

Research Article

Composite Implicit General Iterative Process for a Nonexpansive Semigroup in Hilbert Space

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Let C be nonempty closed convex subset of real Hilbert space H . Consider \mathcal{T} a nonexpansive semigroup $\mathcal{T} = \{T(s) : s \geq 0\}$ with a common fixed point, a contraction f with coefficient $0 < \alpha < 1$, and a strongly positive linear bounded operator A with coefficient $\bar{\gamma} > 0$. Let $0 < \gamma < \bar{\gamma}/\alpha$. It is proved that the sequence $\{x_n\}$ generated iteratively by $x_n = (I - \alpha_n A)(1/t_n) \int_0^{t_n} T(s)y_n ds + \alpha_n \gamma f(x_n)$, $y_n = (I - \beta_n A)x_n + \beta_n \gamma f(x_n)$ converges strongly to a common fixed point $x^* \in F(\mathcal{T})$ which solves the variational inequality $\langle (\gamma f - A)x^*, z - x^* \rangle \leq 0$ for all $z \in F(\mathcal{T})$.

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1. Introduction and preliminaries

Let C be a closed convex subset of a Hilbert space H , recall that $T : C \rightarrow C$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Denote by $F(T)$ the set of fixed points of T , that is, $F(T) := \{x \in C : Tx = x\}$.

Recall that a family $\mathcal{T} = \{T(s) \mid 0 \leq s < \infty\}$ of mappings from C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

- (i) $T(0)x = x$ for all $x \in C$;
- (ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;
- (iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;
- (iv) for all $x \in C, s \mid \rightarrow T(s)x$ is continuous.

We denote by $F(\mathcal{T})$ the set of all common fixed points of \mathcal{T} , that is, $F(\mathcal{T}) = \bigcap_{0 \leq s < \infty} F(T(s))$. It is known that $F(\mathcal{T})$ is closed and convex.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems (see, e.g., [1–5] and the references therein). A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H :

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \quad (1.1)$$

where C is the fixed point set of a nonexpansive mapping T on H , and b is a given point in H . Assume that A is strongly positive, that is, there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2 \quad \forall x \in H. \quad (1.2)$$

It is well known that $F(T)$ is closed convex (cf. [6]). In [3] (see also [4]), it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n b, \quad n \geq 0 \quad (1.3)$$

converges strongly to the unique solution of the minimization problem (1.1) provided that the sequence $\{\alpha_n\}$ satisfies certain conditions.

On the other hand, Moudafi [7] introduced the viscosity approximation method for nonexpansive mappings (see [8] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H . Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \sigma_n)Tx_n + \sigma_n f(x_n), \quad n \geq 0, \quad (1.4)$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$. It is proved [7, 8] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.4) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad x \in C. \quad (1.5)$$

Recently, Marino and Xu [9] combined the iterative method (1.3) with the viscosity approximation method (1.4) considering the following general iteration process:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \quad (1.6)$$

and proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$

generated by (1.6) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in C, \quad (1.7)$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x), \quad (1.8)$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$, for $x \in H$).

In this paper, motivated and inspired by the idea of Marino and Xu [9], we introduce the composite implicit general iteration process (1.9) as follows:

$$\begin{aligned} x_n &= (I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} T(s) y_n ds + \alpha_n \gamma f(x_n), \\ y_n &= (I - \beta_n A) x_n + \beta_n \gamma f(x_n), \end{aligned} \quad (1.9)$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$, and investigate the problem of approximating common fixed point of nonexpansive semigroup $\{T(s) : s \geq 0\}$ which solves some variational inequality. The results presented in this paper extend and improve the main results in Marino and Xu [9], and the methods of proof given in this paper are also quite different.

In what follows, we will make use of the following lemmas. Some of them are known; others are not hard to derive.

Lemma 1.1 (Marino and Xu [9]). *Assume that A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \bar{\gamma}$.*

Lemma 1.2 (Shimizu and Takashi [10]). *Let C be a nonempty bounded closed convex subset of H and let $\mathfrak{T} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C , then for any $h \geq 0$,*

$$\lim_{t \rightarrow \infty} \sup_{x \in C} \left\| \frac{1}{t} \int_0^t T(s)x ds - T(h) \left(\frac{1}{t} \int_0^t T(s)x ds \right) \right\| = 0. \quad (1.10)$$

Lemma 1.3. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $\mathfrak{T} = \{T(t) : 0 \leq t < \infty\}$ be a nonexpansive semigroup on C . If $\{x_n\}$ is a sequence in C satisfying the following properties:*

- (i) $x_n \rightharpoonup z$;
- (ii) $\limsup_{t \rightarrow \infty} \limsup_{n \rightarrow \infty} \|T(t)x_n - x_n\| = 0$,

where $x_n \rightharpoonup z$ denote that $\{x_n\}$ converges weakly to z , then $z \in F(\mathfrak{T})$.

