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Research Article

Monotone Generalized Nonlinear Contractions in Partially Ordered Metric Spaces

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A concept of g-monotone mapping is introduced, and some fixed and common fixed point theorems for g-non-decreasing generalized nonlinear contractions in partially ordered complete metric spaces are proved. Presented theorems are generalizations of very recent fixed point theorems due to Agarwal et al. (2008).

1. Introduction

The Banach fixed point theorem for contraction mappings has been extended in many directions (cf. [1–28]). Very recently Agarwal et al. [1] presented some new results for generalized nonlinear contractions in partially ordered metric spaces. The main idea in [1, 20, 26] involve combining the ideas of iterative technique in the contraction mapping principle with those in the monotone technique.

Recall that if $(X, \leq)$ is a partially ordered set and $F : X \to X$ is such that for $x, y \in X$, $x \leq y$ implies $F(x) \leq F(y)$, then a mapping F is said to be non-decreasing. The main result of Agarwal et al. in [1] is the following fixed point theorem.

Theorem 1.1 (see [1, Theorem 2.2]). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a non-decreasing function $\psi : [0, +\infty) \to [0, +\infty)$ with $\lim_{n\to\infty}\psi^n(t) = 0$ for each $t > 0$ and also suppose F is a non-decreasing* *mapping with*

$$d(F(x),F(y)) \leq \psi\Big(\max\Big\{d(x,y), d(x,F(x)), d(y,F(y)), \frac{1}{2}[d(x,F(y)) + d(y,F(x))]\Big\}\Big) \tag{1.1}$$

for all $x \geq y$. Also suppose either

(a) *F is continuous or*

(b) *if $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to x$ in X, then $x_n \leq x$ for all n hold.*

If there exists an $x_0 \in X$ with $x_0 \leq F(x_0)$ then F has a fixed point.

Agarwal et al. [1] observed that in certain circumstances it is possible to remove the condition that ψ is non-decreasing in Theorem 1.1. So they proved the following fixed point theorem.

Theorem 1.2 (see [1, Theorem 2.3]). *Let $(X,\leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Assume there is a continuous function $\psi : [0,+\infty) \to [0,+\infty)$ with $\psi(t) < t$ for each $t > 0$ and also suppose F is a non-decreasing mapping with*

$$d(F(x),F(y)) \leq \psi(\max\{d(x,y), d(x,F(x)), d(y,F(y))\}) \quad \forall x \geq y. \tag{1.2}$$

Also suppose either (a) or (b) holds. If there exists an $x_0 \in X$ with $x_0 \leq F(x_0)$ then F has a fixed point.

The problem to extend the result of Theorem 1.2 to mappings which satisfy (1.1) remained open. The aim of this note is to solve this problem by using more refined technique of proofs. Moreover, we introduce a concept of g-monotone mapping and prove some fixed and common fixed point theorems for g-non-decreasing generalized nonlinear contractions in partially ordered complete metric spaces.

2. Main results

Definition 2.1. Suppose $(X,\leq)$ is a partially ordered set and $F, g : X \to X$ are mappings of X into itself. One says F is g-non-decreasing if for $x, y \in X$,

$$g(x) \leq g(y) \text{ implies } F(x) \leq F(y). \tag{2.1}$$

Now we present the main result in this paper.

Theorem 2.2. *Let $(X,\leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Assume there is a continuous function φ: $[0,+\infty) \to [0,+\infty)$*

with $\varphi(t) < t$ *for each* $t > 0$ *and also suppose* $F, g : X \to X$ *are such that* $F(X) \subseteq g(X)$, F *is a* g*-non-decreasing mapping and*

$$d(F(x), F(y)) \le \max\Big\{\varphi(d(g(x), g(y))), \varphi(d(g(x), F(x))), \varphi(d(g(y), F(y))), \varphi\Big(\frac{d(g(x), F(y)) + d(g(y), F(x))}{2}\Big)\Big\} \tag{2.2}$$

for all $x, y \in X$ *for which* $g(x) \ge g(y)$. *Also suppose*

$$\begin{aligned} &\text{if } \{g(x_n)\} \subset X \text{ is a non-decreasing sequence with } g(x_n) \longrightarrow g(z) \text{ in } g(X) \\ &\text{then } g(x_n) \le g(z), \quad g(z) \le g(g(z)) \quad \forall\, n \text{ hold.} \end{aligned} \tag{2.3}$$

Also suppose $g(X)$ *is closed. If there exists an* $x_0 \in X$ *with* $g(x_0) \le F(x_0)$, *then* F *and* g *have a coincidence. Further, if* F, g *commute at their coincidence points, then* F *and* g *have a common fixed point.*

