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Research Article

Strict Contractive Conditions and Common Fixed Point Theorems in Cone Metric Spaces

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A lot of authors have proved various common fixed-point results for pairs of self-mappings under strict contractive conditions in metric spaces. In the case of cone metric spaces, fixed point results are usually proved under assumption that the cone is normal. In the present paper we prove common fixed point results under strict contractive conditions in cone metric spaces using only the assumption that the cone interior is nonempty. We modify the definition of property (E.A), introduced recently in the work by Aamri and Moutawakil (2002), and use it instead of usual assumptions about commutativity or compatibility of the given pair. Examples show that the obtained results are proper extensions of the existing ones.

1. Introduction and Preliminaries

Cone metric spaces were introduced by Huang and Zhang in [1], where they investigated the convergence in cone metric spaces, introduced the notion of their completeness, and proved some fixed point theorems for contractive mappings on these spaces. Recently, in [2–6], some common fixed point theorems have been proved for maps on cone metric spaces. However, in [1–3], the authors usually obtain their results for normal cones. In this paper we do not impose the normality condition for the cones.

We need the following definitions and results, consistent with [1], in the sequel.

Let E be a real Banach space. A subset P of E is a *cone* if

(i) P is closed, nonempty and $P \neq \{0\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;

(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define the partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in \operatorname{int} P$ (the interior of P).

There exist two kinds of cones: normal and nonnormal cones. A cone $P \subset E$ is a *normal cone* if

$$\inf\{\|x + y\| : x, y \in P,\ \|x\| = \|y\| = 1\} > 0, \tag{1.1}$$

or, equivalently, if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\|. \tag{1.2}$$

The least positive number satisfying (1.2) is called the normal constant of P. It is clear that $K \geq 1$.

It follows from (1.1) that P is *nonnormal* if and only if there exist sequences $x_n, y_n \in P$ such that

$$0 \leq x_n \leq x_n + y_n, \quad x_n + y_n \longrightarrow 0 \quad \text{but } x_n \nrightarrow 0. \tag{1.3}$$

So, in this case, the Sandwich theorem does not hold. (In fact, validity of this theorem is equivalent to the normality of the cone, see [7].)

Example 1.1 (see [7]). Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0 \text{ on } [0,1]\}$. This cone is not normal. Consider, for example,

$$x_n(t) = \frac{1 - \sin nt}{n+2}, \qquad y_n(t) = \frac{1 + \sin nt}{n+2}. \tag{1.4}$$

Then $\|x_n\| = \|y_n\| = 1$ and $\|x_n + y_n\| = 2/(n+2) \to 0$.

Definition 1.2 (see [1]). Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

(d1) $0 \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(d2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(d3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a *cone metric* on X, and (X, d) is called a *cone metric space*.

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space with $E = \mathbb{R}$ and $P = [0, +\infty[$ (see [1, Example 1]).

Let $\{x_n\}$ be a sequence in X, and $x \in X$. If, for every c in E with $0 \ll c$, there is an $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \ll c$, then it is said that x_n converges to x, and this is denoted by $\lim_{n\to\infty} x_n = x$, or $x_n \to x$, $n \to \infty$. Completeness is defined in the standard way.

It was proved in [1] that if P is a normal cone, then $x_n \in X$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$, $n \to \infty$.

Let (X, d) be a cone metric space. Then the following properties are often useful (particularly when dealing with cone metric spaces in which the cone may be nonnormal):

(p1) if $0 \leq u \ll c$ for each $c \in \operatorname{int} P$ then $u = 0$,

(p2) if $c \in \operatorname{int} P$, $0 \leq a_n$ and $a_n \to 0$, then there exists n_0 such that $a_n \ll c$ for all $n > n_0$.

It follows from (p2) that the sequence x_n converges to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$. In the case when the cone is not necessarily normal, we have only one half of the statements of Lemmas 1 and 4 from [1]. Also, in this case, the fact that $d(x_n, y_n) \to d(x, y)$ if $x_n \to x$ and $y_n \to y$ is not applicable.

2. Compatible and Noncompatible Mappings in Cone Metric Spaces

In the sequel we assume only that E is a Banach space and that P is a cone in E with $\operatorname{int} P \neq \emptyset$. The last assumption is necessary in order to obtain reasonable results connected with convergence and continuity. In particular, with this assumption the limit of a sequence is uniquely determined. The partial ordering induced by the cone P will be denoted by $\leq$.

If (f, g) is a pair of self-maps on the space X then its well known properties, such as commutativity, weak-commutativity [8], R-commutativity [9, 10], weak compatibility [11], can be introduced in the same way in metric and cone metric spaces. The only difference is that we use vectors instead of numbers. As an example, we give the following.