Proof. This lemma is the continuous version of Lemma 2.3 of Tan and Xu [11]. This proof given in [11] is easily extended to the continuous case. \square

2. Main results

Lemma 2.1. *Let H be a Hilbert space, C a closed convex subset of H , let $\mathfrak{T} = \{T(s) : s \geq 0\}$ be a nonexpansive semigroup on C , $\{t_n\} \subset (0, \infty)$ is a sequence, then $I - (1/t_n)\int_0^{t_n} T(s)ds$ is monotone.*

Proof. In fact, for all $x, y \in H$,

$$\begin{aligned} & \left\langle x - y, \left(I - \frac{1}{t_n} \int_0^{t_n} T(s)ds \right) x - \left(I - \frac{1}{t_n} \int_0^{t_n} T(s)ds \right) y \right\rangle \\ &= \|x - y\|^2 - \left\langle x - y, \frac{1}{t_n} \int_0^{t_n} T(s)x ds - \frac{1}{t_n} \int_0^{t_n} T(s)y ds \right\rangle \\ &\geq \|x - y\|^2 - \|x - y\| \frac{1}{t_n} \int_0^{t_n} \|T(s)x - T(s)y\| ds \\ &\geq \|x - y\|^2 - \|x - y\|^2 = 0. \end{aligned} \tag{2.1}$$

□

Theorem 2.2. *Let C be nonempty closed convex subset of real Hilbert space H , suppose that $f : C \rightarrow C$ is a fixed contractive mapping with coefficient $0 < \alpha < 1$, and $\mathfrak{T} = \{T(s) : s \geq 0\}$ is a nonexpansive semigroup on C such that $F(\mathfrak{T})$ is nonempty, and A is a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$, $\{t_n\} \subset (0, \infty)$ are real sequences such that*

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \beta_n = o(\alpha_n), \quad \lim_{n \rightarrow \infty} t_n = \infty, \tag{2.2}$$

then for any $0 < \gamma < \bar{\gamma}/\alpha$, there is a unique $\{x_n\} \in C$ such that

$$\begin{aligned} x_n &= (I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} T(s)y_n ds + \alpha_n \gamma f(x_n), \\ y_n &= (I - \beta_n A)x_n + \beta_n \gamma f(x_n), \end{aligned} \tag{2.3}$$

and the iteration process $\{x_n\}$ converges strongly to the unique solution $x^* \in F(\mathfrak{T})$ of the variational inequality $\langle (\gamma f - A)x^*, z - x^* \rangle \leq 0$ for all $z \in F(\mathfrak{T})$.

Proof. Our proof is divided into five steps.

Since $\alpha_n \rightarrow 0$, $\beta_n \rightarrow 0$ as $n \rightarrow \infty$, we may assume, with no loss of generality, that $\alpha_n < \|A\|^{-1}$, $\beta_n < \|A\|^{-1}$ for all $n \geq 1$.

(i) $\{x_n\}$ is bounded.

Firstly, we will show that the mapping $T_n^f : C \rightarrow C$ defined by

$$T_n^f = (I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} T(s) [(I - \beta_n A) + \beta_n \gamma f] ds + \alpha_n \gamma f \tag{2.4}$$

