \geq 0$. From (A3), we have $F(w,y) \geq 0$ for all $y \in H$ and hence $w \in EP(F)$.

Next, we show that $w \in \cap_{n=1}^{\infty} F(T_n)$. By Lemma 2.6, we have $F(W) = \cap_{n=1}^{\infty} F(T_n)$. Assume $w \notin F(W)$. Since $u_{n_i} \rightharpoonup w$ and $w \neq Ww$, it follows by the Opial's condition that

$$\begin{aligned}
\liminf_{i\to\infty}\|u_{n_i} - w\| &< \liminf_{i\to\infty}\|u_{n_i} - Ww\| \\
&\leq \liminf_{i\to\infty}\{\|u_{n_i} - Wu_{n_i}\| + \|Wu_{n_i} - Ww\|\} \\
&\leq \liminf_{i\to\infty}\|u_{n_i} - w\|,
\end{aligned} \tag{3.64}$$

which derives a contradiction. Thus, we have $w \in F(W) = \cap_{n=1}^{\infty} F(T_n)$. By the same argument as that in the proof of [35, Theorem 2.1, Pages 10–11], we can show that $w \in VI(C, B)$. Hence $w \in \Theta$. Since $z = P_{\Theta}(I - A + \gamma f)(z)$, it follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle (A-\gamma f)z, z-x_n\rangle &= \limsup_{n\to\infty}\langle (A-\gamma f)z, z-u_n\rangle \\ &= \lim_{i\to\infty}\langle (A-\gamma f)z, z-u_{n_i}\rangle \\ &= \langle (A-\gamma f)z, z-w\rangle \le 0.\end{aligned} \tag{3.65}$$

It follows from the last inequality, (3.36), and (3.54) that

$$\limsup_{n\to\infty}\langle \gamma f(z) - Az, W_n k_n - z\rangle \le 0. \tag{3.66}$$

Step 5. Finally, we show that $\{x_n\}$and $\{u_n\}$ converge strongly to $z = P_{\Theta}(I-A+\gamma f)(z)$. Indeed, from (1.25) , we have

$$\begin{aligned}&\|x_{n+1} - z\|^2 \\ &\quad= \left\|\epsilon_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A)W_n k_n - z\right\|^2 \\ &\quad= \left\|((1-\beta_n)I - \epsilon_n A)(W_n k_n - z) + \beta_n(x_n - z) + \epsilon_n(\gamma f(x_n) - Az)\right\|^2 \\ &\quad= \left\|((1-\beta_n)I - \epsilon_n A)(W_n k_n - z) + \beta_n(x_n - z)\right\|^2 + \epsilon_n^2\left\|\gamma f(x_n) - Az\right\|^2 \\ &\qquad+ 2\beta_n\epsilon_n\langle x_n - z, \gamma f(x_n) - Az\rangle \\ &\qquad+ 2\epsilon_n\langle ((1-\beta_n)I - \epsilon_n A)(W_n k_n - z), \gamma f(x_n) - Az\rangle \\ &\quad\le \left[(1-\beta_n-\epsilon_n\overline{\gamma})\|W_n k_n - z\| + \beta_n\|x_n - z\|\right]^2 + \epsilon_n^2\left\|\gamma f(x_n) - Az\right\|^2 \\ &\qquad+ 2\beta_n\epsilon_n\gamma\langle x_n - z, f(x_n) - f(z)\rangle + 2\beta_n\epsilon_n\langle x_n - z, \gamma f(z) - Az\rangle \\ &\qquad+ 2(1-\beta_n)\gamma\epsilon_n\langle W_n k_n - z, f(x_n) - f(z)\rangle + 2(1-\beta_n)\epsilon_n\langle W_n k_n - z, \gamma f(z) - Az\rangle \\ &\qquad- 2\epsilon_n^2\langle A(W_n k_n - z), \gamma f(z) - Az\rangle \\ &\quad\le \left[(1-\beta_n-\epsilon_n\overline{\gamma})\|W_n k_n - z\| + \beta_n\|x_n - z\|\right]^2 + \epsilon_n^2\left\|\gamma f(x_n) - Az\right\|^2 \\ &\qquad+ 2\beta_n\epsilon_n\gamma\|x_n - z\|\left\|f(x_n) - f(z)\right\| + 2\beta_n\epsilon_n\langle x_n - z, \gamma f(z) - Az\rangle \\ &\qquad+ 2(1-\beta_n)\gamma\epsilon_n\|W_n k_n - z\|\left\|f(x_n) - f(z)\right\| + 2(1-\beta_n)\epsilon_n\langle W_n k_n - z, \gamma f(z) - Az\rangle \\ &\qquad- 2\epsilon_n^2\langle A(W_n k_n - z), \gamma f(z) - Az\rangle\end{aligned}$$