Definition 2.1 (see [9]). A pair of self-mappings (f, g) on a cone metric space (X, d) is said to be *R-weakly commuting* if there exists a real number $R > 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$ for all $x \in X$, whereas the pair (f, g) is said to be *pointwise R-weakly commuting* if for each $x \in X$ there exists $R > 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$.

Here it may be noted that at the points of coincidence, R-weak commutativity is equivalent to commutativity and it remains a necessary minimal condition for the existence of a common fixed point of contractive type mappings.

Compatible mappings in the setting of metric spaces were introduced by Jungck [11, 12]. The property (E.A) was introduced in [13]. We extend these concepts to cone metric spaces and investigate their properties in this paper.

Definition 2.2. A pair of self-mappings (f, g) on a cone metric space (X, d) is said to be *compatible* if for arbitrary sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t \in X$, and for arbitrary $c \in P$ with $c \in \operatorname{int} P$, there exists $n_0 \in \mathbb{N}$ such that $d(fgx_n, gfx_n) \ll c$ whenever $n > n_0$. It is said to be *weakly compatible* if $fx = gx$ implies $fgx = gfx$.

It is clear that, as in the case of metric spaces, the pair (f, i_X) (i_X—the identity mapping) is both compatible and weakly compatible, for each self-map f.

If $E = \mathbb{R}$, $\|\cdot\| = |\cdot|$, $P = [0, +\infty[$, then these concepts reduce to the respective concepts of Jungck in metric spaces. It is known that in the case of metric spaces compatibility implies weak compatibility but that the converse is not true. We will prove that the same holds in the case of cone metric spaces.

Proposition 2.3. *If the pair (f, g) of self-maps on the cone metric space (X, d) is compatible, then it is also weakly compatible.*

Proof. Let $fu = gu$ for some $u \in X$. We have to prove that $fgu = gfu$. Take the sequence $\{x_n\}$ with $x_n = u$ for each $n \in \mathbb{N}$. It is clear that $fx_n, gx_n \to fu = gu$. If $c \in P$ with $c \in \operatorname{int} P$, then the compatibility of the pair (f, g) implies that $d(fgx_n, gfx_n) = d(fgu, gfu) \ll c$. It follows by property (p1) that $d(fgu, gfu) = 0$, that is, $fgu = gfu$. □

Example 2.4. We show in this example that the converse in the previous proposition does not hold, neither in the case when the cone P is normal nor when it is not.

Let $X = [0, 2]$ and

(1) $E_1 = \mathbb{R}^2$, $P_1 = \{(a, b) : a \geq 0, b \geq 0\}$ (a normal cone), let $d_1(x, y) = (|x-y|, \alpha|x-y|)$, ($\alpha \geq 0$ fixed), (X, d_1) is a complete cone metric space,

(2) $E_2 = C^1_{\mathbb{R}}[0, 1]$, $P_2 = \{\varphi : \varphi(t) \geq 0, t \in [0, 1]\}$ (a nonnormal cone). Let $d_2(x, y) = |x-y|\varphi$ for some fixed $\varphi \in P_2$, for example, $\varphi(t) = 2^t$. (X, d_2) is also a complete cone metric space.

Consider the pair of mappings (f, g) defined as

$$fx = \begin{cases} 2 - x, & 0 \leq x < 1, \\ 2, & 1 \leq x \leq 2, \end{cases} \qquad gx = \begin{cases} 2x, & 0 \leq x < 1, \\ x, & 1 \leq x \leq 2,\ x \neq \dfrac{4}{3}, \\ 2, & x = \dfrac{4}{3}, \end{cases} \tag{2.1}$$

and the sequence $x_n = 2/3 + 1/n \in X$. It is $fx_n = 2 - (2/3 + 1/n) = 4/3 - 1/n$, $gx_n = 2(2/3 + 1/n) = 4/3 + 2/n$.

In both of the given cone metrics $fx_n, gx_n \to 4/3$ holds. Namely, in the first case, $d_1(fx_n, 4/3) = d_1(4/3 - 1/n, 4/3) = (1/n, \alpha(1/n)) \to (0, 0)$ in the standard norm of the space $\mathbb{R}^2$. Also, $d_1(gx_n, 4/3) = d_1(4/3 + 2/n, 4/3) = (2/n, \alpha(2/n)) \to (0, 0)$ in the same norm (since in this case the cone is normal, we can use that the cone metric d_1 is continuous).