is a contraction. Indeed, from Lemma 1.1, we have for any $x, y \in C$ that

$$\begin{aligned}
\|T_n^f x - T_n^f y\| &\leq \|I - \alpha_n A\| \frac{1}{t_n} \int_0^{t_n} \|T(s) [(I - \beta_n A)x + \beta_n \gamma f(x)] \\
&\quad - T(s) [(I - \beta_n A)y + \beta_n \gamma f(y)]\| ds + \alpha_n \gamma \|f(x) - f(y)\| \\
&\leq (1 - \alpha_n \bar{\gamma}) \|[(I - \beta_n A)x + \beta_n \gamma f(x)] - [(I - \beta_n A)y + \beta_n \gamma f(y)]\| + \alpha_n \gamma \alpha \|x - y\| \\
&\leq (1 - \alpha_n \bar{\gamma}) [\|I - \beta_n A\| \|x - y\| + \beta_n \gamma \alpha \|x - y\|] + \alpha_n \gamma \alpha \|x - y\| \\
&\leq (1 - \alpha_n \bar{\gamma}) [1 - \beta_n (\bar{\gamma} - \gamma \alpha)] \|x - y\| + \alpha_n \gamma \alpha \|x - y\| \\
&= \{(1 - \alpha_n \bar{\gamma}) [1 - \beta_n (\bar{\gamma} - \gamma \alpha)] + \alpha_n \gamma \alpha\} \|x - y\| \\
&= \{[1 - \alpha_n (\bar{\gamma} - \gamma \alpha)] - (1 - \alpha_n \bar{\gamma}) \beta_n (\bar{\gamma} - \gamma \alpha)\} \|x - y\| \\
&< [1 - \alpha_n (\bar{\gamma} - \gamma \alpha)] \|x - y\| < \|x - y\|.
\end{aligned} \tag{2.5}$$

Let $x_n \in C$ be the unique fixed point of T_n^f . Thus,

$$\begin{aligned}
x_n &= (I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} T(s) y_n ds + \alpha_n \gamma f(x_n), \\
y_n &= (I - \beta_n A) x_n + \beta_n \gamma f(x_n)
\end{aligned} \tag{2.6}$$

is well defined. Next, we will show that $\{x_n\}$ is bounded.

Pick any $z \in F(\mathcal{J})$ to obtain

$$\begin{aligned}
\|x_n - z\| &= \left\| (I - \alpha_n A) \left(\frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - z \right) + \alpha_n (\gamma f(x_n) - Az) \right\| \\
&\leq \|I - \alpha_n A\| \frac{1}{t_n} \int_0^{t_n} \|T(s) y_n - z\| ds + \alpha_n [\gamma \|f(x_n) - f(z)\| + \|\gamma f(z) - Az\|] \\
&\leq (1 - \alpha_n \bar{\gamma}) \|y_n - z\| + \alpha_n [\gamma \|f(x_n) - f(z)\| + \|\gamma f(z) - Az\|],
\end{aligned} \tag{2.7}$$

$$\|x_n - z\| \leq (1 - \alpha_n \bar{\gamma}) \|y_n - z\| + \alpha_n \gamma \alpha \|x_n - z\| + \alpha_n \|\gamma f(z) - Az\|. \tag{2.8}$$

Also

$$\begin{aligned}
\|y_n - z\| &\leq \|I - \beta_n A\| \|x_n - z\| + \beta_n \|\gamma f(x_n) - Az\| \\
&\leq (1 - \beta_n \bar{\gamma}) \|x_n - z\| + \beta_n \gamma \alpha \|x_n - z\| + \beta_n \|\gamma f(z) - Az\| \\
&= [1 - \beta_n (\bar{\gamma} - \gamma \alpha)] \|x_n - z\| + \beta_n \|\gamma f(z) - Az\|.
\end{aligned} \tag{2.9}$$

Substituting (2.9) into (2.8), we obtain that

$$\begin{aligned}
\|x_n - z\| &\leq (1 - \alpha_n \bar{\gamma}) [1 - \beta_n (\bar{\gamma} - \gamma \alpha)] \|x_n - z\| + (1 - \alpha_n \bar{\gamma}) \beta_n \|\gamma f(z) - Az\| \\
&\quad + \alpha_n \gamma \alpha \|x_n - z\| + \alpha_n \|\gamma f(z) - Az\| \\
&= \{(1 - \alpha_n \bar{\gamma}) [1 - \beta_n (\bar{\gamma} - \gamma \alpha)] + \alpha_n \gamma \alpha\} \|x_n - z\| \\
&\quad + [(1 - \alpha_n \bar{\gamma}) \beta_n + \alpha_n] \|\gamma f(z) - Az\| \\
&= \{1 - (\bar{\gamma} - \gamma \alpha) [\alpha_n + (1 - \alpha_n \bar{\gamma}) \beta_n]\} \|x_n - z\| \\
&\quad + [\alpha_n + (1 - \alpha_n \bar{\gamma}) \beta_n] \|\gamma f(z) - Az\|, \\
(\bar{\gamma} - \gamma \alpha) [\alpha_n + (1 - \alpha_n \bar{\gamma}) \beta_n] \|x_n - z\| &\leq [\alpha_n + (1 - \alpha_n \bar{\gamma}) \beta_n] \|\gamma f(z) - Az\|, \\
\|x_n - z\| &\leq \frac{1}{\bar{\gamma} - \gamma \alpha} \|\gamma f(z) - Az\|.
\end{aligned} \tag{2.10}$$

