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*Research Article*

# Some Similarity between Contractions and Kannan Mappings

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Contractions are always continuous and Kannan mappings are not necessarily continuous. This is a very big difference between both mappings. However, we know that relaxed both mappings are quite similar. In this paper, we discuss both mappings from a new point of view.



## 1. Introduction

Let $(X, d)$ be a metric space and let $T$ be a mapping on $X$. Then $T$ is called a *contraction* if there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq r d(x, y) \tag{1.1}$$

for all $x, y \in X$. $T$ is called *Kannan* if there exists $\alpha \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty) \tag{1.2}$$

for all $x, y \in X$. We know that if $X$ is complete, then every contraction and every Kannan mapping have a unique fixed point, see [1, 2]. We know that both conditions are independent, that is, there exist a contraction, which is not Kannan, and a Kannan mapping, which is not a contraction. Thus we cannot compare both conditions directly. So we compare both indirectly.

*Fact* 1

Banach fixed-point theorem, which is often called the Banach contraction principle, is very important because it is a very forceful tool in nonlinear analysis. We think that Kannan fixed-point theorem is also very important because Subrahmanyam [3] proved that Kannan theorem characterizes the metric completeness of underlying spaces, that is, a metric space $X$ is complete if and only if every Kannan mapping on $X$ has a fixed point. On the other hand, Connell [4] gave an example of a metric space $X$ such that $X$ is not complete and every contraction on $X$ has a fixed point. Thus the Banach theorem cannot characterize the metric completeness of $X$. Therefore, we consider that the notion of contractions is stronger from this point of view.

*Fact* 2

Using the notion of $\tau$-distances, Suzuki [5] considered some weaker contractions and Kannan mappings and proved the following.

(i) If $T$ is a contraction with respect to a $\tau$-distance, then $T$ is Kannan with respect to another $\tau$-distance.

(ii) If $T$ is Kannan with respect to a $\tau$-distance, then $T$ is a contraction with respect to another $\tau$-distance.

That is, both conditions are completely the same.

Recently, Suzuki [6] proved the following theorem, see also [7].

**Theorem 1.1** (see [6]). *Define a nonincreasing function $\theta$ from $[0,1)$ onto $(1/2,1]$ by*

$$\theta(r)=\begin{cases}1 & \text{if } 0\le r\le \frac{1}{2}(\sqrt{5}-1),\\ \dfrac{1-r}{r^2} & \text{if } \frac{1}{2}(\sqrt{5}-1)\le r\le \dfrac{1}{\sqrt{2}},\\ \dfrac{1}{1+r} & \text{if } \dfrac{1}{\sqrt{2}}\le r<1.\end{cases} \tag{1.3}$$

*Then for a metric space $(X,d)$, the following are equivalent:*

(i) $X$ *is complete,*

(ii) *every mapping $T$ on $X$, satisfying the following, has a fixed point: there exists $r\in[0,1)$ such that $\theta(r)d(x,Tx)\le d(x,y)$ implies $d(Tx,Ty)\le rd(x,y)$ for all $x,y\in X$.*

*Remark 1.2.* $\theta(r)$ is the best constant for every $r$.

The purpose of this paper is to prove a Kannan version of Theorem 1.1. Then we compare the theorem (Theorem 2.2) with Theorem 1.1 and attempt to judge which is stronger from our new point of view.

## 2. Kannan mappings

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

In this section, we prove our main result. We begin with the following lemma.

**Lemma 2.1.** *Let $(X, d)$ be a metric space and let $T$ be a mapping on $X$. Let $x \in X$ satisfy $d(Tx, T^2x) \leq rd(x, Tx)$ for some $r \in [0, 1)$. Then for $y \in X$, either*

$$\frac{1}{1+r}d(x, Tx) \leq d(x, y) \quad \text{or} \quad \frac{1}{1+r}d(Tx, T^2x) \leq d(Tx, y) \tag{2.1}$$

*holds.*

*Proof.* We assume

$$\frac{1}{1+r}d(x, Tx) > d(x, y), \qquad \frac{1}{1+r}d(Tx, T^2x) > d(Tx, y). \tag{2.2}$$

Then we have

$$\begin{aligned} d(x, Tx) &\leq d(x, y) + d(y, Tx) \\ &< \frac{1}{1+r}\left(d(x, Tx) + d(Tx, T^2x)\right) \\ &\leq \frac{1}{1+r}\left(d(x, Tx) + rd(x, Tx)\right) = d(x, Tx). \end{aligned} \tag{2.3}$$

This is a contradiction. □

The following theorem is a Kannan version of Theorem 1.1.