However, $d_1(fgx_n, gfx_n) = d_1(f(4/3 + 2/n), g(4/3 - 1/n)) = d_1(2, 8/3 - 2/n) = (|2/3 - 2/n|, \alpha|2/3 - 2/n|)$. So, taking the fixed vector $(2/3, \alpha(2/3)) \in P_1$, we see that $d_1(fgx_n, gfx_n) \ll c$ does not hold for each $c \in \operatorname{int} P$, for otherwise by (p2) this vector would reduce to $(0, 0)$. Hence, the pair (f, g) is not compatible.

In the case (2) of a nonnormal cone we have $d_2(fx_n, 4/3) = d_2(4/3 - 1/n, 4/3) = |4/3 - 1/n - 4/3|\varphi = (1/n)2^t \to 0$ in the norm of space E_2; $d_2(gx_n, 4/3) = d_2(4/3 + 2/n, 4/3) = |4/3 + 2/n - 4/3|\varphi = (2/n)\varphi \to 0$ in the same norm.

However, $d_2(fgx_n, gfx_n) = d_2(2, 8/3 - 2/n) = |2/3 - 2/n|\varphi = (2/3 - 2/n)2^t$, $n \geq 4$. If we put $u_n(t) = (2/3 - 2/n)2^t$, then $u_n(t) \ll c$ is impossible since $(2/3)2^t = u_n(t) + (2/n)2^t \ll c/2 + c/2 = c$ and $(2/3)2^t \neq 0$ (null function). This means that it is not $u_n(t) \ll c$, and so the pair (f, g) is not compatible.

Since $f(4/3) = g(4/3)$ and $f2 = g2$, in both cases $fg(4/3) = gf(4/3)$ and $fg2 = gf2$.

Clearly, a pair of self-mappings (f, g) on a cone metric space (X, d) is not compatible if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t \in X$ for some $t \in X$ but $\lim_{n\to\infty} d(fgx_n, gfx_n)$ is either nonzero or nonexistent.

Definition 2.5. A pair of self-mappings (f, g) on a cone metric space (X, d) is said to enjoy *property (E.A)* if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t$ for some $t \in X$.

Clearly, each noncompatible pair satisfies property (E.A). The converse is not true. Indeed, let $X = [0,1]$, $E = \mathbb{R}^2$, $P = \{(a,b) : a \geq 0, b \geq 0\}$, $d(x,y) = (|x-y|, \alpha|x-y|)$, $\alpha \geq 0$ fixed, $fx = 2x$, $gx = 3x$, $x_n = 1/n$. Then in the given cone metric both sequences fx_n and gx_n tend to 0, but

$$d(fgx_n, gfx_n) = d(6x_n, 6x_n) = (0,0) \ll c \tag{2.2}$$

for each point $c = (c_1, c_2)$ of $\operatorname{int} P = \{(a,b) : a > 0, b > 0\}$, that is, the pair (f,g) is compatible. In other words, the set of pairs with property (E.A) contains all noncompatible pairs, and also some of the compatible ones.

3. Strict Contractive Conditions and Existence of Common Fixed Points on Cone Metric Spaces

Let (X,d) be a complete cone metric space, let (f,g) be a pair of self-mappings on X and $x, y \in X$. Let us consider the following sets:

$$\begin{aligned}
M_0^{f,g}(x,y) &= \{d(gx,gy), d(gx,fx), d(gy,fy), d(gx,fy), d(gy,fx)\},\\
M_1^{f,g}(x,y) &= \left\{d(gx,gy), d(gx,fx), d(gy,fy), \frac{d(gx,fy)+d(gy,fx)}{2}\right\},\\
M_2^{f,g}(x,y) &= \left\{d(gx,gy), \frac{d(gx,fx)+d(gy,fy)}{2}, \frac{d(gx,fy)+d(gy,fx)}{2}\right\},
\end{aligned} \tag{3.1}$$

and define the following conditions:

(1°) for arbitrary $x, y \in X$ there exists $u_0(x,y) \in M_0^{f,g}(x,y)$ such that

$$d(fx,fy) < u_0(x,y); \tag{3.2}$$

(2°) for arbitrary $x, y \in X$ there exists $u_1(x,y) \in M_1^{f,g}(x,y)$ such that

$$d(fx,fy) < u_1(x,y); \tag{3.3}$$

(3°) for arbitrary $x, y \in X$ there exists $u_2(x,y) \in M_2^{f,g}(x,y)$ such that

$$d(fx,fy) < u_2(x,y). \tag{3.4}$$

These conditions are called *strict contractive conditions*. Since in metric spaces the following inequalities hold:

$$\frac{d(gx,fy)+d(gy,fx)}{2} \leq \max\{d(gx,fy),d(gy,fx)\},$$
$$\frac{d(gx,fx)+d(gy,fy)}{2} \leq \max\{d(gx,fx),d(gy,fy)\}, \tag{3.5}$$

in this setting, condition (2°) is a special case of (1°) and (3°) is a special case of (2°). This is not the case in the setting of cone metric spaces, since for $a,b \in P$, if a and b are incomparable, then also $(a+b)/2$ is incomparable, both with a and with b.