$$
\begin{aligned}
&\leq \left[(1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-z\|+\beta_n\|x_n-z\|\right]^2+\epsilon_n^2\left\|\gamma f(x_n)-Az\right\|^2\\
&\quad+\ 2\beta_n\epsilon_n\gamma\alpha\|x_n-z\|^2+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&\quad+\ 2(1-\beta_n)\gamma\epsilon_n\alpha\|x_n-z\|^2+2(1-\beta_n)\epsilon_n\langle W_nk_n-z,\gamma f(z)-Az\rangle\\
&\quad-\ 2\epsilon_n^2\langle A(W_nk_n-z),\gamma f(z)-Az\rangle\\
&=\left[(1-\epsilon_n\overline{\gamma})^2+2\beta_n\epsilon_n\gamma\alpha+2(1-\beta_n)\gamma\epsilon_n\alpha\right]\|x_n-z\|^2+\epsilon_n^2\left\|\gamma f(x_n)-Az\right\|^2\\
&\quad+\ 2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle+2(1-\beta_n)\epsilon_n\langle W_nk_n-z,\gamma f(z)-Az\rangle\\
&\quad-\ 2\epsilon_n^2\langle A(W_nk_n-z),\gamma f(z)-Az\rangle\\
&\leq\left[1-2(\overline{\gamma}-\alpha\gamma)\epsilon_n\right]\|x_n-z\|^2+\overline{\gamma}^2\epsilon_n^2\|x_n-z\|^2+\epsilon_n^2\left\|\gamma f(x_n)-Az\right\|^2\\
&\quad+\ 2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle+2(1-\beta_n)\epsilon_n\langle W_nk_n-z,\gamma f(z)-Az\rangle\\
&\quad+\ 2\epsilon_n^2\|A(W_nk_n-z)\|\left\|\gamma f(z)-Az\right\|\\
&=\left[1-2(\overline{\gamma}-\alpha\gamma)\epsilon_n\right]\|x_n-z\|^2+\epsilon_n\\
&\quad\times\Big\{\epsilon_n\Big[\overline{\gamma}^2\|x_n-z\|^2+\left\|\gamma f(x_n)-Az\right\|^2\\
&\qquad+\ 2\|A(W_nk_n-z)\|\left\|\gamma f(z)-Az\right\|\Big]+2\beta_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&\qquad+\ 2(1-\beta_n)\langle W_nk_n-z,\gamma f(z)-Az\rangle\Big\}.
\end{aligned}
\tag{3.67}
$$

Since $\{x_n\}$, $\{f(x_n)\}$, and $\{W_nk_n\}$ are bounded, we can take a constant $M>0$ such that

$$\overline{\gamma}^2\|x_n-z\|^2+\left\|\gamma f(x_n)-Az\right\|^2+2\|A(W_nk_n-z)\|\left\|\gamma f(z)-Az\right\|\leq M \tag{3.68}$$

for all $n\geq 0$. It then follows that

$$\|x_{n+1}-z\|^2\leq\left[1-2(\overline{\gamma}-\alpha\gamma)\epsilon_n\right]\|x_n-z\|^2+\epsilon_n\sigma_n, \tag{3.69}$$

where

$$\sigma_n=2\beta_n\langle x_n-z,\gamma f(z)-Az\rangle+2(1-\beta_n)\langle W_nk_n-z,\gamma f(z)-Az\rangle+\epsilon_nM. \tag{3.70}$$