**Theorem 2.2.** *Define a nonincreasing function $\varphi$ from $[0, 1)$ into $(1/2, 1]$ by*

$$\varphi(r) = \begin{cases} 1 & \text{if } 0 \leq r < \dfrac{1}{\sqrt{2}}, \\ \dfrac{1}{1+r} & \text{if } \dfrac{1}{\sqrt{2}} \leq r < 1. \end{cases} \tag{2.4}$$

*Let $(X, d)$ be a complete metric space and let $T$ be a mapping on $X$. Let $\alpha \in [0, 1/2)$ and put $r := \alpha/(1-\alpha) \in [0, 1)$. Assume that*

$$\varphi(r)d(x, Tx) \leq d(x, y) \quad \text{implies } d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty) \tag{2.5}$$

*for all $x, y \in X$, then $T$ has a unique fixed point $z$ and $\lim_n T^n x = z$ holds for every $x \in X$.*

*Proof.* Since $\varphi(r) \leq 1$, $\varphi(r)d(x, Tx) \leq d(x, Tx)$ holds. From the assumption, we have

$$d(Tx, T^2x) \leq \alpha d(x, Tx) + \alpha d(Tx, T^2x), \tag{2.6}$$

and hence

$$d(Tx, T^2x) \leq rd(x, Tx) \tag{2.7}$$

for $x \in X$. Let $u \in X$. Put $u_0 = u$ and $u_n = T^n u$ for all $n \in \mathbb{N}$. From (2.7), we have

$$\sum_{n=1}^{\infty} d(u_n, u_{n+1}) \leq \sum_{n=1}^{\infty} r^n d(u_0, u_1) < \infty. \tag{2.8}$$

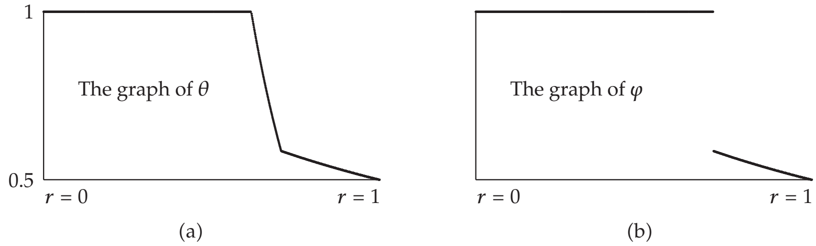


**Figure** 1

So $\{u_n\}$ is a Cauchy sequence in $X$ and by the completeness of $X$, there exists a point $z$ such that $u_n \to z$.

We next show

$$d(z, Tx) \le \alpha d(x, Tx), \quad \forall x \in X \text{ with } x \ne z. \tag{2.9}$$

Since $u_n \to z$, there exists $n_0 \in \mathbb{N}$ such that $d(u_n, z) \le (1/3)d(x, z)$ for all $n \in \mathbb{N}$ with $n \ge n_0$. Then we have

$$\begin{aligned} \varphi(r) d(u_n, Tu_n) &\le d(u_n, Tu_n) = d(u_n, u_{n+1}) \\ &\le d(u_n, z) + d(u_{n+1}, z) \\ &\le \frac{2}{3} d(x, z) = d(x, z) - \frac{1}{3} d(x, z) \\ &\le d(x, z) - d(u_n, z) \le d(u_n, x), \end{aligned} \tag{2.10}$$

and hence

$$d(Tu_n, Tx) \le \alpha d(u_n, Tu_n) + \alpha d(x, Tx) \quad \text{for } n \in \mathbb{N} \text{ with } n \ge n_0. \tag{2.11}$$

Therefore, we obtain

$$\begin{aligned} d(z, Tx) &= \lim_{n \to \infty} d(u_{n+1}, Tx) = \lim_{n \to \infty} d(Tu_n, Tx) \\ &\le \lim_{n \to \infty} (\alpha d(u_n, Tu_n) + \alpha d(x, Tx)) \\ &= \alpha d(x, Tx) \end{aligned} \tag{2.12}$$

for $x \in X$ with $x \ne z$.