The following theorem was proved for metric spaces in [13].

Theorem 3.1. *Let the pair of weakly compatible mappings (f,g) satisfy property (E.A). If condition (3°) is satisfied, $fX \subset gX$, and at least one of fX and gX is complete, then f and g have a unique common fixed point.*

Conditions (1°) and (2°) are not mentioned in [13]. We give an example of a pair of mappings (f,g) satisfying (1°) and (2°), but which have no common fixed points, neither in the setting of metric nor in the setting of cone metric spaces.

Example 3.2. Let $X = [0,1]$ with the standard metric. Take $0 < a < b < 1$ and consider the functions:

$$fx = \begin{cases} ax, & x \in (0,1), \\ a, & x = 0, \\ 0, & x = 1, \end{cases} \qquad gx = bx \text{ for } x \in [0,1]. \tag{3.6}$$

We have to show that for each $(x,y) \in X^2$ there exists $u_0(x,y) \in M_0^{f,g}(x,y)$ such that $d(fx,fy) < u_0(x,y)$ for $x \neq y$.

It is not hard to prove that in all possible five cases one can find a respective $u_0(x,y)$:

(1°) $x,y \in (0,1) \Rightarrow u_0(x,y) = d(gx,gy)$;

(2°) $x = 0$, $y \in (0,1) \Rightarrow u_0(0,y) = d(f0,g0)$;

(3°) $x = 1$, $y \in (0,1) \Rightarrow u_0(1,y) = d(f1,g1)$;

(4°) $x = 0$, $y = 1 \Rightarrow u_0(0,1) = d(g0,g1)$;

(5°) $x = 1$, $y = 0 \Rightarrow u_0(1,0) = d(g1,g0)$.

Let now $x_n = 1/n$. Then $f(x_n) = a/n \to 0$ and $g(x_n) = b/n \to 0$. It is clear that $fX = [0,a] \subset gX = [0,b] \subset X = [0,1]$ and all of them are complete metric spaces, so all the conditions of Theorem 3.1 except (3°) are fulfilled, but there exists no coincidence point of mappings f and g.

Using the previous example, it is easy to construct the respective example in the case of cone metric spaces.

Let $X = [0,1]$, $E = \mathbb{R}^2$, $P = \{(x,y) : x \geq 0, y \geq 0\}$, and let $d : X \times X \to E$ be defined as $d(x,y) = (|x-y|, \alpha|x-y|)$, for fixed $\alpha \geq 0$. Let f, g be the same mappings as in the previous case. Now we have the following possibilities:

(1°) $x, y \in (0,1) \Rightarrow u_0(x,y) = d(gx, gy) = (|bx - by|, \alpha|bx - by|)$;

(2°) $x = 0$, $y \in (0,1) \Rightarrow u_0(0,y) = d(f0, g0) = d(a,0) = (a, \alpha a)$;

(3°) $x = 1$, $y \in (0,1) \Rightarrow u_0(1,y) = d(f1, g1) = d(0,b) = (b, \alpha b)$;

(4°) $x = 0$, $y = 1 \Rightarrow u_0(0,1) = d(g0, g1) = d(0,b) = (b, \alpha b)$;

(5°) $x = 1$, $y = 0 \Rightarrow u_0(1,0) = d(g1, g0) = d(b,0) = (b, \alpha b)$.

Conclusion is the same as in the metric case.

We will prove the following theorem in the setting of cone metric spaces.

Theorem 3.3. *Let f and g be two weakly compatible self-mappings of a cone metric space (X, d) such that*

(i) *(f, g) satisfies property (E.A);*

(ii) *for all $x, y \in X$ there exists $u(x,y) \in M_2^{f,g}(x,y)$ such that $d(fx, fy) < u(x,y)$,*

(iii) *$fX \subset gX$.*

If gX or fX is a complete subspace of X, then f and g have a unique common fixed point.