Using (i), (3.65), and (3.66), we get $\limsup_{n\to\infty}\sigma_n\leq 0$. Applying Lemma 2.9 to (3.69), we conclude that $x_n\to z$ in norm. Finally, noticing

$$\|u_n-z\|=\|T_{r_n}x_n-T_{r_n}z\|\leq\|x_n-z\|, \tag{3.71}$$

we also conclude that $u_n\to z$ in norm. This completes the proof. □

Corollary 3.2 ([28, Theorem 3.1]). *Let C be nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $\{T_n\}$ be an infinitely many nonexpansive of C into itself such that $\Theta := \cap_{n=1}^{\infty} F(T_n) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0,1)$, and let A be a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Let $\{x_n\}$ and $\{u_n\}$ are the sequences generated by*

$$\begin{gathered} x_1 = x \in C \quad \text{chosen arbitrary}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + ((1-\beta)I - \epsilon_n A)W_n u_n, \quad \forall n \geq 1, \end{gathered} \tag{3.72}$$

where $\{W_n\}$ is the sequence generated by (1.23), $\beta \in (0,1)$, $\{\epsilon_n\}$ is a sequences in $(0,1)$, and $\{r_n\}$ is a real sequence in $(0,\infty)$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\epsilon_n = 0$;

(ii) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Theta$ which is the unique solution of the variational inequality

$$\langle (A - \gamma f)z, z - x\rangle \geq 0, \quad \forall x \in \Theta. \tag{3.73}$$

Equivalently, one has $z = P_\Theta(I - A + \gamma f)(z)$.

Proof. Put $B = 0$, $\{\beta_n\} = \beta$, and $\{\alpha_n\} = 0$ in Theorem 3.1., then $y_n = k_n = u_n$. The conclusion of Corollary 3.2 can obtain the desired result easily. □

Corollary 3.3. *Let C be nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let B be an β-inverse-strongly monotone mapping of C into H such that $\Theta := EP(F) \cap VI(C,B) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0,1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Let $\{x_n\}$, $\{y_n\}$, $\{k_n\}$, and $\{u_n\}$ be sequences generated by*

$$\begin{aligned} & x_1 = x \in C \quad \text{chosen arbitrary}, \\ & F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ & y_n = P_C(u_n - \lambda_n B u_n), \\ & k_n = \alpha_n u_n + (1-\alpha_n)P_C(u_n - \lambda_n B y_n), \\ & x_{n+1} = \epsilon_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A)k_n, \quad \forall n \geq 1, \end{aligned} \tag{3.74}$$

where $\{\epsilon_n\}$, $\{\alpha_n\}$, *and* $\{\beta_n\}$ *are three sequences in* $(0,1)$, *and* $\{r_n\}$ *is a real sequence in* $(0,\infty)$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty}\epsilon_n = 0$ *and* $\sum_{n=1}^{\infty}\epsilon_n = \infty$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$;

(v) $\{\lambda_n/\beta\} \subset (\tau, 1-\delta)$ *for some* $\tau, \delta \in (0,1)$ *and* $\lim_{n\to\infty}\lambda_n = 0$.

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to a point* $z \in \Theta$ *which is the unique solution of the variational inequality*

$$\langle (A - \gamma f)z, z - x\rangle \ge 0, \quad \forall x \in \Theta. \tag{3.75}$$

Equivalently, one has $z = P_{\Theta}(I - A + \gamma f)(z)$.

Proof. Put $T_n x = x$ for all $n \in \mathbb{N}$ and for all $x \in C$. Then $W_n x = x$ for all $x \in C$. The conclusion follows from Theorem 3.1. □

Corollary 3.4. *Let* C *be nonempty closed convex subset of a real Hilbert space* H, *let* $\{T_n\}$ *be an infinitely many nonexpansive of* C *into itself, and let* B *be a* β*-inverse-strongly monotone mapping of* C *into* H *such that* $\Theta := \cap_{n=1}^{\infty}F(T_n) \cap VI(C,B) \ne \emptyset$. *Let* f *be a contraction of* H *into itself with* $\alpha \in (0,1)$ *and let* A *be a strongly positive linear bounded operator on* H *with coefficient* $\overline{\gamma} > 0$ *and* $0 < \gamma < \overline{\gamma}/\alpha$. *Let* $\{x_n\}$, $\{y_n\}$, *and* $\{k_n\}$ *be sequences generated by*

