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Research Article

Extensions of Minimization Theorems and Fixed Point Theorems on a Quasimetric Space

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We introduce the new concepts of e-distance, e-type mapping with respect to some e-distance and S-complete quasimetric space, and prove minimization theorems, fixed point theorems, and variational principles on an S-complete quasimetric space. We also give some examples of quasimetrics, e-distances, and e-type mapping with respect to some e-distance. Our results extend, improve, and unify many known results due to Caristi, Ekeland, Ćirić, Kada-Suzuki-Takahashi, Ume, and others.

1. Introduction

Caristi [1] proved a fixed point theorem on complete metric spaces which generalized the Banach contraction principle. Ekeland [2] also obtained a minimization theorem, often called the ε-variational principle for a proper lower semicontinuous function, bounded from below, on complete metric spaces. The two theorems are very useful tools in nonlinear analysis, control theory, economic theory, and global analysis. Later, Takahashi [3] proved the following minimization theorem. Let X be a complete metric space and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Suppose that, for each $u \in X$ with $f(u) > \inf_{x\in X} f(x)$, there exists $v \in X$ such that $v \neq u$ and $f(v) + d(u,v) \le f(u)$. Then, there exists $x_0 \in X$ such that $f(x_0) = \inf_{x\in X} f(x)$. Many authors [3–5] have generalized and extended this minimization theorem in complete metric spaces. In 1996, Kada et al. [4] introduced the concept of w-distance on a metric space as follows. Let X be a metric space with metric d. Then, a function $p : X \times X \to [0,\infty)$ is called a w-distance on X if the following are satisfied:

(1) $p(x,z) \le p(x,y) + p(y,z)$ for any $x,y,z \in X$,

(2) for any $x \in X$, $p(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous,

(3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $d(x,y) \leq \epsilon$.

By using the w-distance, they improved the Caristi fixed point theorem, Ekelend variational principle, and Takahashi's minimization theorem on complete metric spaces.

In this paper, we introduce the new concepts of e-distance, e-type mapping with respect to some e-distance and S-complete quasimetric space, and prove minimization theorems, fixed point theorems, and variational principles on an S-complete quasimetric space. We also give some examples of quasimetrics, e-distances, and e-type mapping with respect to some e-distance. Our results extend, improve, and unify many known results due to Caristi, Ekeland, Ćirić, Kada-Suzuki-Takahashi, Ume, and others.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers, by $\mathbb{R}^+$ the set of all nonnegative real numbers, and by $\mathbb{R}$ the set of all real numbers.

Definition 2.1. A pair (X,d) of a set X and a mapping d from $X \times X$ into $\mathbb{R}$ is said to be a quasimetric space iff for all $x,y,z \in X$,

(1) $0 \leq d(x,y)$ and $d(x,y) = 0$ iff $x = y$,

(2) $d(x,z) \leq d(x,y) + d(y,z)$.

Definition 2.2. Let (X,d) be a quasimetric space and let S be a mapping from $X \times X \times X$ to $\mathbb{R}^+$. Then, S is said to be an e-distance on X iff

(A1) $S(x,y,z) \leq S(x,y,a) + S(x,z,a) + S(y,z,a)$ for all $x,y,z,a \in X$,

(A2) for each $x \in X$, $S(x,y,y)$ is a lower semicontinuous at y in X,

(A3) $S(x,y,z) = 0$ implies $x = y$.

Definition 2.3. Let (X,d) be a quasimetric space and let $S : X \times X \times X \to \mathbb{R}^+$ be an e-distance on X. Then, a function $H : X \times X \times X \to \mathbb{R}^+$ is called an e-type mapping on X with respect to S if the followings are satisfied:

(B1) $H(x,y,z) \leq H(x,y,a) + H(x,z,a) + H(y,z,a)$ for all $x,y,z,a \in X$,

(B2) for each $x \in X$, $H(x,y,y)$ is a lower semicontinuous at y in X.

(B3) for an arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that $H(b,x,y) \leq \delta$, $H(b,x,z) \leq \delta$, and $H(b,y,z) \leq \delta$ imply $S(x,y,z) \leq \varepsilon$.

Remark 2.4. If S is an e-distance on X and

$$S(x,y,z) = S(x,z,y) = S(y,x,z) = S(y,z,x) = S(z,x,y) = S(z,y,x) \tag{2.1}$$

for all $x,y,z \in X$, then clearly S is an e-type mapping on X with respect to S.

Definition 2.5. Let (X,d) be a quasimetric space and let $S : X \times X \times X \to \mathbb{R}^+$ be an e-distance on X.

(1) A sequence $\{x_n\}$ in X is said to be an S-Cauchy sequence iff for every $\varepsilon > 0$, there exists M in $\mathbb{N}$ such that $S(x_n,x_m,x_p) < \varepsilon$ for all $p > m > n > M$.

(2) A quasimetric space X is S-complete iff S-Cauchy sequence in X is convergent.

We give some examples of quasimetrics, e-distances, and e-type mapping with respect to some e-distance.

Example 2.6. Let $q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a mapping defined as follows:

$$q(x,y) = 0 \quad \text{if } x = y, \qquad q(x,y) = \frac{1}{2} \quad \text{if } y < x, \qquad q(x,y) = \frac{1}{3} \quad \text{if } x < y. \tag{2.2}$$

Then, clearly q is a quasimetric but not a metric.