Proof. It follows from (i) that there exists a sequence $\{x_n\}$ satisfying

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t, \quad \text{for some } t \in X. \tag{3.7}$$

Suppose that gX is complete. Then $\lim_{n\to\infty} gx_n = ga$ for some $a \in X$. Also $\lim_{n\to\infty} fx_n = ga$.

We will show that $fa = ga$. Suppose that $fa \neq ga$. Condition (ii) implies that there are the following three cases.

(1°) $d(fx_n, fa) < d(gx_n, ga) \ll c$, that is, $d(fx_n, fa) \ll c$; it follows that $\lim_{n\to\infty} fx_n = fa$ and so $fa = ga$;

(2°) $d(fx_n, fa) < (d(fx_n, gx_n) + d(fa, ga))/2$; it follows that $2d(fx_n, fa) < d(fx_n, gx_n) + d(fa, fx_n) + d(fx_n, ga)$, hence $d(fx_n, fa) < d(fx_n, gx_n) + d(fx_n, ga) \ll c/2 + c/2 = c$, that is, $\lim_{n\to\infty} fx_n = fa$ and so $fa = ga$;

(3°) $d(fx_n, fa) < (d(fa, gx_n) + d(fx_n, ga))/2$; it follows that $2d(fx_n, fa) < d(fa, fx_n) + d(fx_n, gx_n) + d(fx_n, ga)$, hence $d(fx_n, fa) < d(fx_n, gx_n) + d(fx_n, ga) \ll c/2 + c/2 = c$, that is, $\lim_{n\to\infty} fx_n = fa$ and so $fa = ga$.

Hence, we have proved that f and g have a coincidence point $a \in X$ and a point of coincidence $\omega \in X$ such that $\omega = fa = ga$. If ω_1 is another point of coincidence, then there is $a_1 \in X$ with $\omega_1 = fa_1 = ga_1$. Now,

$$d(\omega, \omega_1) = d(fa, fa_1) < u_2(a, a_1), \tag{3.8}$$

where

$$u_2 \in \left\{ d(ga, ga_1), \frac{d(ga, fa) + d(ga_1, fa_1)}{2}, \frac{d(ga, fa_1) + d(ga_1, fa)}{2} \right\}$$
$$= \{d(\omega, \omega_1), 0\}. \tag{3.9}$$

Hence, $d(\omega, \omega_1) = 0$, that is, $\omega = \omega_1$.

Since $\omega = fa = ga$ is the unique point of coincidence of f and g, and f and g are weakly compatible, ω is the unique common fixed point of f and g by [4, Proposition 1.12].

The proof is similar when fX is assumed to be a complete subspace of X since $fX \subset gX$. □

Example 3.4. Let $X = \mathbb{R}$, $E = C^1_{\mathbb{R}}[0,1]$, $P = \{\varphi : \varphi(t) \geq 0, t \in [0,1]\}$, $d(x,y) = |x-y|\varphi$, φ is a fixed function from P, for example, $\varphi(t) = 2^t$.

Consider the mappings $f, g : \mathbb{R} \to \mathbb{R}$ given by $fx = \alpha x$, $gx = \beta x$, $0 < \alpha < \beta < 1$. Then

$$\begin{aligned} d(fx, fy) &= |fx - fy|\varphi = |\alpha x - \alpha y|\varphi = \alpha|x-y|\varphi \\ &= \frac{\alpha}{\beta}|\beta x - \beta y|\varphi = \frac{\alpha}{\beta}|gx - gy|\varphi = \frac{\alpha}{\beta} d(gx, gy) < d(gx, gy), \end{aligned} \tag{3.10}$$

so the conditions of strict contractivity are fulfilled. Further, $gf0 = fg0 = 0$ and it is easy to verify that the sequence $x_n = 1/n$ satisfies the conditions $fx_n \to 0$, $gx_n \to 0$ (even in the setting of cone metric spaces). All the conditions of the theorem are fulfilled. Taking $E = \mathbb{R}$, $P = [0, +\infty[$, $\|\cdot\| = |\cdot|$ we obtain a theorem from [13]. Note that this theorem cannot be applied directly, since the cone may not be normal in our case. So, our theorem is a proper generalization of the mentioned theorem from [13].

Example 3.5. Let $X = [1, +\infty[$, $E = \mathbb{R}^2$, $P = \{(x,y) : x \geq 0, y \geq 0\}$, $d(x,y) = (|x-y|, \alpha|x-y|)$, $\alpha \geq 0$.

