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Research Article

Viscosity Approximation Methods for Generalized Mixed Equilibrium Problems and Fixed Points of a Sequence of Nonexpansive Mappings

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We introduce an iterative scheme by the viscosity approximation method for finding a common element of the set of common solutions for generalized mixed equilibrium problems and the set of common fixed points of a sequence of nonexpansive mappings in Hilbert spaces. We show a strong convergence theorem under some suitable conditions.

1. Introduction

Equilibrium problems theory provides us with a unified, natural, innovative, and general framework to study a wide class of problems arising in finance, economics, network analysis, transportation, elasticity, and optimization, which has been extended and generalized in many directions using novel and innovative techniques; see [1–8]. Inspired and motivated by the research and activities going in this fascinating area, we introduce and consider a new class of equilibrium problems, which is known as the generalized mixed equilibrium problems.

Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to 2^H$ a multivalued mapping. Let $\varphi : C \times C \to R$ be a real-valued function and $\Phi : H \times C \times C \to R$ an equilibrium-like function, that is,

$$\Phi(w,u,v) + \Phi(w,v,u) = 0, \quad \forall (w,u,v) \in H \times C \times C. \tag{1.1}$$

We consider the problem of finding $u \in C$ and $w \in T(u)$ such that

$$\Phi(w,u,v) + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in C, \tag{1.2}$$

which is called the generalized mixed equilibrium problem (for short, GMEP). If T is a single-valued mapping, then problem (1.2) is equivalent to finding $u \in C$ such that

$$\Phi(T(u), u, v) + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in C. \tag{1.3}$$

We denote Ω for the set of solutions of GMEP (1.2). This class is a quite general and unifying one and includes several classes of equilibrium problems and variational inequalities as special cases. In recent years, several numerical techniques including projection, resolvent, and auxiliary principle have been developed and analyzed for solving variational inequalities. It is well known that projection- and resolvent-type methods cannot be extended for equilibrium problems. To overcome this drawback, one usually uses the auxiliary principle technique. Glowinski et al. [9] have used this technique to study the existence of a solution of mixed variational inequalities. The viscosity approximation method is one of the important methods for approximation fixed points of nonexpansive type mappings. It was first discussed by Moudafi [10]. Recently, Hirstoaga [11] and S. Takahashi and W. Takahashi [12] applied viscosity approximation technique for finding a common element of set of solutions of an equilibrium problem (EP) and set of fixed points of a nonexpansive mapping. Very recently, Yao et al. [13] introduced and studied an iteration process for finding a common element of the set of solutions of the EP and the set of common fixed points of infinitely many nonexpansive mappings in H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. For any $n \geq 1$, define a mapping S_n of C into itself as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n) I, \\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1}) I, \\
&\vdots \\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k) I, \\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1}) I, \\
&\vdots \\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2) I, \\
S_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1-\lambda_1) I.
\end{aligned} \tag{1.4}$$

Such a mapping S_n is called the S-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$, see [14].

The purpose of this paper is to develop an iterative algorithm for finding a common element of set of solutions of GMEP (1.2) and set of common fixed points of a sequence of nonexpansive mappings in Hilbert spaces. The result presented in this paper improves and extends the main result of S. Takahashi and W. Takahashi [12].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let C be a closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\left\| x - P_C(x) \right\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.1}$$

P_C is called metric projection of H onto C. It is well known that P_C is nonexpansive. Furthermore, for $x \in H$ and $u \in C$,

$$u = P_C(x) \Longleftrightarrow \langle x - u, u - y \rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

We denote by $F(T)$ the set of fixed points of a self-mapping T on C, that is, $F(T) = \{x \in C : Tx = x\}$. It is well known that if $C \subset H$ is nonempty, bounded, closed, and convex and T is nonexpansive, then $F(T)$ is nonempty; see [15]. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself, where C is a nonempty closed convex subset of a real Hilbert space H. Given a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $[0,1]$, we define a sequence $\{S_n\}_{n=1}^{\infty}$ of self-mappings on C by (1.4). Then we have the following lemmas which are important to prove our results.

Lemma 2.1 (see [14]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $(0, b]$ for some $b \in (0,1)$. Then, for every $x \in C$ and $k \in N$ the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.1, one can define mapping S of C into itself as follows:

$$Sx = \lim_{n\to\infty} S_n x = \lim_{n\to\infty} U_{n,1}x, \tag{2.3}$$

for every $x \in C$. Such a mapping S is called the S-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Throughout this paper, we will assume that $0 < \lambda_n \leq b < 1$ for every $n \geq 1$. Since S_n is nonexpansive, $S : C \to C$ is also nonexpansive.

Lemma 2.2 (see [14]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $(0, b]$ for some $b \in (0,1)$. Then, $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$.*

Let C be a convex subset of a real Hilbert space H and $\kappa : C \to R$ a Fréchet differential function. Then κ is said to be η-convex strongly convex if there exists a constant $\mu > 0$ such that

$$\kappa(y) - \kappa(x) - \langle \kappa'(x), \eta(y,x) \rangle \geq \frac{\mu}{2}\|x - y\|^2, \quad \forall x, y \in C. \tag{2.4}$$

If $\mu = 0$, then κ is said to be η-convex. In particular, if $\eta(y,x) = y - x$ for all $y, x \in C$, then κ is said to be strongly convex.