$$\begin{aligned} &x_1 = x \in C \quad \text{chosen arbitrary}, \\ &y_n = P_C(x_n - \lambda_n B x_n), \\ &k_n = \alpha_n x_n + (1-\alpha_n) P_C(x_n - \lambda_n B y_n), \\ &x_{n+1} = \epsilon_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A) W_n k_n, \quad \forall n \ge 1, \end{aligned} \tag{3.76}$$

where $\{W_n\}$ *is the sequences generated by* (1.23), *and* $\{\epsilon_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ *are three sequences in* $(0,1)$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty}\epsilon_n = 0$ *and* $\sum_{n=1}^{\infty}\epsilon_n = \infty$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\{\lambda_n/\beta\} \subset (\tau, 1-\delta)$ *for some* $\tau, \delta \in (0,1)$ *and* $\lim_{n\to\infty}\lambda_n = 0$.

Then, $\{x_n\}$ *converges strongly to a point* $z \in \Theta$ *which is the unique solution of the variational inequality*

$$\langle (A - \gamma f)z, z - x\rangle \ge 0, \quad \forall x \in \Theta. \tag{3.77}$$

Equivalently, one has $z = P_{\Theta}(I - A + \gamma f)(z)$.

Proof. Put $F(x,y)=0$ for all $x,y\in C$ and $r_n=1$ for all $n\in\mathbb{N}$ in Theorem 3.1. Then, we have $u_n=P_Cx_n=x_n$. So, by Theorem 3.1, we can conclude the desired conclusion easily. □

If $A=I,\gamma\equiv 1$ and $\gamma_n=1-\epsilon_n-\beta_n$ in Theorem 3.1, then we can obtain the following result immediately.

Corollary 3.5. *Let C be nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)–(A4), let $\{T_n\}$ be an infinitely many nonexpansive of C into itself, and let B be an β-inverse-strongly monotone mapping of C into H such that $\Theta:=\cap_{n=1}^{\infty}F(T_n)\cap EP(F)\cap VI(C,B)\neq\emptyset$. Let f be a contraction of H into itself with $\alpha\in(0,1)$. Let $\{x_n\}$, $\{y_n\}$, $\{k_n\}$, and $\{u_n\}$ be sequences generated by*

$$\begin{aligned}
&x_1=x\in C \quad \text{chosen arbitrary},\\
&F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\quad \forall y\in C,\\
&y_n=P_C(u_n-\lambda_nBu_n),\\
&k_n=\alpha_nu_n+(1-\alpha_n)P_C(u_n-\lambda_nBy_n),\\
&x_{n+1}=\epsilon_nf(x_n)+\beta_nx_n+\gamma_nW_nk_n,\quad \forall n\geq 1,
\end{aligned}\tag{3.78}$$

where $\{W_n\}$ is the sequences generated by (1.23), $\{\epsilon_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are three sequences in $(0,1)$ and $\{r_n\}$ is a real sequence in $(0,\infty)$ satisfying the following conditions:

(i) $\epsilon_n+\beta_n+\gamma_n=1$;

(ii) $\lim_{n\to\infty}\epsilon_n=0$ *and* $\sum_{n=1}^{\infty}\epsilon_n=\infty$;

(iii) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(iv) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$;

(v) $\liminf_{n\to\infty}r_n>0$ *and* $\lim_{n\to\infty}|r_{n+1}-r_n|=0$,

(vi) $\{\lambda_n/\beta\}\subset(\tau,1-\delta)$ *for some* $\tau,\delta\in(0,1)$ *and* $\lim_{n\to\infty}\lambda_n=0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z\in\Theta$ which is the unique solution of the variational inequality

$$\langle z-f(z),z-x\rangle\geq 0,\quad \forall x\in\Theta.\tag{3.79}$$

Equivalently, one has $z=P_{\Theta}f(z)$.