Example 2.7. Define a mapping $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ as follows:

$$q(x,y) = \begin{cases} 2x + y, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases} \tag{2.3}$$

Then, clearly q is a quasimetric but not a metric. Also for each $x \in \mathbb{R}^+$, $q(x,y)$ is a lower semicontinuous at y in $\mathbb{R}^+$ and for each $y \in \mathbb{R}^+$, $q(x,y)$ is a lower semicontinuous at x in $\mathbb{R}^+$.

Example 2.8. Let q be as in Example 2.7. Define mappings $S, H : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ as follows:

$$\begin{gathered} S(x,y,z) = \max\{q(x,y), q(x,z), q(y,z)\}, \\ H(x,y,z) = \max\{q(x,y), q(y,x), q(x,z), q(z,x), q(y,z), q(z,y)\}. \end{gathered} \tag{2.4}$$

Then, clearly S and H are e-distance on $\mathbb{R}^+$ and H is an e-type mapping on $\mathbb{R}^+$ with respect to S. Also, H is an e-type mapping on $\mathbb{R}^+$ with respect to H.

Example 2.9. Suppose that $X = \mathbb{R}$ is a metric space with the usual metric. Let $q : X \times X \to \mathbb{R}^+$ be a mapping such that

$$q(x,y) = \max\left\{\left|\frac{1}{3}x - y\right|, \frac{1}{3}|x - y|\right\}. \tag{2.5}$$

Let mappings $S, H : X \times X \times X \to \mathbb{R}^+$ be defined by

$$\begin{gathered} S(x,y,z) = \max\{|x-y|, |x-z|, |y-z|\}, \\ H(x,y,z) = \max\{q(x,y), q(x,z)\}. \end{gathered} \tag{2.6}$$

Then, clearly, q is a w-distance, but q is neither a quasimetric nor a metric, S and H are e-distance on X, and H is an e-type mapping on X with respect to S.

Example 2.10. Let $q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ and $S, H : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ be defined by

$$\begin{gathered} q(x,y) = (x-y)^2, \qquad S(x,y,z) = \max\{q(x,y), q(x,z), q(y,z)\}, \\ H(x,y,z) = q(x,y) + q(x,z) + q(y,z). \end{gathered} \tag{2.7}$$

Then, clearly, q is neither metric nor w-distance, but S and H are e-distance on R and H is an e-type mapping on R with respect to S.

Example 2.11. Let (X, d) be a metric space with a w-distance p in [4]. Define $H, S : X \times X \times X \to \mathbb{R}^+$ as follows:

$$S(x, y, z) = \max\{d(x, y), d(x, z), d(y, z)\},$$
$$H(x, y, z) = \max\{p(x, y), p(x, z)\}. \tag{2.8}$$

Then, clearly S is an e-distance on X, S is an e-type mapping on X with respect to S, and H is an e-type mapping on X with respect to S.

Example 2.12. Let $X = \mathbb{R}^+$ be a metric space with the usual metric. Define $h : X \times X \to X$ and $H, S, G : X \times X \times X \to X$ as follows:

$$h(x, y) = x, \qquad H(x, y, z) = h(y, z), \qquad S(x, y, z) = \max\{h(x, y), h(y, z)\},$$
$$G(x, y, z) = \max\{h(x, y), h(y, z), h(z, x)\}. \tag{2.9}$$

Then,

(1) h is neither metric nor quasimetric since $h(0, 1) = h(0, 2) = 0$ and $1 \neq 2$. Also, h is not w-distance on X. In fact, if h is w-distance on X, then for $\varepsilon = 1$, there exists $\delta > 0$ such that $h(z, x) = z \le \delta$ and $h(z, y) = z \le \delta$ imply $d(x, y) = |x - y| \le 1$. Putting $z = \delta/2$, $x = 1$, and $y = 4$ in the above inequalities, we have $1 \ge |1 - 4| = 3 > 1$, which is a contradiction. Thus, h is not w-distance on X.

(2) S is an e-distance on X, S is an e-type mapping on X with respect to S, and H is an e-type mapping on X with respect to S; but H is not an e-distance on X since H is not satisfied (A3) of Definition 2.2.

(3) G is an e-distance on X, G is an e-type mapping on X with respect to G, and G is an e-type mapping on X with respect to S.

(4) X is a quasimetric and S-complete.

Remark 2.13. If (X, d) is a metric space, then a mapping S generated by d in Example 2.11 is an e-distance and an e-type mapping with respect to S. Thus, for a given metric d on X, we can find an e-distance and an e-type mapping with respect to some e-distance; but there exists a function h in Example 2.12 which is not all of a metric, quasimetric, and w-distance such that mapping S generated by a function h is an e-distance and H generated by a function h is an e-type mapping with respect to S. In this sense, an e-distance and an e-type mapping with respect to some e-distance are proper extension of metric.

The following lemma plays important role to prove minimization theorems, fixed point theorems, and variational inequalities.

Lemma 2.14. *Let X be a quasimetric space, let S be an e-distance on X, let X be a S-complete, and let H be an e-type mapping on X with respect to S. Suppose that $g : X \times X \to X$ is a function such that*

(i) $\max\{H(x, z, g(x, z)), H(x, y, g(x, z)), H(y, z, g(x, z))\} \le H(x, y, g(x, y)) + H(y, z, g(y, z))$ *for all* $x, y, z \in X$,

(ii) *for each* $x \in X$, $H(x, y, g(x, y))$ *is a lower semicontinuous at* y,

(iii) $H(x, z, g(x, z)) = H(x, y, g(x, z)) = H(y, z, g(x, z)) = 0$ *imply* $y = z$.