Let C be a nonempty subset of a real Hilbert space H. A bifunction $\varphi(\cdot,\cdot) : C \times C \to R$ is said to be skew-symmetric if

$$\varphi(u,v) + \varphi(v,u) - \varphi(u,u) - \varphi(v,v) \leq 0, \quad \forall u,v \in C. \tag{2.5}$$

If the skew-symmetric bifunction $\varphi(\cdot,\cdot)$ is linear in both arguments, then

$$\varphi(u,u) \geq 0, \quad \forall u \in C. \tag{2.6}$$

We denote $\rightharpoonup$ for weak convergence and $\to$ for strong convergence. A function $\psi : C \times C \to R$ is called weakly sequentially continuous at $(x_0, y_0) \in C \times C$, if $\psi(x_n, y_n) \to \psi(x_0, y_0)$ as $n \to \infty$ for each sequence $\{(x_n, y_n)\}$ in $C \times C$ converging weakly to (x_0, y_0). The function $\psi(\cdot,\cdot)$ is called weakly sequentially continuous on $C \times C$ if it is weakly sequentially continuous at each point of $C \times C$.

Let $CB(X)$ denote the set of nonempty closed bounded subsets of X. For $A, B \in CB(X)$, define the Hausdorff metric $\mathscr{H}$ as follows:

$$\mathscr{H}(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(b,a)\right\}. \tag{2.7}$$

Lemma 2.3 (see [16]). *Let $A, B \in CB(X)$ and $a \in A$. Then for $\varrho > 1$, there must exist a point $b \in B$ such that $d(a,b) \leq \varrho\mathscr{H}(A,B)$.*

Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to 2^H$ a multivalued mapping. For $x \in C$, let $w \in T(x)$. Let $\varphi : C \times C \to R$ be a real-valued function satisfying the following:

(φ1) $\varphi(\cdot,\cdot)$ is skew symmetric;

(φ2) for each fixed $y \in C$, $\varphi(\cdot, y)$ is convex and upper semicontinuous;

(φ3) $\varphi(\cdot,\cdot)$ is weakly continuous on $C \times C$.

Let $\kappa : C \to R$ be a differentiable functional with Fréchet derivative $\kappa'(x)$ at x satisfying the following:

(κ1) κ' is sequentially continuous from the weak topology to the strong topology;

(κ2) κ' is Lipschitz continuous with Lipschitz constant $\nu > 0$.

Let $\eta : C \times C \to H$ be a function satisfying the following:

(η1) $\eta(x,y) + \eta(y,x) = 0$ for all $x, y \in C$;

(η2) $\eta(\cdot,\cdot)$ is affine in the first coordinate variable;

(η3) for each fixed $y \in C$, $x \mapsto \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology.

Let us consider the equilibrium-like function $\Phi : H \times C \times C \to R$ which satisfies the following conditions with respect to the multivalued mapping $T : C \to 2^H$:

(Φ1) for each fixed $v \in C$, $(w,u) \mapsto \Phi(w,u,v)$ is an upper semicontinuous function from $H \times C$ to R, that is, $w_n \to w$ and $u_n \to u$ imply $\limsup_{n\to\infty}\Phi(w_n,u_n,v) \leq \Phi(w,u,v)$;

(Φ2) for each fixed $(w,v) \in H \times C$, $u \mapsto \Phi(w,u,v)$ is a concave function;

(Φ3) for each fixed $(w,u) \in H \times C$, $v \mapsto \Phi(w,u,v)$ is a convex function.

Let r be a positive parameter. For a given element $x \in C$ and $w_x \in T(x)$, consider the following auxiliary problem for GMEP(1.2): find $u \in C$ such that

$$\Phi(w_x,u,v) + \varphi(v,u) - \varphi(u,u) + \frac{1}{r}\langle \kappa'(u) - \kappa'(x), \eta(v,u)\rangle \geq 0, \quad \forall v \in C. \tag{2.8}$$

It is easy to see that if $u = x$, then u is a solution of GMEP(1.2).

Lemma 2.4 (see [6]). *Let C be a nonempty closed convex bounded subset of a real Hilbert space H and $\varphi : C \times C \to R$ a real-valued function satisfying the conditions (φ1)–(φ3). Let $T : C \to 2^H$ be a multivalued mapping and $\Phi : H \times C \times C \to R$ the equilibrium-like function satisfying the conditions (Φ1)–(Φ3). Assume that $\eta : C \times C \to H$ is a Lipschitz function with Lipschitz constant $\lambda > 0$ which satisfies the conditions (η1)–(η3). Let $\kappa : C \to R$ be an η-strongly convex function with constant $\mu > 0$ which satisfies the conditions (κ1) and (κ2). For each $x \in C$, let $w_x \in T(x)$. For $r > 0$, define a mapping $T_r : C \to C$ by*

$$T_r(x) = \left\{u \in C : \Phi(w_x,u,v) + \varphi(v,u) - \varphi(u,u) + \frac{1}{r}\langle \kappa'(u) - \kappa'(x), \eta(v,u)\rangle \geq 0,\ \forall v \in C\right\}. \tag{2.9}$$

Then one has the following:

(a) *the auxiliary problem* (2.8) *has a unique solution;*

(b) *T_r is single valued;*

(c) *if $\lambda v/\mu$ and $\Phi(w_1,T_r(x_1),T_r(x_2)) + \Phi(w_2,T_r(x_2),T_r(x_1)) \leq 0$ for all $x_1,x_2 \in C$ and all $w_1 \in T(x_1)$, $w_2 \in T(x_2)$, it follows that T_r is nonexpansive;*

(d) *$F(T_r) = \Omega$;*

(e) *Ω is closed and convex.*

We also need the following lemmas for our main results.