Take the mappings $f, g : X \to X$ given by $fx = x^2$, $gx = x^3$. Then, since $x, y \geq 1$, for $x \neq y$ it is

$$d(fx, fy) = \left(\left| x^2 - y^2 \right|, \alpha \left| x^2 - y^2 \right| \right) < \left(\left| x^3 - y^3 \right|, \alpha \left| x^3 - y^3 \right| \right) = d(gx, gy), \tag{3.11}$$

that is, the conditions of strict contractivity are fulfilled. Taking $x_n = 1 + 1/n$ we have that in the cone metric space (X, d), $fx_n \to 1$, $gx_n \to 1$, and $fg1 = gf1 = 1$. Indeed,

$$\begin{aligned} d(fx_n, 1) &= \left(\left| \frac{1}{n^2} - 1 \right|, \alpha \left| \frac{1}{n^2} - 1 \right| \right) \longrightarrow (1,1), \\ d(gx_n, 1) &= \left(\left| \frac{1}{n^3} - 1 \right|, \alpha \left| \frac{1}{n^3} - 1 \right| \right) \longrightarrow (1,1), \end{aligned} \tag{3.12}$$

(in the norm of space E), which means that the pair of mappings (f, g) of the cone metric space (X, d) satisfies condition (E.A). The conditions of the theorem are fulfilled in the case of a normal cone P.

Corollary 3.6. *If all the conditions of Theorem 3.3 are fulfilled, except that (ii) is substituted by either of the conditions*

$$d(fx,fy) < d(gx,gy),$$
$$d(fx,fy) < \frac{1}{2}(d(gx,fx)+d(gy,fy)), \tag{3.13}$$
$$d(fx,fy) < \frac{1}{2}(d(gx,gy)+d(gy,fx)),$$

then f and g have a unique common fixed point.

Proof. Formulas in (3.13) are clearly special cases of (ii). □

Note that formulas in (3.13) are strict contractive conditions which correspond to the contractive conditions of Theorems 1, 2, and 3 from [2].

3.1. Cone Metric Version of Das-Naik's Theorem

The following theorem was proved by Das and Naik in [14].

Theorem 3.7. *Let (X,d) be a complete metric space. Let f be a continuous self-map on X and g be any self-map on X that commutes with f. Further, let $fX \subset gX$ and there exists a constant $\lambda \in (0,1)$ such that for all $x,y \in X$:*

$$d(fx,fy) \leq \lambda \cdot u_0(x,y), \tag{3.14}$$

where $u_0(x,y) = \max M_0^{f,g}(x,y)$. Then f and g have a unique common fixed point.

Now we recall the definition of g-quasi-contractions on cone metric spaces. Such mappings are generalizations of Das-Naik's quasi-contractions.

Definition 3.8 (see [3]). Let (X,d) be a cone metric space, and let $f,g : X \to X$. Then f is called a g-quasicontraction, if for some constant $\lambda \in (0,1)$ and for every $x,y \in X$, there exists $u(x,y) \in M_0^{f,g}(x,y)$ such that

$$d(fx,fy) \leq \lambda \cdot u(x,y). \tag{3.15}$$

The following theorem was proved in [3].

Theorem 3.9. *Let (X,d) be a complete cone metric space with a normal cone. Let $f,g : X \to X$, f is a g-quasicontraction that commutes with g, one of the mappings f and g is continuous, and they satisfy $fX \subset gX$. Then f and g have a unique common fixed point in X.*

Using property (E.A) of the pair (f,g) instead of commutativity and continuity, we can prove the existence of a common fixed point without normality condition. Then, Theorem 3.7 for metric spaces follows as a consequence.

Theorem 3.10. *Let f and g be two weakly compatible self-mappings of a cone metric space (X, d) such that*

(i) (f, g) *satisfies property (E.A);*

(ii) f *is a g-quasicontraction;*

(iii) $fX \subset gX$.

If gX or fX is a complete subspace of X, then f and g have a unique common fixed point.

Proof. Let $x_n \in X$ be such that $fx_n \to t \in X$, $gx_n \to t$. It follows from (iii) and the completeness of one of fX, gX that there exists $a \in X$ such that $ga = t$. Hence, $fx_n, gx_n \to ga$. We will prove first that $fa = ga$. Putting $x = x_n$ and $y = a$ in (3.15) we obtain that

$$d(fx_n, fa) \le \lambda \cdot u(x_n, a), \tag{3.16}$$

for some $u(x_n, a) \in \{d(gx_n, ga), d(gx_n, fx_n), d(gx_n, fa), d(ga, fx_n), d(ga, fa)\}$. We have to consider the following cases:

(1°) $d(fx_n, fa) \le \lambda \cdot d(gx_n, ga) \ll \lambda \cdot c/\lambda = c$;