Thus $\{x_n\}$ is bounded.

$$(ii) \lim_{n \rightarrow \infty} \|x_n - T(s)x_n\| = 0.$$

Denote that $z_n := (1/t_n) \int_0^{t_n} T(s) y_n ds$, since $\{x_n\}$ is bounded, $\|z_n - z\| \leq \|y_n - z\|$ and $\{Az_n\}, \{f(x_n)\}$ are also bounded, From (2.6) and $\lim_{n \rightarrow \infty} \alpha_n = 0$, we have

$$\|x_n - z_n\| = \alpha_n \|\gamma f(x_n) - Az_n\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{2.11}$$

Let $K = \{w \in C : \|w - z\| \leq (1/(\bar{\gamma} - \gamma \alpha)) \|\gamma f(z) - Az\|\}$, then K is a nonempty bounded closed convex subset of C and $T(s)$ -invariant. Since $\{x_n\} \subset K$ and K is bounded, there exists $r > 0$ such that $K \subset B_r$, it follows from Lemma 1.2 that

$$\lim_{n \rightarrow \infty} \|z_n - T(s)z_n\| = 0 \quad \forall s \geq 0. \tag{2.12}$$

From (2.11) and (2.12), we have

$$\lim_{n \rightarrow \infty} \|x_n - T(s)x_n\| = 0. \tag{2.13}$$

(iii) There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in F(\mathcal{J})$ and x^* is the unique solution of the following variational inequality:

$$\langle (A - \gamma f)x^*, x^* - z \rangle \leq 0 \quad \forall z \in F(\mathcal{J}). \tag{2.14}$$

Firstly since

$$\|y_n - x_n\| = \beta_n \|\gamma f(x_n) - A(x_n)\|. \tag{2.15}$$

From condition $\beta_n \rightarrow 0$ and the boundedness of $\{x_n\}$, we obtain that $\|y_n - x_n\| \rightarrow 0$. Again by boundedness of $\{x_n\}$, we know that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \rightharpoonup x^*$. Then $y_{n_k} \rightharpoonup x^*$. From Lemma 1.3 and step (ii), we have that $x^* \in F(\mathcal{J})$.

Next we will prove that x^* solves the variational inequality (2.14). Since

$$x_n = (I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} T(s) y_n ds + \alpha_n \gamma f(x_n), \quad (2.16)$$

we derive that

$$(A - \gamma f)x_n = -\frac{1}{\alpha_n} (I - \alpha_n A) \left(I - \frac{1}{t_n} \int_0^{t_n} T(s) ds \right) y_n + \frac{1}{\alpha_n} [(I - \alpha_n A)y_n - (I - \alpha_n A)x_n]. \quad (2.17)$$

It follows that, for all $z \in F(\mathcal{J})$,

$$\begin{aligned} \langle (A - \gamma f)x_n, y_n - z \rangle &= -\frac{1}{\alpha_n} \left\langle (I - \alpha_n A) \left(I - \frac{1}{t_n} \int_0^{t_n} T(s) ds \right) y_n, y_n - z \right\rangle \\ &\quad + \frac{1}{\alpha_n} \langle (I - \alpha_n A)y_n - (I - \alpha_n A)x_n, y_n - z \rangle \\ &= -\frac{1}{\alpha_n} \left\langle \left(I - \frac{1}{t_n} \int_0^{t_n} T(s) ds \right) y_n - \left(I - \frac{1}{t_n} \int_0^{t_n} T(s) ds \right) z, y_n - z \right\rangle \\ &\quad + \left\langle A \left(I - \frac{1}{t_n} \int_0^{t_n} T(s) ds \right) y_n, y_n - z \right\rangle \\ &\quad + \frac{1}{\alpha_n} \langle (I - \alpha_n A)y_n - (I - \alpha_n A)x_n, y_n - z \rangle. \end{aligned} \quad (2.18)$$

Using Lemma 2.1, we have from (2.18) that

$$\begin{aligned} \langle (A - \gamma f)x_n, y_n - z \rangle &\leq \left\langle A \left(I - \frac{1}{t_n} \int_0^{t_n} T(s) ds \right) y_n, y_n - z \right\rangle + \frac{1}{\alpha_n} \langle (I - \alpha_n A)(y_n - x_n), y_n - z \rangle \\ &\leq \left\langle A \left(I - \frac{1}{t_n} \int_0^{t_n} T(s) ds \right) y_n, y_n - z \right\rangle + \frac{1}{\alpha_n} \beta_n \|\gamma f(x_n) - A(x_n)\| \|y_n - z\|. \end{aligned} \quad (2.19)$$