Lemma 2.5 (see [17]). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three sequences of nonnegative numbers such that*

$$a_{n+1} \leq b_n a_n + c_n, \quad \forall n = 1,2,\ldots. \tag{2.10}$$

If $b_n \geq 1$, $\sum_{n=1}^{\infty}(b_n - 1) < \infty$, and $\sum_{n=1}^{\infty}c_n < \infty$, then $\lim_{n\to\infty}a_n$ exists.

Lemma 2.6. *Let $\{a_n\}$ and $\{c_n\}$ be sequences of nonnegative numbers such that*

$$a_{n+1} \leq \Theta a_n + c_n, \quad \forall n = 1,2,\ldots. \tag{2.11}$$

If $\Theta \in (0,1)$ and $\sum_{n=1}^{\infty}c_n < \infty$, then $\lim_{n\to\infty}a_n = 0$.

Proof. It is easy to see that inequality (2.11) is equivalent to

$$a_{n+1} \leq \Theta b_n a_n + c_n, \quad \forall n = 1, 2, \ldots, \tag{2.12}$$

where $\Theta \in (0,1)$, $b_n = 1$ and $\sum_{n=1}^{\infty} c_n < \infty$. It follows that

$$a_{n+1} \leq b_n a_n + c_n \quad \forall n = 1, 2, \ldots. \tag{2.13}$$

Note that Lemma 2.5 implies that $\lim_{n\to\infty} a_n$ exists. Suppose $\lim_{n\to\infty} a_n = d$ for some $d > 0$. It is obvious that $\lim_{n\to\infty} c_n = 0$ and so inequality (2.12) implies that $d \leq \Theta d$, which is a contradiction. Thus, $\lim_{n\to\infty} a_n = d = 0$. This completes the proof. □

Lemma 2.7 (see [6]). *Let $\{x_n\}$ be a sequence in a normed space $(X, \|\cdot\|)$ such that*

$$\|x_{n+1} - x_{n+2}\| \leq \Theta \|x_n - x_{n+1}\| b_n + c_n, \quad \forall n = 1, 2, \ldots, \tag{2.14}$$

where $\Theta \in (0,1)$, and $\{b_n\}$ and $\{c_n\}$ are sequences satisfy the following conditions:

(i) $b_n \geq 1$ *for all* $n = 1, 2, \ldots$ *and* $\sum_{n=1}^{\infty}(b_n - 1) < \infty$;

(ii) $c_n \geq 0$ *for all* $n = 1, 2, \ldots$ *and* $\sum_{n=1}^{\infty} c_n < \infty$.

Then $\{x_n\}$ is a Cauchy sequence.

Lemma 2.8 (see [18]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \lambda_n) a_n + \lambda_n \sigma_n + \xi_n, \quad \forall n = 1, 2, \ldots, \tag{2.15}$$

where $\{\lambda_n\}$, $\{\sigma_n\}$ and $\{\xi_n\}$ are sequences of real numbers satisfying the following conditions:

(i) $\{\lambda_n\} \subset [0,1]$, $\lim_{n\to\infty} \lambda_n = 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$;

(iii) $\xi_n \geq 0$ *for all* $n = 1, 2, \ldots$ *and* $\sum_{n=1}^{\infty} \xi_n < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

3. Iterative algorithm and convergence theorem

Let C be a nonempty closed convex subset of a real Hilbert space H, $T : C \to CB(H)$ a multivalued mapping, $f : C \to C$ a contraction mapping with constant $\alpha \in [0,1)$, and $S_n : C \to C$ an S-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$, where sequence $\{T_n\}$ is nonexpansive. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ and $\{r_n\}$ a sequence in $(0,\infty)$. We can develop Algorithm 3.1 for finding a common element of a set of fixed points of S-mapping S_n and a set of solutions of GMEP(1.2).

Algorithm 3.1. For given $x_1 \in C$ and $w_1 \in T(x_1)$, there exist sequences $\{x_n\}$, $\{u_n\}$ in C and $\{w_n : w_n \in T(x_n)\}$ in H such that for all $n = 1, 2, \ldots,$

$$\begin{gathered} \|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n}\right)\mathscr{H}(T(x_n), T(x_{n+1})); \\ \Phi(w_n, u_n, v) + \varphi(v, u_n) - \varphi(u_n, u_n) + \frac{1}{r_n}\langle \kappa'(u_n) - \kappa'(x_n), \eta(v, u_n)\rangle \geq 0, \quad \forall v \in C; \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S_n(u_n). \end{gathered} \tag{3.1}$$

We now prove the strong convergence of iterative sequence $\{x_n\}$, $\{u_n\}$, and $\{w_n\}$ generated by Algorithm 3.1.