Let us prove that $z$ is a fixed point of $T$. In the case where $0 \le r < 1/\sqrt{2}$, arguing by contradiction, we assume that $Tz \ne z$. Then we have, from (2.9),

$$d(z, T^2 z) \le \alpha d(Tz, T^2 z) \le \alpha r d(z, Tz), \tag{2.13}$$

and hence

$$\begin{aligned} d(z, Tz) &\le d(z, T^2 z) + d(Tz, T^2 z) \\ &\le \alpha r d(z, Tz) + r d(z, Tz) = \frac{r + 2r^2}{1 + r} d(z, Tz) \\ &< \frac{r + 1}{1 + r} d(z, Tz) = d(z, Tz). \end{aligned} \tag{2.14}$$

This is a contradiction. Therefore, we obtain $Tz = z$. In the case where $1/\sqrt{2} \le r < 1$, from Lemma 2.1, either

$$\varphi(r)d(u_{2n}, u_{2n+1}) \le d(u_{2n}, z) \quad \text{or} \quad \varphi(r)d(u_{2n+1}, u_{2n+2}) \le d(u_{2n+1}, z) \tag{2.15}$$

holds for $n \in \mathbb{N}$. Thus there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$\varphi(r)d(u_{n_j}, u_{n_j+1}) \le d(u_{n_j}, z) \tag{2.16}$$

for $j \in \mathbb{N}$. From the assumption, we have

$$d(z, Tz) = \lim_{j\to\infty} d(u_{n_j+1}, Tz) \le \lim_{j\to\infty} (\alpha d(u_{n_j}, u_{n_j+1}) + \alpha d(z, Tz)) = \alpha d(z, Tz). \tag{2.17}$$

Since $\alpha < 1/2$, we have $Tz = z$. Therefore, we have shown $Tz = z$ in both cases.

From (2.9), we obtain that the fixed point $z$ is unique. □

*Remark 2.3.* Since $\theta(r) \le \varphi(r)$ for every $r$, we can consider that Kannan is stronger from our new point of view. Though $\theta$ and $\varphi$ are different, we remark that the graphs of $\theta$ and $\varphi$ are quite similar.

The following theorem shows that $\varphi(r)$ is the best constant for every $r$.

**Theorem 2.4.** *Define a function $\varphi$ as in Theorem 2.2. For every $\alpha \in [0, 1/2)$, putting $r = \alpha/(1-\alpha)$ , there exist a complete metric space $(X, d)$ and a mapping $T$ on $X$ such that $T$ has no fixed points and*

$$\varphi(r)d(x, Tx) < d(x, y) \quad \textit{implies } d(Tx, Ty) \le \alpha d(x, Tx) + \alpha d(y, Ty) \tag{2.18}$$

*for all $x, y \in X$.*

*Proof.* In the case where $0 \le r < 1/\sqrt{2}$, define a complete subset $X$ of the Euclidean space $\mathbb{R}$ by $X = \{-1, 1\}$. We also define a mapping $T$ on $X$ by $Tx = -x$ for $x \in X$. Then $T$ dose not have a fixed point and

$$\varphi(r)d(x, Tx) = 2 \ge d(x, y) \tag{2.19}$$

for all $x, y \in X$. In the case where $1/\sqrt{2} \le r < 1$, define a complete subset $X$ of the Euclidean space $\mathbb{R}$ by

$$X = \{0, 1\} \cup \{x_n : n \in \mathbb{N} \cup \{0\}\}, \tag{2.20}$$

where $x_n = (1-r)(-r)^n$ for all $n \in \mathbb{N} \cup \{0\}$. Define a mapping $T$ on $X$ by $T0 = 1$, $T1 = 1 - r$, and $Tx_n = x_{n+1}$ for $n \in \mathbb{N} \cup \{0\}$. Then the following are obvious:

(i) $d(T0, T1) = r = \alpha d(0, T0) + \alpha d(1, T1)$,

(ii) $\varphi(r)d(0, T0) \ge \varphi(r)d(x_n, Tx_n) = d(0, x_n)$ for $n \in \mathbb{N} \cup \{0\}$.