Now replacing n in (2.19) with n_k and letting $k \rightarrow \infty$, we notice that

$$\left(I - \frac{1}{t_{n_k}} \int_0^{t_{n_k}} T(s) ds \right) y_{n_k} \rightarrow 0, \quad (2.20)$$

and from condition $\beta_n = o(\alpha_n)$ and boundedness of $\{x_n\}$, we have

$$\frac{1}{\alpha_{n_k}} \beta_{n_k} \|\gamma f(x_{n_k}) - A(x_{n_k})\| \|y_{n_k} - z\| \longrightarrow 0. \quad (2.21)$$

For $x^* \in F(\mathcal{J})$, we obtain

$$\langle (A - \gamma f)x^*, x^* - z \rangle \leq 0. \quad (2.22)$$

From [9, Theorem 3.2], we know that the solution of the variational inequality (2.14) is unique. That is, $x^* \in F(\mathcal{J})$ is a unique solution of (2.14).

(iv)

$$\limsup_{n \rightarrow \infty} \left\langle \frac{1}{t_n} \int_0^{t_n} T(s)y_n ds - x^*, \gamma f(x^*) - Ax^* \right\rangle \leq 0, \quad (2.23)$$

where x^* is obtained in step (iii).

To see this, there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left\langle \frac{1}{t_n} \int_0^{t_n} T(s)y_n ds - x^*, \gamma f(x^*) - Ax^* \right\rangle \\ &= \lim_{i \rightarrow \infty} \left\langle \frac{1}{t_{n_i}} \int_0^{t_{n_i}} T(s)y_{n_i} ds - x^*, \gamma f(x^*) - Ax^* \right\rangle, \end{aligned} \quad (2.24)$$

we may also assume that $x_{n_i} \rightharpoonup z$, then $(1/t_{n_i}) \int_0^{t_{n_i}} T(s)y_{n_i} ds \rightharpoonup z$, note from step (ii) that $z \in F(\mathcal{J})$ in virtue of Lemma 1.2. It follows from the variational inequality (2.14) that

$$\limsup_{n \rightarrow \infty} \left\langle \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds - x^*, \gamma f(x^*) - Ax^* \right\rangle = \langle z - x^*, \gamma f(x^*) - Ax^* \rangle \leq 0. \quad (2.25)$$

So (2.23) holds thank to (2.14).

(v) $x_n \rightarrow x^*$ ($n \rightarrow \infty$).

Finally, we will prove $x_n \rightarrow x^*$. Since

$$\begin{aligned} \|y_n - x^*\|^2 &= \|(I - \beta_n A)(x_n - x^*) + \beta_n(\gamma f(x_n) - Ax^*)\|^2 \\ &\leq \|I - \beta_n A\| \|x_n - x^*\|^2 + \beta_n \|\gamma f(x_n) - Ax^*\|^2 \\ &\leq (1 - \beta_n \bar{\gamma}) \|x_n - x^*\|^2 + \beta_n \|\gamma f(x_n) - Ax^*\|^2. \end{aligned} \quad (2.26)$$

Next, we calculate

$$\begin{aligned}
\|x_n - x^*\|^2 &= \left\| (I - \alpha_n A) \left(\frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^* \right) + \alpha_n (\gamma f(x_n) - Ax^*) \right\|^2 \\
&= \left\| (I - \alpha_n A) \left(\frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^* \right) \right\|^2 + \alpha_n^2 \|\gamma f(x_n) - Ax^*\|^2 \\
&\quad + 2\alpha_n \left\langle (I - \alpha_n A) \left(\frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^* \right), \gamma f(x_n) - Ax^* \right\rangle \tag{2.27} \\
&\leq (1 - \alpha_n \bar{\gamma})^2 \|y_n - x^*\|^2 + \alpha_n^2 \|\gamma f(x_n) - Ax^*\|^2 \\
&\quad + 2\alpha_n \left\langle (I - \alpha_n A) \left(\frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^* \right), \gamma f(x_n) - Ax^* \right\rangle.
\end{aligned}$$