Theorem 3.2. *Let C be a nonempty closed convex bounded subset of a real Hilbert space H, $T : C \to CB(H)$ a multivalued $\mathscr{H}$-Lipschitz continuous mapping with constant $L > 0$, $f : C \to C$ a contraction mapping with constant $\alpha \in [0, 1)$. Let $\varphi : C \times C \to R$ be a real-valued function satisfying the conditions $(\varphi 1)$–$(\varphi 3)$ and let $\Phi : H \times C \times C \to R$ be an equilibrium-like function satisfying conditions $(\Phi 1)$–$(\Phi 3)$ and $(\Phi 4)$:*

- $(\Phi 4)$ *$\Phi(w, T_r(x), T_s(y)) + \Phi(\widetilde{w}, T_s(y), T_r(x)) \leq -\gamma\|T_r(x) - T_s(y)\|^2$ for all $x, y \in C$ and $r, s \in (0, \infty)$, where $\gamma > 0$, $w \in T(x)$ and $\widetilde{w} \in T(y)$.*

Assume that $\eta : C \times C \to H$ is a Lipschitz function with Lipschitz constant $\lambda > 0$ which satisfies the conditions $(\eta 1)$~$(\eta 3)$. Let $\kappa : C \to R$ be an η-strongly convex function with constant $\mu > 0$ which satisfies conditions $(\kappa 1)$ and $(\kappa 2)$ with $\lambda\nu/\mu \leq 1$. Let $S_n : C \to C$ be an S-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$ and $\bigcap_{n=1}^{\infty} F(T_n) \cap \Omega \neq \varnothing$, where sequence $\{T_n\}$ is nonexpansive. Let $\{x_n\}$, $\{u_n\}$ and $\{w_n\}$ be sequences generated by Algorithm 3.1, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{r_n\}$ in $(0, \infty)$ satisfying the following conditions:

- (C1) *$\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n+1}| < \infty$;*
- (C2) *$\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty}|r_n - r_{n+1}| < \infty$;*
- (C3) *$\sum_{n=1}^{\infty}(1 - \alpha_n)\varepsilon_n < \infty$ where $\varepsilon_n = \sup_{x\in C}\|S_n(x) - S_{n+1}(x)\|$.*

Then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega$, and $\{w_n\}$ converges strongly to $w^* \in T(x^*)$, where $x^* = P_{\cap_{n=1}^{\infty} F(T_n)\cap\Omega} f(x^*)$.

Proof. It is easy to see from $(\Phi 4)$ that

$$\Phi(w, T_r(x), T_s(y)) + \Phi(\widetilde{w}, T_s(y), T_r(x)) \leq 0 \tag{3.2}$$

for all $x, y \in C$ and $r, s \in (0, \infty)$, where $\gamma > 0$, $w \in T(x)$, and $\widetilde{w} \in T(y)$. All the conclusions (a)–(e) of Lemma 2.4 hold.

Let $Q = P_{\cap_{n=1}^{\infty} F(T_n)\cap\Omega}$. Then Qf is a contraction of C into itself. In fact,

$$\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| < \alpha\|x - y\|, \quad \forall x, y \in C. \tag{3.3}$$

Hence there exists a unique element $q \in C$ such that $q = Qf(q)$. Noting that $f(q) \in C$ and $Qf(q) \in \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega$, we get that $q \in \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega$.

Now, we prove that $\|x_n - x_{n+1}\| \to 0$ and $\|u_n - u_{n+1}\| \to 0$ as $n \to \infty$. Observe that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + (1-\alpha_n)S_n(u_n) - \alpha_{n-1}f(x_{n-1}) - (1-\alpha_{n-1})S_{n-1}(u_{n-1})\| \\
&= \|\alpha_n f(x_n) - \alpha_n f(x_{n-1}) + \alpha_n f(x_{n-1}) - \alpha_{n-1}f(x_{n-1}) \\
&\quad + (1-\alpha_n)S_n(u_n) - (1-\alpha_n)S_{n-1}(u_{n-1}) \\
&\quad + (1-\alpha_n)S_{n-1}(u_{n-1}) - (1-\alpha_{n-1})S_{n-1}(u_{n-1})\| \\
&\le \alpha_n \|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|S_{n-1}(u_{n-1})\|) \\
&\quad + (1-\alpha_n)\|S_n(u_n) - S_{n-1}(u_{n-1})\| \\
&\le \alpha\alpha_n\|x_n - x_{n-1}\| + 2|\alpha_n - \alpha_{n-1}|\operatorname{diam}(C) \\
&\quad + (1-\alpha_n)(\|S_n(u_n) - S_{n-1}(u_n)\| + \|S_{n-1}(u_n) - S_{n-1}(u_{n-1})\|) \\
&\le \alpha\alpha_n\|x_n - x_{n-1}\| + 2|\alpha_n - \alpha_{n-1}|\operatorname{diam}(C) + (1-\alpha_n)(\|u_n - u_{n-1}\| + \varepsilon_{n-1}).
\end{aligned} \tag{3.4}$$