Assume that $f : X \to (-\infty, \infty]$ *is a proper lower semicontinuous function bounded from below. Suppose that for any* $u \in X$ *with* $\inf_{x \in X} f(x) < f(u)$, *there exists* $v \in X$ *with* $v \neq u$ *such that*

$$f(v) + H(u, v, g(u, v)) \le f(u). \tag{2.10}$$

Then there exists $x_0 \in X$ *such that* $\inf_{x \in X} f(x) = f(x_0)$.

Proof. Suppose $\inf_{x \in X} f(x) < f(y)$ for every $y \in X$. For each $y \in X$, let

$$S(y) = \{v \in X \mid H(y, v, g(y, v)) \le f(y) - f(v)\}. \tag{2.11}$$

Then, by hypothesis and (2.11), $S(y)$ is nonempty for each $y \in X$. From condition (i) and (2.11), we obtain

$$S(v) \subseteq S(y) \quad \text{for each } v \in S(y). \tag{2.12}$$

For each $y \in X$, let

$$c(y) = \inf\{f(v) \mid v \in S(y)\}. \tag{2.13}$$

Since f is proper, there exists $u \in X$ such that $f(u) < \infty$. Thus, from (2.12) and (2.13) there exists a sequence $\{u_n\}_{n=1}^{\infty}$ in X such that $u_1 = u$, $u_{n+1} \in S(u_n)$, $S(u_n) \subseteq S(u)$ and

$$f(u_{n+1}) < c(u_n) + \frac{1}{n} \tag{2.14}$$

for all $n \in \mathbb{N}$. From (2.11), (2.13), and (2.14), we have

$$H(u_n, u_{n+1}, g(u_n, u_{n+1})) \le f(u_n) - f(u_{n+1}), \tag{2.15}$$

$$f(u_{n+1}) - \frac{1}{n} < c(u_n) \le f(u_{n+1}). \tag{2.16}$$

By (2.15), $\{f(u_n)\}_{n=1}^{\infty}$ is a nonincreasing sequence of real numbers and so it converges. Therefore, from (2.16), there is some $\beta \in \mathbb{R}$ such that

$$\beta = \lim_{n \to \infty} c(u_n) = \lim_{n \to \infty} f(u_n). \tag{2.17}$$

Let n, p, t, and $r \in \mathbb{N}$. Then, by condition (i) and (2.15), we obtain

$$H(u_n, u_{n+p}, g(u_n, u_{n+p+t})) \le f(u_n) - f(u_{n+p+t}), \tag{2.18}$$

$$H(u_n, u_{n+p+t}, g(u_n, u_{n+p+t})) \le f(u_n) - f(u_{n+p+t}), \tag{2.19}$$

$$H(u_{n+p}, u_{n+p+t}, g(u_n, u_{n+p+t})) \le f(u_n) - f(u_{n+p+t}). \tag{2.20}$$

From (2.18), (2.19), and (2.20), we have

$$\begin{aligned}
&H(u_n, u_{n+p}, u_{n+p+t}) \\
&\quad \le H(u_n, u_{n+p}, g(u_n, u_{n+p+t})) + H(u_n, u_{n+p+t}, g(u_n, u_{n+p+t})) + H(u_{n+p}, u_{n+p+t}, g(u_n, u_{n+p+t})) \\
&\quad \le 3\{f(u_n) - f(u_{n+p+t})\}.
\end{aligned} \tag{2.21}$$

From (2.21), we obtain the following inequalities:

$$\begin{aligned}
H(u_n, u_{n+p}, u_{n+p+t+r}) &\le 3\{f(u_n) - f(u_{n+p+t+r})\}, \\
H(u_n, u_{n+p+t}, u_{n+p+t+r}) &\le 3\{f(u_n) - f(u_{n+p+t+r})\}.
\end{aligned} \tag{2.22}$$

By (2.21), (2.22), as well as Definitions 2.3 and 2.5, $\{u_n\}$ is an S-Cauchy in X. Since X is an S-complete, there exists $u_0 \in X$ such that

$$\lim_{n\to\infty} u_n = u_0. \tag{2.23}$$

From (2.17), (2.18), (2.23), condition (ii), and hypothesis, we have

$$H(u_n, u_0, g(u_n, u_0)) \le \liminf_{p\to\infty} H(u_n, u_{n+p}, g(u_n, u_{n+p+t+r})), \tag{2.24}$$

$$f(u_0) \le \lim_{p\to\infty} f(u_{n+p+t+r}) = \beta. \tag{2.25}$$

From (2.18), (2.24), and (2.25), we have

$$\begin{aligned}
f(u_0) \le \beta &= \lim_{p\to\infty} \sup f(u_{n+p+t+r}) \\
&\le \lim_{p\to\infty} \sup \{f(u_n) - H(u_n, u_{n+p}, g(u_n, u_{n+p+t}))\} \\
&= f(u_n) + \lim_{p\to\infty} \sup \{-H(u_n, u_{n+p}, g(u_n, u_{n+p+t}))\} \\
&= f(u_n) - \lim_{p\to\infty} \inf \{H(u_n, u_{n+p}, g(u_n, u_{n+p+t}))\} \\
&\le f(u_n) - H(u_n, u_0, g(u_n, u_0)).
\end{aligned} \tag{2.26}$$