(2°) $d(fx_n, fa) \le \lambda \cdot d(gx_n, fx_n) \le \lambda d(gx_n, fa) + \lambda d(fa, fx_n)$ which implies $d(fx_n, fa) \le (\lambda/(1-\lambda)) d(gx_n, fa) \ll (\lambda/(1-\lambda))(c/(\lambda/(1-\lambda))) = c$;

(3°) $d(fx_n, fa) \le \lambda \cdot d(gx_n, fa) \le \lambda \cdot d(gx_n, fx_n) + \lambda \cdot d(fx_n, fa)$ which implies $d(fx_n, fa) \le \lambda \cdot d(ga, fx_n) \ll \lambda \cdot c/\lambda = c$;

(4°) $d(fx_n, fa) \le \lambda d(fx_n, ga) \ll \lambda(c/\lambda) = c$, since $fx_n \to ga$;

(5°) $d(fx_n, fa) \le \lambda \cdot d(ga, fa) \le \lambda d(ga, fx_n) + \lambda d(fa, fx_n)$ which implies $d(fx_n, fa) \le (\lambda/(1-\lambda)) d(fx_n, ga) \ll (\lambda/(1-\lambda))(c/(\lambda/(1-\lambda))) = c$.

Thus, in all possible cases, $d(fx_n, fa) \ll c$ for each $c \in \operatorname{int} P$ and so $fx_n \to fa$. The uniqueness of limits (which is a consequence of the condition $\operatorname{int} P \ne \emptyset$ without using normality of the cone) implies that $fa = ga$.

Since f and g are weakly compatible it follows that $fga = gfa = ffa = gga$. Let us prove that $fa = ga$ is a common fixed point of the pair (f, g). Suppose $ffa \ne fa$. Putting in (3.15) $x = fa$, $y = a$, we obtain that

$$d(ffa, fa) \le \lambda u(fa, a), \tag{3.17}$$

where $u(fa, a) \in \{d(gfa, ga), d(gfa, ffa), d(gfa, fa), d(ga, ffa), d(ga, fa)\} = \{d(ffa, fa), d(ffa, ffa), d(ffa, fa), d(fa, ffa), d(fa, fa)\} = \{d(ffa, fa), 0\}$. So, we have only two possible cases:

(1°) $d(ffa, fa) \le \lambda d(ffa, fa)$ implying $d(ffa, fa) = 0$ and $ffa = fa$;

(2°) $d(ffa, fa) \le \lambda \cdot 0 = 0$ implying $d(ffa, fa) = 0$ and $ffa = fa$.

The uniqueness follows easily. The theorem is proved. □

Note that in Theorems 3.3 and 3.10 condition that one of the subspaces fX, gX is complete can be replaced by the condition that one of them is closed in the cone metric space (X, d).

Corollary 3.11. *The conclusion in Theorem 3.7 remains valid if the conditions of commutativity and continuity of one of the mappings f, g are replaced by the condition (E.A) for the pair (f, g).*

Proof. This follows easily by taking $E = \mathbb{R}$, $\|\cdot\| = |\cdot|$, $P = [0, +\infty[$. □

Taking into account [15, Theorem 2.1] and results from [5], it can be seen that the question of existence of fixed points for quasicontractions on complete cone metric spaces without normality condition is still open in the case when $\lambda \in [1/2, 1[$. Theorem 3.10 answers this question when property (E.A) is fulfilled.

Note that the common fixed point problem for a weak compatible pair with property (E.A) under strict conditions in symmetric spaces was investigated in [16–21]. As an example we state the following result.

Theorem 3.12 (see [19]). *Let (X, d) be a symmetric (semimetric) space that enjoys property (W_3) (the Hausdorffness of the topology $\tau(d)$). Let f and g be two self-mappings on X such that*

(i) *(f, g) satisfies property (E.A),*

(ii) *for all $x, y \in X$, $x \neq y$,*

$$d(gx, gy) < \max\left\{d(fx, fy), \frac{k}{2}(d(gx, fx) + d(gy, fy)), \frac{k}{2}(d(gy, fx) + d(gx, fy))\right\}, \tag{3.18}$$

for some k, $1 \leq k < 2$. If fX is a d-closed ($\tau(d)$-closed) subset of X, then f and g have a point of coincidence.

This result can be proved in the setting of cone metric spaces putting "$\in$" instead of "max," and also for the symmetric space (X, D) associated with a complete cone metric space with a normal cone, introduced in [22].