Noting that $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$, it follows from (3.1) that

$$\Phi(w_n, u_n, v) + \varphi(v, u_n) - \varphi(u_n, u_n) + \frac{1}{r_n}\langle \kappa'(u_n) - \kappa'(x_n), \eta(v, u_n)\rangle \ge 0, \tag{3.5}$$

$$\Phi(w_{n+1}, u_{n+1}, v) + \varphi(v, u_{n+1}) - \varphi(u_{n+1}, u_{n+1}) + \frac{1}{r_{n+1}}\langle \kappa'(u_{n+1}) - \kappa'(x_{n+1}), \eta(v, u_{n+1})\rangle \ge 0 \quad \forall v \in C. \tag{3.6}$$

Putting $v = u_{n+1}$ in (3.5) and $v = u_n$ in (3.6), respectively, we have

$$\begin{gathered}
\Phi(w_n, u_n, u_{n+1}) + \varphi(u_{n+1}, u_n) - \varphi(u_n, u_n) + \frac{1}{r_n}\langle \kappa'(u_n) - \kappa'(x_n), \eta(u_{n+1}, u_n)\rangle \ge 0, \\
\Phi(w_{n+1}, u_{n+1}, u_n) + \varphi(u_n, u_{n+1}) - \varphi(u_{n+1}, u_{n+1}) + \frac{1}{r_{n+1}}\langle \kappa'(u_{n+1}) - \kappa'(x_{n+1}), \eta(u_n, u_{n+1})\rangle \ge 0.
\end{gathered} \tag{3.7}$$

Adding up those inequalities, we obtain from (2.5), $(\eta 1)$, and $(\Phi 4)$ that

$$-\frac{1}{r_n}\langle \kappa'(u_n) - \kappa'(x_n), \eta(u_n, u_{n+1})\rangle + \frac{1}{r_{n+1}}\langle \kappa'(u_{n+1}) - \kappa'(x_{n+1}), \eta(u_n, u_{n+1})\rangle \ge \gamma\|u_n - u_{n+1}\|^2. \tag{3.8}$$

It follows that

$$
\begin{aligned}
&\gamma r_n \|u_n - u_{n+1}\|^2 \\
&\leq \left\langle \kappa'(u_n) - \kappa'(x_n) - \frac{r_n}{r_{n+1}}\left(\kappa'(u_{n+1}) - \kappa'(x_{n+1})\right), \eta(u_{n+1}, u_n) \right\rangle \\
&\leq \left\langle \kappa'(u_n) - \kappa'(u_{n+1}), \eta(u_{n+1}, u_n) \right\rangle \\
&\quad + \left\langle \kappa'(u_{n+1}) - \kappa'(x_{n+1}) + \kappa'(x_{n+1}) - \kappa'(x_n) - \frac{r_n}{r_{n+1}}\left(\kappa'(u_{n+1}) - \kappa'(x_{n+1})\right), \eta(u_{n+1}, u_n) \right\rangle \\
&\leq -\mu \|u_n - u_{n+1}\|^2 + \left(\|\kappa'(x_{n+1}) - \kappa'(x_n)\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|\kappa'(u_{n+1}) - \kappa'(x_{n+1})\| \right) \|\eta(u_{n+1}, u_n)\| \\
&\leq -\mu \|u_n - u_{n+1}\|^2 + \lambda\nu \left(\|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_{n+1}} \|u_{n+1} - x_{n+1}\| \right) \|u_{n+1} - u_n\|,
\end{aligned}
\tag{3.9}
$$

since η and κ' are Lipschitz continuous wiht Lipschitz constants λ and ν, respectively. Noting that $\liminf_{n\to\infty} r_n > 0$, without loss of generality, we assume that there exists a real number $\bar{r} > 0$ such that $r_n \geq \bar{r} > 0$ for all $n = 1, 2, \ldots$. Thus,

$$
\gamma \bar{r} \|u_{n+1} - u_n\| \leq -\mu \|u_{n+1} - u_n\| + \lambda\nu \left(\|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{\bar{r}} \|u_{n+1} - x_{n+1}\| \right), \tag{3.10}
$$

which implies that

$$
\left(1 + \frac{\gamma \bar{r}}{\mu}\right) \|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{\bar{r}} \mathrm{diam}(C), \tag{3.11}
$$

and hence

$$
\|u_{n+1} - u_n\| \leq \delta \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{\bar{r}} \delta \, \mathrm{diam}(C), \tag{3.12}
$$

where $\delta = 1/(1 + \gamma\bar{r}/\mu) \in (0, 1)$. Set $\Theta := \max\{\alpha, \delta\} \in (0, 1)$. Combining (3.4) and (3.12) yields