Thus it follows from (2.26) that

$$\begin{aligned}
\|x_n - x^*\|^2 &\leq (1 - \alpha_n \bar{\gamma})^2 (1 - \beta_n \bar{\gamma}) \|x_n - x^*\|^2 + (1 - \alpha_n \bar{\gamma})^2 \beta_n \|\gamma f(x_n) - Ax^*\|^2 \\
&\quad + \alpha_n^2 \|\gamma f(x_n) - Ax^*\|^2 + 2\alpha_n \left\langle \frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^*, \gamma f(x_n) - Ax^* \right\rangle \\
&\quad - 2\alpha_n^2 \left\langle A \left(\frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^* \right), \gamma f(x_n) - Ax^* \right\rangle \\
&\leq (1 - \alpha_n \bar{\gamma})^2 (1 - \beta_n \bar{\gamma}) \|x_n - x^*\|^2 + (1 - \alpha_n \bar{\gamma})^2 \beta_n \|\gamma f(x_n) - Ax^*\|^2 \\
&\quad + \alpha_n^2 \|\gamma f(x_n) - Ax^*\|^2 + 2\alpha_n \gamma \left\langle \frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^*, f(x_n) - f(x^*) \right\rangle \\
&\quad + 2\alpha_n \left\langle \frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^*, \gamma f(x^*) - Ax^* \right\rangle \\
&\quad - 2\alpha_n^2 \left\langle A \left(\frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^* \right), \gamma f(x_n) - Ax^* \right\rangle \\
&\leq (1 - \alpha_n \bar{\gamma})^2 (1 - \beta_n \bar{\gamma}) \|x_n - x^*\|^2 + 2\alpha_n \gamma \alpha \|y_n - x^*\| \|x_n - x^*\| \\
&\quad + (1 - \alpha_n \bar{\gamma})^2 \beta_n \|\gamma f(x_n) - Ax^*\|^2 \\
&\quad + \alpha_n \left[2 \left\langle \frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^*, \gamma f(x^*) - Ax^* \right\rangle \right. \\
&\quad \left. + \alpha_n \left(\|\gamma f(x_n) - Ax^*\|^2 + 2 \left\| A \left(\frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^* \right) \right\| \cdot \|\gamma f(x_n) - Ax^*\| \right) \right]
\end{aligned}$$