4. Strict Contractivity and the Hardy-Rogers Theorem

It was proved in [23] (see also [24]) that a self-map f of a complete metric space X has a unique fixed point if for some nonnegative scalars a_i, $i = \overline{1,5}$ with $\sum_{i=1}^{5} a_i < 1$ and for all $x, y \in X$, the inequality

$$d(fx, fy) \leq a_1 d(x, y) + a_2 d(x, fx) + a_3 d(y, fy) + a_4 d(x, fy) + a_5 d(y, fx) \tag{4.1}$$

holds. In [4, Theorem 2.8], this result was proved in the setting of cone metric spaces, but in a generalized version—for a pair of self-mappings satisfying certain conditions.

Assuming property (E.A), we can prove the following theorem.

Theorem 4.1. *Let (X, d) be a cone metric space and let (f, g) be a weakly compatible pair of self-mappings on X satisfying condition (E.A). Suppose that there exist nonnegative scalars a_i, $i = \overline{1,5}$ such that $\sum_{i=1}^{5} a_i < 1$ and that for each $x, y \in X$,*

$$d(fx, fy) < a_1 d(gx, gy) + a_2 d(gx, fx) + a_3 d(gy, fy) + a_4 d(gx, fy) + a_5 d(gy, fx). \tag{4.2}$$

If $fX \subset gX$ and if at least one of fX and gX is a complete subspace of X, then f and g have a unique common fixed point.

Proof. There exists a sequence $x_n \in X$ such that $fx_n \to t$, $gx_n \to t$ in the cone metric d, for some $t \in X$. Let, for example, gX be complete. Then there exists $a \in X$ such that fx_n and gx_n converge to $ga = t$. Let us prove that $fa = ga$. Putting in (4.2) x_n and a instead of x and y, respectively, we obtain

$$\begin{aligned} d(fx_n, fa) &< a_1 d(gx_n, ga) + a_2 d(gx_n, fx_n) + a_3 d(ga, fa) \\ &\quad + a_4 d(gx_n, fa) + a_5 d(ga, fx_n) \\ &\le a_1 d(gx_n, ga) + a_2 d(gx_n, fx_n) + a_3 d(ga, fx_n) + a_3 d(fx_n, fa) \\ &\quad + a_4 d(gx_n, fx_n) + a_4 d(fx_n, fa) + a_5 d(ga, fx_n). \end{aligned} \tag{4.3}$$

Hence,

$$\begin{aligned} (1 - a_3 - a_4) d(fx_n, fa) &< a_1 d(gx_n, ga) + a_2 d(gx_n, fx_n) + a_3 d(ga, fx_n) \\ &\quad + a_4 d(gx_n, fx_n) + a_5 d(ga, fx_n), \end{aligned} \tag{4.4}$$

that is, denoting $k_i = a_i/(1 - a_3 - a_4)$, $i = \overline{1,5}$,

$$\begin{aligned} d(fx_n, fa) &\le k_1 d(gx_n, ga) + k_2 d(gx_n, fx_n) + k_3 d(ga, fx_n) \\ &\quad + k_4 d(gx_n, fa) + k_5 d(ga, fx_n) \\ &\ll k_1 \frac{c}{5k_1} + k_2 \frac{c}{5k_2} + k_3 \frac{c}{5k_3} + k_4 \frac{c}{5k_4} + k_5 \frac{c}{5k_5} \\ &= c. \end{aligned} \tag{4.5}$$

Thus, $d(fx_n, fa) \ll c$, that is, $fx_n \to fa$. The uniqueness of limit in cone metric spaces (when the cone has nonempty interior) implies that $fa = ga$.

Since the mappings f, g are weakly compatible, this implies that $fga = gfa = ffa = gga$. Hence, we obtain that $fa = ga$ is the unique common fixed point of the mappings f and g. Namely, suppose that $fa \ne ffa$. Putting in (4.2) a and fa instead of x, y, respectively, we obtain

$$\begin{aligned} d(fa, ffa) &< a_1 d(fa, ffa) + a_2 d(fa, fa) + a_3 d(ffa, ffa) \\ &\quad + a_4 d(fa, ffa) + a_5 d(fa, ffa) \\ &= (a_1 + a_4 + a_5) d(fa, ffa) < d(fa, ffa), \end{aligned} \tag{4.6}$$

a contradiction.

Since $fX \subset gX$, the proof is the same if we assume that fX is complete. □

The version of Hardy-Rogers' theorem for metric spaces from [24] is obtained taking $E = \mathbb{R}$, $\|\cdot\| = |\cdot|$, $P = [0, +\infty[$, $g = i_X$.

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