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Research Article

A Fixed Point Approach to the Stability of a Functional Equation of the Spiral of Theodorus

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Cădariu and Radu applied the fixed point method to the investigation of Cauchy and Jensen functional equations. In this paper, we adopt the idea of Cădariu and Radu to prove the stability of a functional equation of the spiral of Theodorus, $f(x+1) = (1 + i/\sqrt{x+1})f(x)$.

1. Introduction

In 1940, Ulam [1] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms: *let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?*

The case of approximately additive functions was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. Indeed, he proved that each solution of the inequality $\|f(x+y) - f(x) - f(y)\| \le \varepsilon$, for all x and y, can be approximated by an exact solution, say an additive function. Later, the result of Hyers was significantly generalized for additive mappings by Aoki [3] (see also [4]) and for linear mappings by Rassias [5]. It should be remarked that we can find in the books [6–8] a lot of references concerning the stability of functional equations (see also [9–11]).

Recently, Jung and Sahoo [12] proved the generalized Hyers-Ulam stability of the functional equation $f(\sqrt{r^2+1}) = f(r) + \arctan(1/r)$ which is closely related to the square root spiral, for the case that $f(1) = 0$ and $f(r)$ is monotone increasing for $r > 0$ (see also [13, 14]).

In 2003, Cădariu and Radu [15] applied the fixed point method to the investigation of Jensen's functional equation (see [16–19]). Using such a clever idea, they could present a short and simple proof for the stability of the Cauchy functional equation.

In [20], Gronau investigated the solutions of the Theodorus functional equation

$$f(x+1) = \left(1 + \frac{i}{\sqrt{x+1}}\right) f(x), \tag{1.1}$$

where $i = \sqrt{-1}$. The function $T : (-1, \infty) \to \mathbb{C}$ defined by

$$T(x) = \prod_{k=1}^{\infty} \frac{1 + i/\sqrt{k}}{1 + i/\sqrt{x+k}} \tag{1.2}$$

is called the Theodorus function.

Theorem 1.1. *The unique solution $f : (-1, \infty) \to \mathbb{C}$ of (1.1) satisfying the additional condition that*

$$\lim_{n \to \infty} \frac{f(x+n)}{f(n)} = 1 \tag{1.3}$$

for all $x \in (0,1)$ is the Theodorus function.

Theorem 1.2. *If $f : (-1, \infty) \to \mathbb{C}$ is a solution of (1.1) such that $f(0) = 1$, $|f(x)|$ is monotonic and $\arg(f(x))$ is monotonic and continuous, then f is the Theodorus function.*

Theorem 1.3. *If $f : (-1, \infty) \to \mathbb{C}$ is a solution of (1.1) such that $f(0) = 1$, $|f(x)|$ and $\arg(f(x))$ are monotonic and such that $\arg(f(n+1)) = \arg(f(n)) + \arg(1 + i/\sqrt{n+1})$ for any $n \in \{0, 1, 2, \ldots\}$, then f is the Theodorus function.*

In this paper, we will adopt the idea of Cădariu and Radu and apply a fixed point method for proving the Hyers-Ulam-Rassias stability of the Theodorus functional equation (1.1).

2. Preliminaries

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies

(M_1) $d(x, y) = 0$ if and only if $x = y$;

(M_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(M_3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

We now introduce one of fundamental results of fixed point theory. For the proof, refer to [21].

Theorem 2.1. *Let (X, d) be a generalized complete metric space. Assume that $\Lambda : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1} f, \Lambda^k f) < \infty$ for some $f \in X$, then the following are true.*

(a) *The sequence* $\{\Lambda^n f\}$ *converges to a fixed point* F *of* Λ*;*

(b) F *is the unique fixed point of* Λ *in*

$$X^* = \{g \in X \mid d(\Lambda^k f, g) < \infty\}; \tag{2.1}$$

(c) *If* $h \in X^*$*, then*

$$d(h, F) \le \frac{1}{1-L} d(\Lambda h, h). \tag{2.2}$$

3. Main results

In the following theorem, by using the idea of Cădariu and Radu (see [15, 16]), we will prove the Hyers-Ulam-Rassias stability of the functional equation (1.1) for the spiral of Theodorus.

Theorem 3.1. *Given a constant* $a > 0$*, suppose* $\varphi : [a, \infty) \to [0, \infty)$ *is a function and there exists a constant* L*,* $0 < L < 1$*, such that*

$$\varphi(x+1) + \frac{1}{\sqrt{x+1}} \varphi(x) \le L\varphi(x) \tag{3.1}$$

for all $x \ge a$*. If a function* $f : [a, \infty) \to \mathbb{C}$ *satisfies the inequality*

$$\left| f(x+1) - \left(1 + \frac{i}{\sqrt{x+1}}\right) f(x) \right| \le \varphi(x) \tag{3.2}$$

for all $x \ge a$*, then there exists a unique solution* $F : [a, \infty) \to \mathbb{C}$ *of* (1.1)*, which satisfies*

$$|F(x) - f(x)| \le \frac{1}{1-L} \varphi(x) \tag{3.3}$$

for all $x \ge a$*. More precisely,* F *is defined by*

$$F(x) = \lim_{n\to\infty} \left[\sum_{k=1}^{n} (-i)^k \sum_{1 \le j_1 \le \cdots \le j_k \le n+1-k} \left(\prod_{m=1}^{k} \frac{1}{\sqrt{x + j_m}} \right) f(x+n-k) + f(x+n) \right] \tag{3.4}$$

for all $x \ge a$*.*

Proof. We set $X = \{h \mid h : [a, \infty) \to \mathbb{C} \text{ is a function}\}$ and introduce a generalized metric on X as follows:

$$d(g, h) = \inf\{C \in [0, \infty] \mid |g(x) - h(x)| \le C\varphi(x), \forall x \ge a\}. \tag{3.5}$$