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\leq (\alpha\alpha_n + (1 - \alpha_n)\delta) \|x_n - x_{n-1}\| + (1 - \alpha_n)\varepsilon_{n-1} \\
&\quad + \left(2|\alpha_n - \alpha_{n-1}| + (1 - \alpha_n)\delta \frac{|r_n - r_{n-1}|}{\bar{r}} \right) \mathrm{diam}(C) \\
&\leq \Theta \|x_n - x_{n-1}\| + (1 - \alpha_n)\varepsilon_{n-1} + \left(2|\alpha_n - \alpha_{n-1}| + \frac{|r_n - r_{n-1}|}{\bar{r}} \right) \mathrm{diam}(C).
\end{aligned}
\tag{3.13}
$$

From conditions $(C1)$ and $(C3)$,

$$\begin{aligned}\sum_{n=1}^{\infty}(1-\alpha_{n+1})\varepsilon_n &= \sum_{n=1}^{\infty}((1-\alpha_n)\varepsilon_n + (\alpha_n - \alpha_{n+1})\varepsilon_n) \\ &\le \sum_{n=1}^{\infty}\left((1-\alpha_n)\varepsilon_n + |\alpha_n - \alpha_{n+1}|\sup_{n\in N}\varepsilon_n\right) < \infty.\end{aligned} \tag{3.14}$$

Set $a_n := \|x_n - x_{n-1}\|$ and

$$c_n := (1-\alpha_n)\varepsilon_{n-1} + \left(2|\alpha_n - \alpha_{n-1}| + \frac{|r_n - r_{n-1}|}{\overline{r}}\right)\operatorname{diam}(C). \tag{3.15}$$

Then Lemmas 2.6 and 2.7 imply that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\{x_n\}$ is a Cauchy sequence in C. Hence from (3.12), we get

$$\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0. \tag{3.16}$$

We know from $(C3)$ that $\lim_{n\to\infty}\varepsilon_n = 0$. It follows that

$$\begin{aligned}\|x_{n+1} - S_{n+1}(u_{n+1})\| &\le \|x_{n+1} - S_n(u_n)\| + \|S_n(u_n) - S_n(u_{n+1})\| + \|S_n(u_{n+1}) - S_{n+1}(u_{n+1})\| \\ &\le \alpha_n\|f(x_n) - S_n(u_n)\| + \|u_{n+1} - u_n\| + \varepsilon_n \\ &\le \alpha_n\operatorname{diam}(C) + \|u_{n+1} - u_n\| + \varepsilon_n.\end{aligned} \tag{3.17}$$

Thus, $\lim_{n\to\infty}\|x_n - S_n(u_n)\| = 0$.

Next, we prove that there exists $x^* \in C$, such that $x_n \to x^*$, $u_n \to x^*$, and $w_n \to \widehat{w}$ as $n \to \infty$, where $\widehat{w} \in T(x^*)$.

Let $p \in \bigcap_{n=1}^{\infty}F(T_n)\cap\Omega$. Then

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n}(x_n) - T_{r_n}(p)\|^2 \\ &\le \langle T_{r_n}(x_n) - T_{r_n}(p), x_n - p\rangle \\ &\le \langle u_n - p, x_n - p\rangle \\ &\le \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2),\end{aligned} \tag{3.18}$$

and so

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|u_n - x_n\|^2 \le \|x_n - p\|^2. \tag{3.19}$$

By the convexity of $\|\cdot\|$, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|S_n(u_n)-p\|^2\\ &\leq \alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|u_n-p\|^2\\ &\leq \alpha_n \operatorname{diam}(C)^2+\|x_n-p\|^2-\|u_n-x_n\|^2.\end{aligned} \tag{3.20}$$

It follows that

$$\begin{aligned}\|u_n-x_n\|^2 &\leq \alpha_n \operatorname{diam}(C)^2+\left(\|x_n-p\|^2-\|x_{n+1}-p\|^2\right)\\ &\leq \alpha_n \operatorname{diam}(C)^2+\left(\|x_n-p\|+\|x_{n+1}-p\|\right)\|x_n-x_{n+1}\|\\ &\leq \alpha_n \operatorname{diam}(C)^2+2\|x_n-x_{n+1}\|\operatorname{diam}(C).\end{aligned} \tag{3.21}$$

This implies that

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.22}$$

Since $\{x_n\}$ is a Cauchy sequence in C, there exists an element $x^* \in C$ such that $\lim_{n\to\infty}x_n = x^*$. Now $\lim_{n\to\infty}\|u_n-x_n\|=0$ implies that $\lim_{n\to\infty}u_n = x^*$. From (3.1), we have

$$\begin{aligned}\|w_n-w_{n+1}\| &\leq \left(1+\frac{1}{n}\right)\mathscr{H}(T(x_n),T(x_{n+1}))\\ &\leq 2\mathscr{H}(T(x_n),T(x_{n+1}))\\ &\leq 2L\|x_n-x_{n+1}\|\end{aligned} \tag{3.23}$$

and for $m>n\geq 1$,

$$\|w_m-w_n\| \leq \sum_{i=n}^{m-1}\|w_i-w_{i+1}\| \leq 2L\sum_{i=n}^{m-1}\|x_i-x_{i+1}\|, \tag{3.24}$$

$$\begin{aligned}\sum_{i=n}^{m-1}\|x_i-x_{i+1}\| &= \sum_{i=n}^{m-1}a_{i+1} \leq \sum_{i=n}^{m-1}(\Theta a_i+c_i)\\ &= \Theta\sum_{i=n}^{m-1}a_i+\sum_{i=n}^{m-1}c_i\\ &= \Theta\sum_{i=n}^{m-1}a_{i+1}+\Theta(a_n-a_m)+\sum_{i=n}^{m-1}c_i\\ &\leq \Theta\sum_{i=n}^{m-1}a_{i+1}+\Theta a_n+\sum_{i=n}^{m-1}c_i.\end{aligned} \tag{3.25}$$