Corollary 3.6. *Let C be nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let $\{T_n\}$ be an infinite family of nonexpansive of C into itself such that $\Theta := \cap_{n=1}^{\infty} F(T_n) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0,1)$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{aligned} &x_1 = x \in C \quad \text{chosen arbitrary}, \\ &F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ &x_{n+1} = \epsilon_n f(x_n) + \beta_n x_n + \gamma_n W_n u_n, \quad \forall n \geq 1, \end{aligned} \tag{3.80}$$

where $\{\epsilon_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are three sequences in $(0,1)$, and $\{r_n\}$ is a real sequence in $(0,\infty)$ satisfying the following conditions:

(i) $\epsilon_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty}\epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Theta$ which is the unique solution of the variational inequality

$$\langle (A - \gamma f)z, z - x \rangle \geq 0, \quad \forall x \in \Theta. \tag{3.81}$$

Equivalently, one has $z = P_{\Theta}(I - A + \gamma f)(z)$.

Proof. Put $B = 0$ and $\{\alpha_n\} = 0$ in Corollary 3.5. then $y_n = k_n = u_n$. The conclusion of Corollary 3.6 can obtain the desired result easily. □

4. Application for Optimization Problem

In this section, we shall utilize the results presented in the paper to study the following optimization problem:

$$\begin{aligned} &\min \quad h(x), \\ &x \in C. \end{aligned} \tag{4.1}$$

where $h(x)$ is a convex and lower semicontinuous functional defined on a closed subset C of a Hilbert space H. We denote by T the set of solution of (4.1). Let F be a bifunction from $C \times C$ to $\mathbb{R}$ defined by $F(x,y) = h(y) - h(x)$. We consider the following equilibrium problem, that is, to find $x \in C$ such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{4.2}$$

It is obvious that $EP(F) = T$, where $EP(F)$ denotes the set of solution of equilibrium problem (4.2). In addition, it is easy to see that $F(x, y)$ satisfies the conditions $(A1)$–$(A4)$ in Section 1. Therefore, from the Corollary 3.6, we know the following iterative sequence $\{x_n\}$ defined by

$$\begin{aligned} &x_1 \in C \quad \text{chosen arbitrary}, \\ &h(y) - h(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C \\ &x_{n+1} = \epsilon_n f(x_n) + \beta_n x_n + \gamma_n u_n, \end{aligned} \tag{4.3}$$

where $\{\epsilon_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0, 1)$, and $\{r_n\}$ is a real sequence in $(0, \infty)$ satisfying the following conditions:

(i) $\epsilon_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty}\epsilon_n = 0$ and $\sum_{n=1}^{\infty} \epsilon_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Then, $\{x_n\}$ converges strongly to a point $z = P_T f(z)$ of optimization problem (4.1).

In special case, we pick $f(x) = 0$ for all $x \in H$ and $\beta_n = 0$, $r_n = 1$, $\epsilon_n = 1/2$ for all $n \in \mathbb{N}$, then $x_{n+1} = 1/2u_n$, and from (4.3) we obtain a special iterative scheme

$$\begin{aligned} &h(y) - h(u_n) + \left\langle y - u_n, u_n - \frac{1}{2}u_{n-1}\right\rangle \geq 0, \quad \forall y \in C, \quad n \geq 2, \\ &h(y) - h(u_1) + \langle y - u_1, u_1 - x_1\rangle \geq 0, \quad \forall y \in C. \end{aligned} \tag{4.4}$$

Then, $\{u_n\}$ converges strongly to a solution $z = P_T 0$ of optimization problem (4.1). In fact, the z is the minimum norm point on the T.

Therefore, we consider a special from of optimization problem (4.1) which is as follows: (i.e., is taking $h(x) = \|x\|$)

$$\begin{aligned} &\min \ \|x\|, \\ &x \in C. \end{aligned} \tag{4.5}$$

In fact, the solution of optimization problem (4.4) is named the minimum norm point on the closed convex set C. From iterative algorithm (4.4) we obtain the following iterative algorithm (4.5), and $\{u_n\}$ is defined by

$$\begin{aligned} &\|y\| - \|u_n\| + \left\langle y - u_n, u_n - \frac{1}{2}u_{n-1}\right\rangle \geq 0, \quad \forall y \in C, \quad n \geq 2, \\ &\|y\| - \|u_1\| + \langle y - u_1, u_1 - x_1\rangle \geq 0, \quad \forall y \in C \end{aligned} \tag{4.6}$$

for any initial guess $x_1 \in H$. Then, $\{u_n\}$ converges strongly to a minimum norm point on the closed convex set C.

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