First, we will verify that (X, d) is a complete space. Let $\{g_n\}$ be a Cauchy sequence in (X, d). According to the definition of Cauchy sequences, there exists, for any given $\varepsilon > 0$, a positive integer N_ε such that $d(g_m, g_n) \le \varepsilon$ for all $m, n \ge N_\varepsilon$. From the definition of the generalized metric d, it follows that

$$\forall \varepsilon > 0 \, \exists N_\varepsilon \in \mathbb{N} \, \forall m, n \ge N_\varepsilon \quad \forall x \ge a : |g_m(x) - g_n(x)| \le \varepsilon\varphi(x). \tag{3.6}$$

If $x \geq a$ is fixed, (3.6) implies that $\{g_n(x)\}$ is a Cauchy sequence in $(\mathbb{C}, |\cdot|)$. Since $(\mathbb{C}, |\cdot|)$ is complete, $\{g_n(x)\}$ converges in $(\mathbb{C}, |\cdot|)$ for each $x \geq a$. Hence we can define a function $g : [a, \infty) \to \mathbb{C}$ by

$$g(x) = \lim_{n \to \infty} g_n(x). \tag{3.7}$$

If we let m increase to infinity, it follows from (3.6) that for any $\varepsilon > 0$, there exists a positive integer N_ε with $|g_n(x) - g(x)| \leq \varepsilon\varphi(x)$ for all $n \geq N_\varepsilon$ and all $x \geq a$, that is, for any $\varepsilon > 0$, there exists a positive integer N_ε such that $d(g_n, g) \leq \varepsilon$ for any $n \geq N_\varepsilon$. This fact leads us to the conclusion that $\{g_n\}$ converges in (X, d). Hence (X, d) is a complete space (cf. the proof of [22, Theorem 3.1] or [16, Theorem 2.5]).

We now define an operator $\Lambda : X \to X$ by

$$(\Lambda h)(x) = h(x+1) - \frac{i}{\sqrt{x+1}} h(x) \quad (x \geq a) \tag{3.8}$$

for any $h \in X$. We assert that Λ is strictly contractive on X. Given $g, h \in X$, let $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq C$, that is,

$$|g(x) - h(x)| \leq C\varphi(x) \tag{3.9}$$

for all $x \geq a$. It then follows from (3.1) and (3.8) that

$$\begin{aligned} \left|(\Lambda g)(x) - (\Lambda h)(x)\right| &\leq \left|g(x+1) - h(x+1)\right| + \frac{1}{\sqrt{x+1}} \left|g(x) - h(x)\right| \\ &\leq C\varphi(x+1) + \frac{C}{\sqrt{x+1}} \varphi(x) \\ &\leq LC\varphi(x) \end{aligned} \tag{3.10}$$

for every $x \geq a$, that is, $d(\Lambda g, \Lambda h) \leq LC$. Hence we conclude that $d(\Lambda g, \Lambda h) \leq L d(g, h)$, for any $g, h \in X$.

Next, we assert that $d(\Lambda f, f) < \infty$. In view of (3.2) and the definition of Λ, we get

$$\left|(\Lambda f)(x) - f(x)\right| \leq \varphi(x) \tag{3.11}$$

for each $x \geq a$, that is,

$$d(\Lambda f, f) \leq 1. \tag{3.12}$$

By using mathematical induction, we now prove that

$$(\Lambda^n f)(x) = \sum_{k=1}^{n} (-i)^k \sum_{1 \leq j_1 \leq \cdots \leq j_k \leq n+1-k} \left(\prod_{m=1}^{k} \frac{1}{\sqrt{x + j_m}} \right) f(x + n - k) + f(x + n) \tag{3.13}$$

for all $n \in \mathbb{N}$ and all $x \geq a$. Since $f \in X$, the definition (3.8) implies that (3.13) is true for $n = 1$. Now, assume that (3.13) holds true for some $n \geq 1$. It then follows from (3.8) and (3.13) that