From (2.11), (2.13), and (2.26), it follows that

$$u_0 \in S(u_n) \text{ and hence } c(u_n) \le f(u_0) \quad \forall\, n \in \mathbb{N}. \tag{2.27}$$

Taking the limit in inequality (2.27) when n tends to infinity, we have

$$\lim_{n\to\infty} c(u_n) \le f(u_0). \tag{2.28}$$

From (2.17), (2.25), and (2.28), we have

$$\beta = f(u_0). \tag{2.29}$$

On the other hand, by hypothesis, (2.11), and (2.23), we have the following property:

$$\exists\, v_1 \in X - \{u_0\} \text{ satisfying } v_1 \in S(u_0). \tag{2.30}$$

From (2.12), (2.13), (2.27), and (2.30), we have

$$v_1 \in S(x_n) \quad \forall\, n \in \mathbb{N}, \tag{2.31}$$

$$c(u_n) \le f(v_1). \tag{2.32}$$

From (2.11), (2.17), (2.29), (2.30), and (2.32), it follows that

$$\beta = f(v_1). \tag{2.33}$$

From (2.11), (2.29), (2.30), and (2.33), we have

$$H(u_0, v_1, g(u_0, v_1)) = 0. \tag{2.34}$$

By method similar to (2.30)~(2.34),

$$\exists\, v_2 \in X - \{v_1\} \text{ such that } H(v_1, v_2, g(v_2, v_1)) = 0. \tag{2.35}$$

From (2.34), (2.35), and conditions (i), (iii), we obtain

$$v_1 = v_2. \tag{2.36}$$

This is a contradiction from (2.35). The proof is complete. □

3. Minimization theorems and its applications

Theorem 3.1. *Let X and S be as in Lemma 2.14 and let $g : X \times X \to X$ be a function such that*

(iv) $\max\{S(x,z,g(x,z)), S(x,y,g(x,z)), S(y,z,g(x,z))\} \le S(x,y,g(x,y)) + S(y,z, g(y,z))$ *for all* $x, y, z \in X$,

(v) *for each* $x \in X$, $S(x,y,g(x,y))$ *is a lower semicontinuous at* $y \in X$,

(vi) $S(x,y,z) = S(x,z,y) = S(y,x,z) = S(y,z,x) = S(z,x,y) = S(z,y,x)$ *for all* $x, y, z \in X$,

(vii) $S(x,z,g(x,z)) = S(x,y,g(x,z)) = S(y,z,g(x,z)) = 0$ *imply* $y = z$.

Assume that $f : X \to (-\infty, \infty]$ *is a proper lower semicontinuous function bounded from below. Suppose that for each* $u \in X$ *with* $\inf_{x\in X} f(x) < f(u)$, *there exists* $v \in X$ *with* $v \neq u$ *such that*

$$f(v) + S(u, v, g(u, v)) \leq f(u). \tag{3.1}$$

Then, there exists $x_0 \in X$ *such that* $\inf_{x\in X} f(x) = f(x_0)$.

Proof. Let $H(x, y, z) = S(x, y, z)$ for all $x, y, z \in X$. Then, all conditions of Theorem 3.1 satisfy the suppositions in Lemma 2.14. Therefore, Theorem 3.1 follows from Lemma 2.14. □

Corollary 3.2 (see [4, Theorem 1]). *Let* (X, d) *be a complete metric space, and let* $f : X \to (-\infty, \infty]$ *be a proper lower semicontinuous function, bounded from below. Assume that there exists a* w*-distance* p *on* X *such that for any* $u \in X$ *with* $\inf_{x\in X} f(x) < f(u)$, *there exists* $v \in X$ *with* $v \neq u$ *and* $f(v) + p(u, v) \leq f(u)$. *Then, there exists* $x_0 \in X$ *such that* $\inf_{x\in X} f(x) = f(x_0)$.

Proof. Let $H(x, y, z) = \max\{p(x, y), p(x, z)\}$ for all $x, y, z \in X$ and

$$S(x, y, z) = \max\{d(x, y), d(x, z), d(y, z)\} \tag{3.2}$$

for all $x, y, z \in X$ and let $g(x, y) = y$ for all $x, y \in X$. Then, X, H, g, and f satisfy the suppositions in Lemma 2.14. Therefore, Corollary 3.2 follows from Lemma 2.14. □

The following example shows that Lemma 2.14 is more general than Corollary 3.2.

Example 3.3. Let X, h, H, and S be as in Example 2.12. Define $g : X \times X \to X$ and $f : X \to (-\infty, \infty]$ as follows:

$$g = h, \qquad f(x) = \begin{cases} 4x + 3, & \text{if } 0 \leq x < 4, \\ -1, & \text{if } x = 4, \\ 5x + 3, & \text{if } 4 < x. \end{cases} \tag{3.3}$$

It is clear that all of the conditions except inequality in Lemma 2.14 are satisfied. To show that inequality in Lemma 2.14 is satisfied, we need to consider several possible cases as follows.