$$\begin{aligned}
&\leq (1 - \alpha_n \bar{\gamma})^2 (1 - \beta_n \bar{\gamma}) \|x_n - x^*\|^2 + 2\alpha_n \gamma \alpha [1 - \beta_n (\bar{\gamma} - \gamma \alpha)] \|x_n - x^*\|^2 \\
&\quad + 2\alpha_n \gamma \alpha \beta_n \|\gamma f(x^*) - Ax^*\| \|x_n - x^*\| + (1 - \alpha_n \bar{\gamma})^2 \beta_n \|\gamma f(x_n) - Ax^*\|^2 \\
&\quad + \alpha_n \left[2 \left\langle \frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^*, \gamma f(x^*) - Ax^* \right\rangle \right. \\
&\quad \quad \left. + \alpha_n (\|\gamma f(x_n) - Ax^*\|^2 + 2 \left\| A \left(\frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^* \right) \right\| \cdot \|\gamma f(x_n) - Ax^*\|) \right] \\
&= (1 - \alpha_n \bar{\gamma})^2 (1 - \beta_n \bar{\gamma}) \|x_n - x^*\|^2 + 2\alpha_n \gamma \alpha (1 - \beta_n \bar{\gamma}) \|x_n - x^*\| \\
&\quad + 2\alpha_n \beta_n \alpha^2 \gamma^2 \|x_n - x^*\|^2 + 2\alpha_n \gamma \alpha \beta_n \|\gamma f(x^*) - Ax^*\| \|x_n - x^*\| \\
&\quad + (1 - \alpha_n \bar{\gamma})^2 \beta_n \|\gamma f(x_n) - Ax^*\|^2 \\
&\quad + \alpha_n \left[2 \left\langle \frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^*, \gamma f(x^*) - Ax^* \right\rangle \right. \\
&\quad \quad \left. + \alpha_n \left(\|\gamma f(x_n) - Ax^*\|^2 + 2 \left\| A \left(\frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^* \right) \right\| \cdot \|\gamma f(x_n) - Ax^*\| \right) \right] \\
&= [(1 - \alpha_n \bar{\gamma})^2 + 2\alpha_n \gamma \alpha] (1 - \beta_n \bar{\gamma}) \|x_n - x^*\|^2 + 2\alpha_n \beta_n \alpha^2 \gamma^2 \|x_n - x^*\|^2 \\
&\quad + 2\alpha_n \gamma \alpha \beta_n \|\gamma f(x^*) - Ax^*\| \|x_n - x^*\| + (1 - \alpha_n \bar{\gamma})^2 \beta_n \|\gamma f(x_n) - Ax^*\|^2 \\
&\quad + \alpha_n \left[2 \left\langle \frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^*, \gamma f(x^*) - Ax^* \right\rangle \right. \\
&\quad \quad \left. + \alpha_n \left(\|\gamma f(x_n) - Ax^*\|^2 + 2 \left\| A \left(\frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^* \right) \right\| \cdot \|\gamma f(x_n) - Ax^*\| \right) \right] \\
&< [(1 - \alpha_n \bar{\gamma})^2 + 2\alpha_n \gamma \alpha] \|x_n - x^*\|^2 + 2\alpha_n \beta_n \alpha^2 \gamma^2 \|x_n - x^*\|^2 \\
&\quad + 2\alpha_n \gamma \alpha \beta_n \|\gamma f(x^*) - Ax^*\| \|x_n - x^*\| + (1 - \alpha_n \bar{\gamma})^2 \beta_n \|\gamma f(x_n) - Ax^*\|^2 \\
&\quad + \alpha_n \left[2 \left\langle \frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^*, \gamma f(x^*) - Ax^* \right\rangle \right. \\
&\quad \quad \left. + \alpha_n \left(\|\gamma f(x_n) - Ax^*\|^2 + 2\alpha_n \left\| A \left(\frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^* \right) \right\| \right. \right. \\
&\quad \quad \quad \left. \left. \cdot \|\gamma f(x_n) - Ax^*\| \right) \right] \\
&= [1 - 2(\bar{\gamma} - \gamma \alpha) \alpha_n] \|x_n - x^*\|^2 + 2\alpha_n \beta_n \alpha^2 \gamma^2 \|x_n - x^*\|^2 \\
&\quad + 2\alpha_n \gamma \alpha \beta_n \|\gamma f(x^*) - Ax^*\| \|x_n - x^*\| + (1 - \alpha_n \bar{\gamma})^2 \beta_n \|\gamma f(x_n) - Ax^*\|^2 \\
&\quad + \alpha_n \left[2 \left\langle \frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^*, \gamma f(x^*) - Ax^* \right\rangle \right. \\
&\quad \quad \left. + \alpha_n \left(\|\gamma f(x_n) - Ax^*\|^2 + 2 \left\| A \left(\frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^* \right) \right\| \right. \right. \\
&\quad \quad \quad \left. \left. \cdot \|\gamma f(x_n) - Ax^*\| + \bar{\gamma}^2 \|x_n - x^*\|^2 \right) \right].
\end{aligned}$$

(2.28)

Thus

$$\begin{aligned}
2(\bar{\gamma} - \gamma\alpha)\|x_n - x^*\|^2 &\leq 2\beta_n\alpha^2\gamma^2\|x_n - x^*\|^2 + 2\gamma\alpha\beta_n\|\gamma f(x^*) - Ax^*\| \cdot \|x_n - x^*\| \\
&\quad + (1 - \alpha_n\bar{\gamma})^2 \frac{\beta_n}{\alpha_n} \|\gamma f(x_n) - Ax^*\|^2 \\
&\quad + \left[2 \left\langle \frac{1}{t_n} \int_0^{t_n} T(s)y_n ds - x^*, \gamma f(x^*) - Ax^* \right\rangle \right. \\
&\quad \left. + \alpha_n \left(\|\gamma f(x_n) - Ax^*\|^2 + 2 \left\| A \left(\frac{1}{t_n} \int_0^{t_n} T(s)y_n ds - x^* \right) \right\| \right. \right. \\
&\quad \quad \left. \left. \cdot \|\gamma f(x_n) - Ax^*\| + \gamma^2 \|x_n - x^*\|^2 \right) \right] \\
&\leq \beta_n [2\alpha^2\gamma^2\|x_n - x^*\|^2 + 2\gamma\alpha\|\gamma f(x^*) - Ax^*\| \cdot \|x_n - x^*\|] \\
&\quad + \frac{\beta_n}{\alpha_n} \|\gamma f(x_n) - Ax^*\|^2 \\
&\quad + \left[2 \left\langle \frac{1}{t_n} \int_0^{t_n} T(s)y_n ds - x^*, \gamma f(x^*) - Ax^* \right\rangle \right. \\
&\quad \left. + \alpha_n \left(\|\gamma f(x_n) - Ax^*\|^2 + 2 \left\| A \left(\frac{1}{t_n} \int_0^{t_n} T(s)y_n ds - x^* \right) \right\| \right. \right. \\
&\quad \quad \left. \left. \cdot \|\gamma f(x_n) - Ax^*\| + \gamma^2 \|x_n - x^*\|^2 \right) \right]. \tag{2.29}
\end{aligned}$$