$$
\begin{aligned}
(\Lambda^{n+1} f)(x) &= (\Lambda^n f)(x+1) - \frac{i}{\sqrt{x+1}}(\Lambda^n f)(x) \\
&= (\Lambda^n f)(x+1) + \sum_{k=1}^{n}(-i)^{k+1} \sum_{1=j_1\leq\cdots\leq j_{k+1}\leq n+1-k} \\
&\quad \times \left(\prod_{m=1}^{k+1} \frac{1}{\sqrt{x+j_m}}\right) f(x+n-k) - \frac{i}{\sqrt{x+1}} f(x+n) \\
&= (\Lambda^n f)(x+1) + \sum_{k=1}^{n-1}(-i)^{k+1} \sum_{1=j_1\leq\cdots\leq j_{k+1}\leq n+1-k} \\
&\quad \times \left(\prod_{m=1}^{k+1} \frac{1}{\sqrt{x+j_m}}\right) f(x+n-k) + (-i)^{n+1}\left(\frac{1}{\sqrt{x+1}}\right)^{n+1} f(x) - \frac{i}{\sqrt{x+1}} f(x+n) \\
&= \sum_{k=1}^{n}(-i)^{k} \sum_{1\leq j_1\leq\cdots\leq j_{k}\leq n+1-k} \left(\prod_{m=1}^{k} \frac{1}{\sqrt{x+(1+j_m)}}\right) f(x+1+n-k) \\
&\quad + f(x+1+n) + \sum_{k=2}^{n}(-i)^{k} \sum_{1=j_1\leq\cdots\leq j_{k}\leq n+2-k} \left(\prod_{m=1}^{k} \frac{1}{\sqrt{x+j_m}}\right) f(x+n+1-k) \\
&\quad - \frac{i}{\sqrt{x+1}} f(x+n) + (-i)^{n+1}\left(\frac{1}{\sqrt{x+1}}\right)^{n+1} f(x) \\
&= \sum_{k=1}^{n}(-i)^{k} \sum_{2\leq j_1\leq\cdots\leq j_{k}\leq n+2-k} \left(\prod_{m=1}^{k} \frac{1}{\sqrt{x+j_m}}\right) f(x+n+1-k) \\
&\quad + f(x+n+1) + \sum_{k=1}^{n}(-i)^{k} \sum_{1=j_1\leq\cdots\leq j_{k}\leq n+2-k} \left(\prod_{m=1}^{k} \frac{1}{\sqrt{x+j_m}}\right) f(x+n+1-k) \\
&\quad + (-i)^{n+1} \sum_{1\leq j_1\leq\cdots\leq j_{n+1}\leq 1} \left(\prod_{m=1}^{n+1} \frac{1}{\sqrt{x+j_m}}\right) f(x) \\
&= \sum_{k=1}^{n+1}(-i)^{k} \sum_{1\leq j_1\leq\cdots\leq j_{k}\leq n+2-k} \left(\prod_{m=1}^{k} \frac{1}{\sqrt{x+j_m}}\right) f(x+n+1-k) + f(x+n+1),
\end{aligned}
\tag{3.14}
$$

which is the case when n is replaced by $n+1$ in (3.13).

Considering (3.12), if we set $k = 0$ in Theorem 2.1, then Theorem 2.1(a) implies that there exists a function $F \in X$, which is a fixed point of Λ, such that $d(\Lambda^n f, F) \to 0$ as $n \to \infty$. Hence, we can choose a sequence $\{C_n\}$ of positive numbers with $C_n \to 0$ as $n \to \infty$ such that $d(\Lambda^n f, F) \leq C_n$ for each $n \in \mathbb{N}$. In view of definition of d, we have

$$\left|(\Lambda^n f)(x) - F(x)\right| \leq C_n \varphi(x) \quad (x \geq a) \tag{3.15}$$

for all $n \in \mathbb{N}$. This implies the pointwise convergence of $\{(\Lambda^n f)(x)\}$ to $F(x)$ for every fixed $x \geq a$. Therefore, using (3.4), we can conclude that (3.4) is true.

Moreover, because F is a fixed point of Λ, definition (3.8) implies that F is a solution to (1.1).

Since $k = 0$ (see (3.12)) and $f \in X^* = \{g \in X \mid d(f,g) < \infty\}$ in Theorem 2.1, by Theorem 2.1(c) and (3.12), we obtain

$$d(f,F) \leq \frac{1}{1-L} d(\Lambda f, f) \leq \frac{1}{1-L}, \tag{3.16}$$

that is, the inequality (3.3) is true for all $x \geq a$.

Assume that inequality (3.3) is also satisfied with another function $G : [a,\infty) \to \mathbb{C}$ which is a solution of (1.1). (As G is a solution of (1.1), G satisfies that $G(x) = G(x+1) - (i/\sqrt{x+1})G(x) = (\Lambda G)(x)$ for all $x \geq a$. In other words, G is a fixed point of Λ.) In view of (3.3) with G and the definition of d, we know that

$$d(f,G) \leq \frac{1}{1-L} < \infty, \tag{3.17}$$

that is, $G \in X^* = \{g \in X \mid d(f,g) < \infty\}$. Thus, Theorem 2.1(b) implies that $F = G$. This proves the uniqueness of F. □

Indeed, Cădariu and Radu proved a general theorem concerning the Hyers-Ulam-Rassias stability of a generalized equation for the square root spiral

$$f\left(p^{-1}(p(x)+k)\right) = f(x) + h(x) \tag{3.18}$$

(see [23, Theorem 3.1]).

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