(1) For $u = 0$ in X, there exists $v = 4$ in X such that

$$f(v) + H(u, v, g(u, v)) = f(v) + v = f(4) + 4 = (-1) + 4 = 3 = f(0) = f(u). \tag{3.4}$$

(2) For $u \in X$ with $0 < u < 4$, there exists $v \in (0, (3/4)u)$ such that

$$\begin{aligned} f(v) + H(u, v, g(u, v)) &= f(v) + v = 4v + 3 + v \\ &< 3u + 3 + u = 4u + 3 = f(u). \end{aligned} \tag{3.5}$$

(3) For $u \in X$ with $4 < u$, there exists $v \in (0,4)$ such that

$$\begin{aligned} f(v) + H(u,v,g(u,v)) &= f(v) + v = 4v + 3 + v \\ &< 5v + 3 < 5u + 3 = f(u). \end{aligned} \tag{3.6}$$

Hence, for $u \in X$ with $\inf_{x\in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$ such that

$$f(v) + H(u,v,g(u,v)) \leq f(u), \tag{3.7}$$

that is, inequality in Lemma 2.14 is satisfied. Thus, all of the conditions in Lemma 2.14 are satisfied and therefore, there exists $4 \in X$ such that $\inf_{x\in X} f(x) = f(4)$.

Remark 3.4. Since $H(u,v,g(u,v)) = H(u,v,u) = h(v,u)$ is not all of metric, quasimetric, and w-distance, Corollary 3.2 cannot be applicable. This means that Lemma 2.14 is a proper extension of Corollary 3.2.

The following theorem extend, improve, and unify many known results due to Caristi [1], Kada et al. [4], Takahashi [3], and Ume [5].

Theorem 3.5. *Let X, S, H, and g be as in Lemma 2.14. Assume that $f : X \to (-\infty,\infty]$ is a proper lower semicontinuous function bounded from below. Let T be a mapping from X into itself. Suppose that*

$$f(Tx) + H(x,Tx,g(x,Tx)) \leq f(x) \tag{3.8}$$

for every $x \in X$. Then, there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $H(x_0,Tx_0,g(x_0,Tx_0)) = 0$.

Proof. Since f is proper, there exists $u \in X$ such that $f(u) < \infty$. Let

$$Y = \{x \in X : f(x) \leq f(u)\}. \tag{3.9}$$

Then, since f is lower semicontinuous, Y is closed. Hence, Y is S-complete. Let $x \in Y$. Then, since

$$f(Tx) + H(x,Tx,g(x,Tx)) \leq f(x) \leq f(u), \tag{3.10}$$

we have $Tx \in Y$. So, Y is invariant under T. Assume that $Tx \neq x$ for every $x \in Y$. Then, by Lemma 2.14, there exists $v_0 \in Y$ such that $f(v_0) = \inf_{x\in Y} f(x)$. Since $f(Tv_0) + H(v_0,Tv_0,g(v_0,Tv_0)) \leq f(v_0)$, and $f(v_0) = \inf_{x\in Y} f(x)$, we have $f(Tv_0) = f(v_0) = \inf_{x\in Y} f(x)$ and $H(v_0,Tv_0,g(v_0,Tv_0)) = 0$. Similarly, we obtain $f(T^2v_0) = f(Tv_0) = \inf_{x\in Y} f(x)$ and $H(Tv_0,T^2v_0,g(Tv_0,T^2v_0)) = 0$. By (i) and (iii) of Lemma 2.14, $Tv_0 = T^2v_0$. This is a contradiction. Therefore, T has a fixed point x_0 in Y. Since $f(x_0) < \infty$ and

$$f(x_0) + H(x_0,x_0,g(x_0,x_0)) = f(Tx_0) + H(x_0,Tx_0,g(x_0,Tx_0)) \leq f(x_0), \tag{3.11}$$

we have $H(x_0,x_0,g(x_0,x_0)) = 0$. □

We give an example to support Theorem 3.5.

Example 3.6. Let X, g, h, H, and S be as in Example 3.3. Define $T : X \to X$ and $f : X \to (-\infty, \infty]$ as follows:

$$Tx = \frac{1}{2}x \quad \forall x \in X, \qquad f(x) = \begin{cases} 4x+1, & \text{if } 0 \leq x \leq 4, \\ 5x+1, & \text{if } 4 < x. \end{cases} \tag{3.12}$$

Clearly, f is a proper lower semicontinuous function, bounded from below. Now, we show that inequality in Theorem 3.5 is satisfied. There are several possible cases which we need to consider.

(1) For $x \in X$ with $0 \leq Tx = (1/2)x \leq 2$, we have

$$\begin{aligned} f(Tx) + H(x, Tx, g(x, Tx)) &= f(Tx) + Tx = f\left(\frac{1}{2}x\right) + \frac{1}{2}x \\ &= 4 \times \frac{1}{2}x + 1 + \frac{1}{2}x = \frac{5}{2}x + 1 \leq 4x + 1 = f(x). \end{aligned} \tag{3.13}$$

(2) For $x \in X$ with $2 < Tx = (1/2)x \leq 4$, we have

$$\begin{aligned} f(Tx) + H(x, Tx, g(x, Tx)) &= f(Tx) + Tx = f\left(\frac{1}{2}x\right) + \frac{1}{2}x \\ &= 4 \times \frac{1}{2}x + 1 + \frac{1}{2}x = \frac{5}{2}x + 1 \leq 5x + 1 = f(x). \end{aligned} \tag{3.14}$$