Thus,

$$\sum_{i=n}^{m-1}\|x_i - x_{i+1}\| \leq \frac{\Theta}{1-\Theta}\|x_n - x_{n-1}\| + \frac{\sum_{i=n}^{m-1} c_i}{1-\Theta}. \tag{3.26}$$

By (3.24) and (3.26), we have

$$\lim_{m,n\to\infty}\|w_m - w_n\| = 0. \tag{3.27}$$

It follows that $\{w_n\}$ is a Cauchy sequence in H and so there exists an element $\widehat{w}$ in H such that $\lim_{n\to\infty} w_n = \widehat{w}$:

$$\begin{aligned} d(\widehat{w}, T(x^*)) &= \inf_{b\in T(x^*)} d(\widehat{w}, b) \\ &\leq \|\widehat{w} - w_n\| + d(w_n, T(x^*)) \\ &\leq \|\widehat{w} - w_n\| + \mathscr{H}(T(x_n), T(x^*)) \\ &\leq \|\widehat{w} - w_n\| + L\|x_n - x^*\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{3.28}$$

that is, $d(\widehat{w}, T(x^*)) = 0$. We conclude that $\widehat{w} \in T(x^*)$ as $T(x^*) \in CB(H)$.

It follows that

$$\begin{aligned} \|x^* - S_n(x^*)\| &\leq \|x^* - u_n\| + \|u_n - x_n\| + \|x_n - S_n(u_n)\| + \|S_n(u_n) - S_n(x^*)\| \\ &\leq 2\|x^* - u_n\| + \|u_n - x_n\| + \|x_n - S_n(u_n)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \end{aligned} \tag{3.29}$$

and so $x^* = \lim_{n\to\infty} S_n(x^*) = S(x^*)$, that is, $x \in F(S) = \bigcap_{n=1}^{\infty} F(T_n)$. Since $x_n \to x^*$ and $u_n \to x^*$, we know that $\kappa'(u_n) - \kappa'(x_n) \to 0$. From (3.1) and ($\Phi$1), we have

$$\Phi(\widehat{w}, x^*, v) + \varphi(v, x^*) - \varphi(x^*, x^*) \geq 0, \tag{3.30}$$

that is, $x^* \in \Omega$. Thus, $x^* \in \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega$.

Since $q = Qf(q)$, we have $\langle f(q) - q, p - q\rangle \leq 0$ for all $p \in \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega$. From $x_n \to x^*$, we have

$$\lim_{n\to\infty}\langle f(q) - q, x_n - q\rangle = \langle f(q) - q, x^* - q\rangle \leq 0 \tag{3.31}$$

and so

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|(1-\alpha_n)(S_n(u_n)-q)+\alpha_n(f(x_n)-q)\|^2 \\
&\le (1-\alpha_n)^2\|S_n(u_n)-q\|^2+2\alpha_n\langle f(x_n)-q, x_{n+1}-q\rangle \\
&\le (1-\alpha_n)^2\|u_n-q\|^2+2\alpha_n\langle f(x_n)-f(q)+f(q)-q, x_{n+1}-q\rangle \\
&\le (1-\alpha_n)^2\|u_n-q\|^2+2\alpha\alpha_n\|x_n-q\|\|x_{n+1}-q\|+2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle \\
&\le (1-\alpha_n)^2\|u_n-q\|^2+\alpha\alpha_n(\|x_n-q\|^2+\|x_{n+1}-q\|^2)+2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle.
\end{aligned} \tag{3.32}$$

It follows from (3.19) that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\le \frac{(1-\alpha_n)^2+\alpha\alpha_n}{1-\alpha\alpha_n}\|x_n-q\|^2+\frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(q)-q, x_{n+1}-q\rangle \\
&\le \left(1-\frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n}\right)\|x_n-q\|^2+\frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n} \\
&\quad\times\left[\frac{\alpha_n}{1-\alpha}\sup_{n\in N}\|x_n-q\|^2+\frac{1}{1-\alpha}\langle f(q)-q, x_{n+1}-q\rangle\right].
\end{aligned} \tag{3.33}$$

Set

$$\begin{aligned}
\lambda_n &:= \frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n}, \qquad \xi_n := 0, \\
\sigma_n &:= \frac{\alpha_n}{1-\alpha}\sup_{n\in N}\|x_n-q\|^2+\frac{1}{1-\alpha}\langle f(q)-q, x_{n+1}-q\rangle.
\end{aligned} \tag{3.34}$$

Then, $\lim_{n\to\infty}\lambda_n = 0$, $\sum_{n=1}^{\infty}\lambda_n = \infty$, and $\limsup_{n\to\infty}\sigma_n \le 0$. It follows from Lemma 2.8 that $\lim_{n\to\infty}x_n = q$ and so $x^* = q$. This completes the proof. □

Remark 3.3. Theorem 3.2 improves and extends the main results of S. Takahashi and W. Takahashi [12].