Since $\{x_n\}$ is bounded, we can take a constant $L_1, L_2, L_3 > 0$ such that

$$\begin{aligned}
L_1 &\geq 2\alpha^2\gamma^2\|x_n - x^*\|^2 + 2\gamma\alpha\|\gamma f(x^*) - Ax^*\| \cdot \|x_n - x^*\|, \\
L_2 &\geq \|\gamma f(x_n) - Ax^*\|^2, \\
L_3 &\geq \|\gamma f(x_n) - Ax^*\|^2 + 2 \left\| A \left(\frac{1}{t_n} \int_0^{t_n} T(s)y_n ds - x^* \right) \right\| \cdot \|\gamma f(x_n) - Ax^*\| + \gamma^2 \|x_n - x^*\|^2 \tag{2.30}
\end{aligned}$$

for all $n \geq 0$. It then follows from (2.29) that

$$2(\bar{\gamma} - \gamma\alpha)\|x_n - x^*\|^2 \leq \beta_n L_1 + \frac{\beta_n}{\alpha_n} L_2 + 2 \left\langle \frac{1}{t_n} \int_0^{t_n} T(s)y_n ds - x^*, \gamma f(x^*) - Ax^* \right\rangle + \alpha_n L_3. \tag{2.31}$$

Then

$$\begin{aligned} & 2(\bar{\gamma} - \gamma\alpha) \limsup_{n \rightarrow \infty} \|x_n - x^*\|^2 \\ & \leq \limsup_{n \rightarrow \infty} \left(\beta_n L_1 + \frac{\beta_n}{\alpha_n} L_2 + \alpha_n L_3 \right) + 2 \limsup_{n \rightarrow \infty} \left\langle \frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - x^*, \gamma f(x^*) - Ax^* \right\rangle. \end{aligned} \quad (2.32)$$

From condition $\alpha_n \rightarrow 0$, $\beta_n = o(\alpha_n)$ and (2.23), we conclude that

$$\limsup_{n \rightarrow \infty} \|x_n - x^*\|^2 \leq 0. \quad (2.33)$$

So $x_n \rightarrow x^*$. This completes the proof of the Theorem 2.2. \square

It follows from the above proof that Theorem 2.2 is valid for nonexpansive mappings. Thus, we have that Corollaries 2.3 and 2.4 are two special cases of Theorem 2.2.

Corollary 2.3. *Let T be a nonexpansive mapping from nonempty closed convex subset C of a Hilbert space H to C , $\{x_n\}$ is generated by the following algorithm:*

$$\begin{aligned} x_n &= (I - \alpha_n A) T y_n + \alpha_n \gamma f(x_n), \\ y_n &= (I - \beta_n A) x_n + \beta_n \gamma f(x_n), \end{aligned} \quad (2.34)$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ are real sequences such that

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \beta_n = o(\alpha_n), \quad (2.35)$$

then for any $0 < \gamma < \bar{\gamma}/\alpha$, the sequence $\{x_n\}$ above converges strongly to the unique solution $x^* \in F(T)$ of the variational inequality $\langle (\gamma f - A)x^*, z - x^* \rangle \leq 0$ for all $z \in F(T)$.

Corollary 2.4. *Let T be a nonexpansive mapping from nonempty closed convex subset C of a Hilbert space H to C , $\{x_n\}$ is generated by the following algorithm:*

$$\begin{aligned} x_n &= (I - \alpha_n A) T y_n + \alpha_n \gamma f(x_n), \\ y_n &= (I - \beta_n A) x_n + \beta_n \gamma f(x_n), \end{aligned} \quad (2.36)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the following condition: $\lim_{n \rightarrow \infty} \alpha_n = 0$, then the sequence $\{x_n\}$ converges strongly to the unique solution $x^* \in F(T)$ of the variational inequality $\langle (I - f)x^*, x^* - z \rangle \leq 0$ for all $z \in F(T)$.

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