(3) For $x \in X$ with $4 < Tx = (1/2)x$, we have

$$\begin{aligned} f(Tx) + H(x, Tx, g(x, Tx)) &= f(Tx) + Tx \\ &= 5 \times \frac{1}{2}x + 1 + \frac{1}{2}x = 3x + 1 \leq 5x + 1 = f(x). \end{aligned} \tag{3.15}$$

Hence $f(Tx) + H(x, Tx, g(x, Tx)) \leqq f(x)$ for all $x \in X$. Thus, all of the conditions in Theorem 3.5 are satisfied and, therefore, there exists $0 \in X$ such that $T0 = 0$ and $H(0, T0, g(0, T0)) = 0$.

Remark 3.7. Since $H(u, v, g(u, v)) = H(u, v, u) = h(v, u)$ is neither metric nor w-distance, fixed point theorems of Caristi [1], Kada et al. [4], Takahashi [3], and Ume [5] cannot be applicable. Therefore, Theorem 3.5 is a proper extension of results due to Caristi [1], Kada et al. [4], Takahashi [3], and Ume [5].

Using methods similar to Theorems 3.1 and 3.5, we have the following corollary.

Corollary 3.8. *Let X, S, f, and g be as in Theorem 3.1. Let T be a mapping X into itself. Suppose that $f(Tx) + S(x, Tx, g(x, Tx)) \leq f(x)$ for every $x \in X$. Then, there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $S(x_0, Tx_0, g(x_0, Tx_0)) = 0$.*

The following theorem is a generalization of the corresponding results in [2–5].

Theorem 3.9. *Let X, S, H, and g be as in Lemma 2.14. Assume that $f : X \to (-\infty, \infty]$ is a proper lower semicontinuous function bounded from below. Then, the following (1) and (2) hold:*

(1) *for any $u \in X$ with $f(u) < \infty$, there exists $v \in X$ such that $f(v) \leq f(u)$ and $f(z) > f(v) - H(v, z, g(v, z))$ for every $z \in X$ with $z \neq v$;*

(2) *for any $\varepsilon > 0$ and $u \in X$ with $H(u, u, g(u, u)) = 0$ and $f(u) \leq \inf_{x \in X} f(x) + \varepsilon$, there exists $v \in X$ such that $f(v) \leq f(u)$, $H(u, v, g(u, v)) \leq 1$, and $f(z) > f(v) - \varepsilon H(v, z, g(v, z))$ for every $z \in X$ with $z \neq v$.*

Proof. (1) Let $u \in X$ be such that $f(u) < \infty$ and let $Y = \{x \in X : f(x) \leq f(u)\}$. Then, Y is nonempty closed and an S-complete. Hence, we may prove that there exists an element $v \in Y$ such that $f(z) > f(v) - H(v, z, g(v, z))$ for every $z \in X$ with $z \neq v$. Suppose not. Then, for every $x \in Y$, there exists $z \in X$ such that $z \neq x$ and $f(z) + H(x, z, g(x, z)) \leq f(x)$. Since $f(z) \leq f(x) \leq f(u)$, $z \in X$ is an element of Y. By Lemma 2.14, there exists $x_0 \in Y$ such that $f(x_0) = \inf_{x \in Y} f(x)$. Then, there exists $x_1 \in Y$ such that $x_1 \neq x_0$ and $f(x_1) + H(x_0, x_1, g(x_0, x_1)) \leq f(x_0)$. Hence, we have $f(x_1) = f(x_0)$ and $H(x_0, x_1, g(x_0, x_1)) = 0$. Similarly, there exists $x_2 \in Y$ such that $x_2 \neq x_1$ and $H(x_1, x_2, g(x_1, x_2)) = 0$. From (i) and (iii) of Lemma 2.14, $x_1 = x_2$. This is a contradiction. Therefore, there exists $v \in Y$ such that $f(z) > f(v) - H(v, z, g(v, z))$ for every $z \in X$ with $z \neq v$.

(2) Let $Z = \{x \in X : f(x) \leq f(u) - \varepsilon H(u, x, g(u, x))\}$. Then, Z is nonempty closed and an S-complete. As in the proof of (1), we have that there exists $v \in Z$ such that $f(z) > f(v) - \varepsilon H(v, z, g(v, z))$ for every $z \in X$ with $z \neq v$. On the other hand, since $v \in Z$, we have $f(v) \leq f(u) - \varepsilon H(u, v, g(u, v)) \leq f(u)$ and

$$H(u, v, g(u, v)) \leq \frac{1}{\varepsilon}\left[f(u) - f(v)\right] \leq \frac{1}{\varepsilon}\left[f(u) - \inf_{x \in X} f(x)\right] \leq \frac{1}{\varepsilon}\varepsilon = 1. \tag{3.16}$$

The proof is complete. □

Using methods similar to Theorems 3.1 and 3.9, we have the following corollary.