We now give some applications of Theorem 3.2. If the set-valued mapping T in Theorem 3.2 is single-valued, then we have the following corollary.

Corollary 3.4. *Let C be a nonempty closed convex bounded subset of a real Hilbert space H, $T : C \to H$ a Lipschitz continuous mapping with constant $L > 0$, $f : C \to C$ a contraction mapping with constant $\alpha \in [0,1)$. Let $\varphi : C\times C \to R$ be a real-valued function satisfying the conditions* (φ1)–(φ3) *and let $\Phi : H\times C\times C \to R$ be an equilibrium-like function satisfying the conditions* (Φ1)–(Φ3) *and* (Φ4)$'$*:*

(Φ4)$'$ $\Phi(T(x),T_r(x),T_s(y))+\Phi(T(y),T_s(y),T_r(x)) \le -\gamma\|T_r(x)-T_s(y)\|^2$ *for all $x,y\in C$ and $r,s\in(0,\infty)$.*

Assume that $\eta : C \times C \to H$ *is a Lipschitz function with Lipschitz constant* $\lambda > 0$ *which satisfies the conditions* $(\eta 1)$~$(\eta 3)$. *Let* $\kappa : C \to R$ *be an* η*-strongly convex function with constant* $\mu > 0$ *which satisfies the conditions* $(\kappa 1)$ *and* $(\kappa 2)$ *with* $\lambda\nu/\mu \leq 1$. *Let* $S_n : C \to C$ *be an S-mapping generated by* $T_1, T_2, \ldots$ *and* $\lambda_1, \lambda_2, \ldots$ *and* $\bigcap_{n=1}^{\infty} F(T_n) \cap \Omega \neq \emptyset$, *where sequence* $\{T_n\}$ *is nonexpansive. Let* $\{x_n\}$, $\{u_n\}$, *and* $\{w_n\}$ *be sequences generated by*

$$\begin{gathered}\Phi(T(x_n), u_n, v) + \varphi(v, u_n) - \varphi(u_n, u_n) + \frac{1}{r_n}\langle \kappa'(u_n) - \kappa'(x_n), \eta(v, u_n)\rangle \geq 0, \quad \forall v \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S_n(u_n), \quad n = 1, 2, \ldots,\end{gathered} \tag{3.35}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{r_n\}$ *in* $(0,\infty)$ *satisfying conditions* $(C1)$–$(C3)$. *Then the sequences* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $x^* \in \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega$, *where* $x^* = P_{\cap_{n=1}^{\infty} F(T_n) \cap \Omega} f(x^*)$.

Corollary 3.5. *Let* C *be a nonempty closed convex bounded subset of a real Hilbert space* H, $T : C \to CB(H)$ *a multivalued* $\mathscr{H}$*-Lipschitz continuous mapping with constant* $L > 0$, $f : C \to C$ *a contraction mapping with constant* $\alpha \in [0,1)$. *Let* $\varphi : C \times C \to R$ *be a real-valued function satisfying the conditions* $(\varphi 1)$–$(\varphi 3)$ *and let* $\Phi : H \times C \times C \to R$ *be an equilibrium-like function satisfying the conditions* $(\Phi 1)$–$(\Phi 4)$ *and* $\Omega \neq \emptyset$. *Assume that* $\eta : C \times C \to H$ *is a Lipschitz function with Lipschitz constant* $\lambda > 0$ *which satisfies the conditions* $(\eta 1)$~$(\eta 3)$. *Let* $\kappa : C \to R$ *be an* η*-strongly convex function with constant* $\mu > 0$ *which satisfies the conditions* $(\kappa 1)$ *and* $(\kappa 2)$ *with* $\lambda\nu/\mu \leq 1$. *Let* $\{x_n\}$, $\{u_n\}$, *and* $\{w_n\}$ *be sequences generated by*

$$\begin{gathered} w_n \in T(x_n), \quad \|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n}\right)\mathscr{H}(T(x_n), T(x_{n+1})), \\ \Phi(w_n, u_n, v) + \varphi(v, u_n) - \varphi(u_n, u_n) + \frac{1}{r_n}\langle \kappa'(u_n) - \kappa'(x_n), \eta(v, u_n)\rangle \geq 0, \quad \forall v \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) u_n, \quad n = 1, 2, \ldots,\end{gathered} \tag{3.36}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{r_n\}$ *in* $(0,\infty)$ *satisfying conditions* $(C1)$ *and* $(C2)$. *Then the sequences* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $x^* \in \Omega$, *and* $\{w_n\}$ *converges strongly to* $w^* \in T(x^*)$, *where* $x^* = P_{\Omega} f(x^*)$.

Proof. Let $T_n = I$ in Theorem 3.2 for $n = 1, 2, \ldots$, where I is an identity mapping. Then $S_n = I$ for $n = 1, 2, \ldots$. Thus, the condition $(C3)$ is satisfied. Now Corollary 3.5 follows from Theorem 3.2. This completes the proof. □

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