Also, we have

$$\begin{aligned} d(Tx_m, Tx_n) &\le d(0, Tx_m) + d(0, Tx_n) = \alpha d(x_m, Tx_m) + \alpha d(x_n, Tx_n), \\ d(T1, Tx_n) - (\alpha d(1, T1) + \alpha d(x_n, Tx_n)) &\le d(0, T1) + d(0, Tx_n) - (\alpha d(1, T1) + \alpha d(x_n, Tx_n)) \\ &= d(0, T1) - \alpha d(1, T1) = \frac{1-2r^2}{1+r} \le 0 \end{aligned} \tag{2.21}$$

for $m, n \in \mathbb{N} \cup \{0\}$. □

## 3. Generalized Kannan mappings

It is a very natural question of whether or not another fixed-point theorem with $\theta$ exists. In this section, we give a positive answer to this problem.

**Theorem 3.1.** *Define a nonincreasing function $\theta$ as in Theorem 1.1. Let $(X,d)$ be a complete metric space and let $T$ be a mapping on $X$. Suppose that there exists $r \in [0,1)$ such that*

$$\theta(r)d(x,Tx) \leq d(x,y) \quad \textit{implies } d(Tx,Ty) \leq r\max\{d(x,Tx),d(y,Ty)\} \tag{3.1}$$

*for all $x,y \in X$. Then $T$ has a unique fixed point $z$ and $\lim_n T^n x = z$ holds for every $x \in X$.*

*Proof.* Since $\theta(r)d(x,Tx) \leq d(x,Tx)$, we have, from the assumption,

$$d(Tx,T^2x) \leq r\max\{d(x,Tx),d(Tx,T^2x)\} \tag{3.2}$$

and hence

$$d(Tx,T^2x) \leq rd(x,Tx) \tag{3.3}$$

for $x \in X$. Let $u \in X$. Put $u_0 = u$ and $u_n = T^n u$ for all $n \in \mathbb{N}$. As in the proof of Theorem 2.2, we can prove that $\{u_n\}$ converges to some $z \in X$.

We next show

$$d(z,Tx) \leq rd(x,Tx) \quad \text{for all } x \in X \text{ with } x \neq z. \tag{3.4}$$

Since $u_n \to z$, we have $\theta(r)d(u_n,Tu_n) \leq d(u_n,x)$ for sufficiently large $n \in \mathbb{N}$. Hence we obtain, from the assumption,

$$\begin{aligned} d(z,Tx) &= \lim_{n\to\infty} d(u_{n+1},Tx) = \lim_{n\to\infty} d(Tu_n,Tx) \\ &\leq \lim_{n\to\infty} r\max\{d(u_n,Tu_n),d(x,Tx)\} = rd(x,Tx) \end{aligned} \tag{3.5}$$

for $x \in X$ with $x \neq z$.

Let us prove that $z$ is a fixed point of $T$. In the case where $0 \leq r < 1/\sqrt{2}$, we note

$$\theta(r) \leq \frac{1-r}{r^2}. \tag{3.6}$$

We will show, by induction,

$$d(T^n z,Tz) \leq rd(z,Tz) \tag{3.7}$$

for $n \in \mathbb{N}$ with $n \geq 2$. When $n = 2$, (3.7) becomes (3.3), thus (3.7) holds. We assume $d(T^n z,Tz) \leq rd(z,Tz)$ for some $n \in \mathbb{N}$ with $n \geq 2$. Since

$$d(z,Tz) \leq d(z,T^n z) + d(T^n z,Tz) \leq d(z,T^n z) + rd(z,Tz), \tag{3.8}$$

we have $d(z,Tz) \le (1/(1-r))d(z,T^n z)$, and hence

$$\begin{aligned}\theta(r)d(T^n z,T^{n+1}z) &\le \frac{1-r}{r^2}d(T^n z,T^{n+1}z) \le \frac{1-r}{r^n}d(T^n z,T^{n+1}z)\\ &\le (1-r)d(z,Tz) \le d(z,T^n z).\end{aligned} \tag{3.9}$$