Corollary 3.10. *Let X, S and g be as in Theorem 3.1. Assume that $f : X \to (-\infty, \infty]$ is a proper lower semicontinuous function bounded from below. Then,*

(1) *for each $u \in X$ with $f(u) < \infty$, there exists $v \in X$ such that $f(v) \leq f(u)$ and $f(z) > f(v) - S(v, z, g(v, z))$ for every $z \in X$ with $z \neq v$;*

(2) *for each $\varepsilon > 0$ and $u \in X$ with $S(u, u, g(u, u)) = 0$ and $f(u) \leq \inf_{x \in X} f(x) + \varepsilon$, there exists $v \in X$ such that $f(v) \leq f(u)$, $S(u, v, g(u, v)) \leq 1$, and $f(z) > f(v) - \varepsilon S(v, z, g(v, z))$ for every $z \in X$ with $z \neq v$.*

The following is an example to support Theorem 3.9.

Example 3.11. Let X, g, h, H, S, and f be as in Example 3.6. Taking $v = 0$ in X, (1) and (2) of Theorem 3.9 hold.

Remark 3.12. Since $H(u, z, g(v, z)) = H(v, z, v) = h(z, v)$ is neither metric nor w-distance, theorems in [2–5] cannot be applicable. Therefore, Theorem 3.9 is a generalization of the corresponding results in [2–5].

4. Fixed point theorems

The following theorem is a generalization of the corresponding results in [4, 6, 7].

Theorem 4.1. *Let X, S, H and g be as in Lemma 2.14. Let T be a mapping from X into itself. Suppose that there exists $r \in [0,1)$ such that*

$$H(Tx, T^2x, g(Tx, T^2x)) \le rH(x, Tx, g(x, Tx)) \tag{4.1}$$

for every $x \in X$ and that

$$\inf\{H(x, y, g(x, y)) + H(x, Tx, g(x, Tx)) : x \in X\} > 0 \tag{4.2}$$

for every $y \in X$ with $y \ne Ty$. Then, there exists $z \in X$ such that $z = Tz$. Moreover, if $v = Tv$, then $H(v, v, g(v, v)) = 0$.

Proof. Let $u \in X$ and define the sequence $\{u_n\}_{n=0}^{\infty}$ satisfying the following: $u_0 = u$ and $u_n = T^n u$ for any $n \in \mathbb{N}$. Then, we have, for any $n \in \mathbb{N}$,

$$\begin{aligned} H(u_n, u_{n+1}, g(u_n, u_{n+1})) &\le rH(u_{n-1}, u_n, g(u_{n-1}, u_n)) \\ &\le r^2 H(u_{n-2}, u_{n-1}, g(u_{n-2}, u_{n-1})) \\ &\;\vdots \\ &\le r^n H(u, u_1, g(u, u_1)). \end{aligned} \tag{4.3}$$

From (i) of Lemma 2.14 and (4.3), we have

$$\begin{aligned} H(u_n, u_{n+p}, g(u_n, u_{n+p})) &\le \sum_{j=0}^{p-1} H(u_{n+j}, u_{n+j+1}, g(u_{n+j}, u_{n+j+1})) \\ &\le \frac{r^n(1-r^p)}{1-r} H(u, u_1, g(u, u_1)), \end{aligned} \tag{4.4}$$

$$\begin{aligned} H(u_n, u_{n+p}, u_{n+p+t}) &\le H(u_n, u_{n+p}, g(u_n, u_{n+p+t})) \\ &\quad + H(u_n, u_{n+p+t}, g(u_n, u_{n+p+t})) \\ &\quad + H(u_{n+p}, u_{n+p+t}, g(u_n, u_{n+p+t})) \\ &\le 2\{H(u_n, u_{n+p}, g(u_n, u_{n+p})) + H(u_{n+p}, u_{n+p+t}, g(u_{n+p}, u_{n+p+t}))\} \\ &\quad + H(u_n, u_{n+p+t}, g(u_n, u_{n+p+t})). \end{aligned} \tag{4.5}$$

Thus,

$$H(u_n, u_{n+p}, u_{n+p+t}) \leq \frac{5r^n}{1-r} H(u, u_1, g(u, u_1)). \tag{4.6}$$

Since X is an S-complete, $\{u_n\}$ converges to some point $z \in X$. By (ii) of Lemma 2.14 and (4.4),

$$H(u_n, z, g(u_n, z)) \leq \frac{r^n}{1-r} H(u, u_1, g(u, u_1)). \tag{4.7}$$

Assume that $z \neq Tz$. Then, by hypothesis, (4.3), and (4.7), we have

$$\begin{aligned}
0 &< \inf\{H(x, z, g(x, z)) + H(x, Tx, g(x, Tx)) : x \in X\} \\
&\leq \inf\{H(u_n, z, g(u_n, z)) + H(u_n, u_{n+1}, g(u_n, u_{n+1})) : n \in N\} \\
&\leq \inf\left\{\frac{r^n}{1-r} H(u, u_1, g(u, u_1)) + r^n H(u, u_1, g(u, u_1)) : n \in N\right\} \\
&= 0.
\end{aligned} \tag{4.8}$$

This is a contradiction. Therefore, we have $z = Tz$. If $v = Tv$, then

$$\begin{aligned}
H(v, v, g(v, v)) &= H(Tv, T^2v, g(Tv, T^2v)) \\
&\leq rH(v, Tv, g(v, Tv)) \\
&= rH(v, v, g(v, v)).
\end{aligned} \tag{4.9}$$

Hence, $H(v, v, g(v, v)) = 0$. □

Using methods similar to Theorems 3.1 and 4.1, we have the following corollary.

Corollary 4.2. *Let X, S, and g be as in Theorem 3.1. Let T be a mapping from X into itself. Suppose that there exists $r \in [0, 1)$ such that*

$$S(Tx, T^2x, g(Tx, T^2x)) \leq rS(x, Tx, g(x, Tx)) \tag{4.10}$$

for every $x \in X$ and that

$$\inf\{S(x, y, g(x, y)) + S(x, Tx, g(x, Tx)) : x \in X\} > 0 \tag{4.11}$$

for every $y \in X$ with $y \neq Ty$. Then, there exists $z \in X$ such that $z = Tz$. Moreover, if $v = Tv$, then $S(v, v, g(v, v)) = 0$.

Corollary 4.3 (see [7]). *Let (X,d) be a complete metric space with a w-distance p and let T be a self-mapping of X. Suppose that there exists $r \in [0,1)$ such that*

$$p(Tx,Ty) \le r \cdot \max\{p(x,y), p(x,Tx), p(y,Ty), p(x,Ty), p(y,Tx)\} \tag{4.12}$$

for every $x, y \in X$ and that

$$\inf\{p(x,y) + p(x,Tx) : x \in X\} > 0 \tag{4.13}$$

for every $y \in X$ with $y \neq Ty$. Then, there exists $z \in X$ such that $z = Tz$. Moreover, if $v = Tv$, then $p(v,v) = 0$.

Proof. By Lemma 2.6 in [7], for every $x \in X$

$$\sup\left[p\left(T^i x, T^j x\right) \mid i, j \in \mathbb{N} \cup \{0\}\right] < \infty. \tag{4.14}$$

Define $H : X \times X \times X \to [0,\infty)$ by

$$H(x,y,z) = \max\left\{ \sup\left[p\left(T^i x, T^j x\right) \mid i, j \in \mathbb{N} \cup \{0\}\right], p(x,y), p(x,z)\right\} \tag{4.15}$$

for all $x, y, z \in X$ and define $S : X \times X \times X \to [0,\infty)$ by

$$S(x,y,z) = \max\{d(x,y), d(x,z), d(y,z)\} \tag{4.16}$$

for all $x, y, z \in X$. Let $g : X \times X \to X$ be a function such that for all $x, y \in X$, $g(x,y) = y$. Then, these hypotheses are satisfisfied by all conditions of Theorem 4.1. Therefore, there exists $z \in X$ such that $z = Tz$. Moreover, if $v = Tv$, then $p(v,v) = 0$. □

Corollary 4.4 (see [6]). *Let (X,d) be a complete metric space and and let $T : X \to X$ be a mapping such that*

$$d(Tx,Ty) \le r \cdot \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \tag{4.17}$$

for all $x, y \in X$ and some $r \in [0,1)$. Then, T has a unique fixed point.

Proof. Since a metric d is a w-distance, (4.17) implies (4.12). By Lemma 2.5 in [7], (4.13) is satisfied. Therefore, by Corollary 4.3, the result follows. □

Corollary 4.5. *Let (X,d) be a complete metric space, let T be a continuous mapping from X into itself, and let $f : X \to (-\infty,\infty]$ be a proper lower semicontinuous function, bounded from below. Assume that there exists a w-distance p on X such that for any $u \in X$ with $\inf_{x\in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$ and*

$$f(v) + \max\{p(Tu,v), p(Tu,Tv)\} \le f(u). \tag{4.18}$$

Then, there exists $x_0 \in X$ such that $\inf_{x\in X} f(x) = f(x_0)$.

Proof. Let $H(x,y,z) = \max\{p(Tx,Ty), p(Tx,Tz), p(Tx,y), p(Tx,z)\}$ for all $x,y,z \in X$ and $S(x,y,z) = \max\{d(x,y), d(x,z), d(y,z)\}$ for all $x,y,z \in X$, and let $g(x,y) = y$ for all $x,y \in X$. Then, X, H, g, and f satisfy the suppositions in Lemma 2.14. Therefore, Corollary 4.5 follows from Lemma 2.14. □

As a consequence of Corollary 4.5, we have the following corollary.

Corollary 4.6 (see [4, Corollary 1]). *Let (X,d) be a complete metric space, let T be a continuous mapping from X into itself, and let $f : X \to (-\infty,\infty]$ be a proper lower semicontinuous function, bounded from below. Assume that for any $u \in X$ with $\inf_{x\in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$ and*

$$f(v) + \max\{d(Tu,v), d(Tu,Tv)\} \leq f(u). \tag{4.19}$$

Then, there exists $x_0 \in X$ such that $\inf_{x\in X} f(x) = f(x_0)$.

The following is an example to support Theorem 4.1.

Example 4.7. Let X, g, h, H, S, and T be as in Example 3.6. Taking $r = 2/3$, all of conditions in Theorem 4.1 are satisfied. Therefore, there exists $0 \in X$ such that $0 = T0$. If $v = Tv = (1/2)v$, then $H(v,v,g(v,v)) = H(v,v,v) = h(v,v) = v = 0$.

Remark 4.8. Since $H(Tx, T^2x, g(Tx,T^2x)) = H(Tx,T^2x,Tx) = h(T^2x,Tx)$ is neither metric nor w-distance, theorems in [4, 6, 7] cannot be applicable. Therefore, Theorem 4.1 is a generalization of the corresponding results in [4, 6, 7].

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