Therefore, by the assumption, we have

$$d(T^{n+1}z,Tz) \le r\max\{d(T^n z,T^{n+1}z),d(z,Tz)\} = rd(z,Tz). \tag{3.10}$$

By induction, (3.7) holds for $n \in \mathbb{N}$ with $n \ge 2$. Arguing, by contradiction, we assume $Tz \ne z$. Then from (3.7), $T^n z \ne z$ holds for all $n \in \mathbb{N}$. Then by (3.4), we have

$$d(T^{n+1}z,z) \le rd(T^n z,T^{n+1}z) \le r^{n+1}d(z,Tz). \tag{3.11}$$

This implies $T^n z \to z$, which contradicts (3.7). Therefore, we obtain $Tz = z$. In the case where $1/\sqrt{2} \le r < 1$, as in the proof of Theorem 2.2, we can show that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\varphi(r)d(u_{n_j},u_{n_j+1}) \le d(u_{n_j},z)$ for $j \in \mathbb{N}$. From the assumption, we have

$$d(z,Tz) = \lim_{j\to\infty} d(u_{n_j+1},Tz) \le \lim_{j\to\infty} r\max\{d(u_{n_j},u_{n_j+1}),d(z,Tz)\} = rd(z,Tz). \tag{3.12}$$

Since $r < 1$, the above inequality implies that $Tz = z$. Therefore, we have shown that $Tz = z$ in both cases.

From (3.4), we obtain that the fixed point $z$ is unique. □

*Remark 3.2.* When the second author was proving Theorem 1.1, he did not feel that $\theta(r)$ was natural. However, since the above proof is easier to understand how $\theta(r)$ works, the authors can faintly feel that $\theta(r)$ is natural.

The following theorem shows that $\theta(r)$ is the best constant for every $r$.

**Theorem 3.3.** *Define a function $\theta$ as in Theorem 1.1. Then for any $r \in [0,1)$, there exist a complete metric space $(X,d)$ and a mapping $T$ on $X$ such that $T$ has no fixed points and*

$$\theta(r)d(x,Tx) < d(x,y) \quad \text{implies } d(Tx,Ty) \le r\max\{d(x,Tx),d(y,Ty)\} \tag{3.13}$$

*for all $x,y \in X$.*

*Proof.* We have already shown the conclusion in the case where $0 \le r \le (1/2)(\sqrt{5}-1)$ or $1/\sqrt{2} \le r < 1$ because $\varphi(r) = \theta(r)$ holds. So let us consider the case where $(1/2)(\sqrt{5}-1) < r < 1/\sqrt{2}$. Define a complete subset $X$ of the Euclidean space $\mathbb{R}$ by $X = \{x_n : n \in \mathbb{N}\}$, where $x_0 = 0$, $x_1 = 1$, $x_2 = 1-r$, and $x_n = (1-r-r^2)(-r)^{n-3}$ for $n \ge 3$. Define a mapping $T$ on $X$ by $Tx_n = x_{n+1}$ for $n \in \mathbb{N}$. Then the following are obvious:

(i) $d(Tx_0,Tx_1) = r = rd(x_0,Tx_0) = r\max\{d(x_0,Tx_0),d(x_1,Tx_1)\}$,

(ii) $\theta(r)d(x_0,Tx_0) \ge \theta(r)d(x_2,Tx_2) = 1-r = d(x_0,x_2)$,

(iii) $\theta(r)d(x_0,Tx_0) \ge \theta(r)d(x_n,Tx_n) = ((1-r^2)/r^2)d(x_0,x_n) \ge d(x_0,x_n)$ for $n \ge 3$,

(iv) $d(Tx_1,Tx_2) = r^2 = rd(x_1,Tx_1)$.

Since

$$x_3 < x_5 < x_7 < \cdots < x_0 < \cdots < x_8 < x_6 < x_4 < x_2 < x_1, \tag{3.14}$$

we have the following:

(i) $d(Tx_1, Tx_n) < d(x_2, x_3) = r^2 = rd(x_1, Tx_1)$ for $n \geq 3$,

(ii) $d(Tx_2, Tx_n) - rd(x_2, Tx_2) \leq d(x_3, x_4) - r^3 = 2r^2 - 1 \leq 0$ for $n \geq 3$,

(iii) $d(Tx_m, Tx_n) \leq d(Tx_m, Tx_{m+1}) = rd(x_m, Tx_m)$ for $3 \leq m < n$.

This completes the proof. □

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