Proof. Let $x_0 \in X$ be such that $g(x_0) \le F(x_0)$. Since $F(X) \subseteq g(X)$, we can choose $x_1 \in X$ so that $g(x_1) = F(x_0)$. Again from $F(X) \subseteq g(X)$ we can choose $x_2 \in X$ such that $g(x_2) = F(x_1)$. Continuing this process we can choose a sequence $\{x_n\}$ in X such that

$$g(x_{n+1}) = F(x_n) \quad \forall n \ge 0. \tag{2.4}$$

Since $g(x_0) \le F(x_0)$ and $F(x_0) = g(x_1)$, we have $g(x_0) \le g(x_1)$. Then from (2.1),

$$F(x_0) \le F(x_1). \tag{2.5}$$

Thus, by (2.4), $g(x_1) \le g(x_2)$. Again from (2.1),

$$F(x_1) \le F(x_2), \tag{2.6}$$

that is, $g(x_2) \le g(x_3)$. Continuing we obtain

$$F(x_0) \le F(x_1) \le F(x_2) \le F(x_3) \le \cdots \le F(x_n) \le F(x_{n+1}) \le \cdots. \tag{2.7}$$

In what follows we will suppose that $d(F(x_n), F(x_{n+1})) > 0$ for all n, since if $F(x_{n+1}) = F(x_n)$ for some n, then by (2.4),

$$F(x_{n+1}) = g(x_{n+1}), \tag{2.8}$$

that is, F and g have a coincidence at $x = x_{n+1}$, and so we have finished the proof. We will show that

$$d(F(x_n), F(x_{n+1})) < d(F(x_{n-1}), F(x_n)) \quad \forall n \ge 1. \tag{2.9}$$

From (2.4) and (2.7) we have that $g(x_n) \leq g(x_{n+1})$ for all $n \geq 0$. Then from (2.2) with $x = x_n$ and $y = x_{n+1}$,

$$\begin{aligned} d(F(x_n), F(x_{n+1})) \leq \max\Big\{ &\varphi(d(g(x_n), g(x_{n+1}))), \varphi(d(g(x_n), F(x_n))), \\ &\varphi(d(g(x_{n+1}), F(x_{n+1}))), \\ &\varphi\left(\frac{d(g(x_n), F(x_{n+1})) + d(g(x_{n+1}), F(x_n))}{2}\right)\Big\}. \end{aligned} \tag{2.10}$$

Thus by (2.4),

$$\begin{aligned} d(F(x_n), F(x_{n+1})) \leq \max\Big\{ &\varphi(d(F(x_{n-1}), F(x_n))), \varphi(d(F(x_{n-1}), F(x_n))), \\ &\varphi(d(F(x_n), F(x_{n+1}))), \varphi\left(\frac{1}{2}d(F(x_{n-1}), F(x_{n+1}))\right)\Big\}. \end{aligned} \tag{2.11}$$

Hence

$$\begin{aligned} d(F(x_n), F(x_{n+1})) \leq \max\Big\{ &\varphi(d(F(x_{n-1}), F(x_n))), \varphi(d(F(x_n), F(x_{n+1}))), \\ &\varphi\left(\frac{1}{2}d(F(x_{n-1}), F(x_{n+1}))\right)\Big\}. \end{aligned} \tag{2.12}$$

If $d(F(x_n), F(x_{n+1})) \leq \varphi(d(F(x_{n-1}), F(x_n)))$, then (2.9) holds, as $\varphi(t) < t$ for $t > 0$.

Since we suppose that $d(F(x_n), F(x_{n+1})) > 0$ and as $\varphi(t) < t$ for $t > 0$, then $d(F(x_n), F(x_{n+1}))) \leq \varphi(d(F(x_n), F(x_{n+1})))$ it is impossible.

If from (2.12) we have $d(F(x_n), F(x_{n+1})) \leq \varphi(d(F(x_{n-1}), F(x_{n+1}))/2)$, and if $d(F(x_{n-1}), F(x_{n+1}))/2 > 0$, then we have

$$\begin{aligned} d(F(x_n), F(x_{n+1})) &\leq \varphi\left(\frac{1}{2}d(F(x_{n-1}), F(x_{n+1}))\right) \\ &< \frac{1}{2}d(F(x_{n-1}), F(x_{n+1})) \\ &\leq \frac{1}{2}d(F(x_{n-1}), F(x_n)) + \frac{1}{2}d(F(x_n), F(x_{n+1})). \end{aligned} \tag{2.13}$$

Hence

$$d(F(x_n), F(x_{n+1})) < d(F(x_{n-1}), F(x_n)). \tag{2.14}$$

Therefore, we proved that (2.9) holds.

From (2.9) it follows that the sequence $\{d(F(x_n), F(x_{n+1}))\}$ of real numbers is monotone decreasing. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n\to\infty} d(F(x_n), F(x_{n+1})) = \delta. \tag{2.15}$$

Now we will prove that $\delta = 0$. By the triangle inequality,

$$\frac{1}{2}d(F(x_{n-1}), F(x_{n+1})) \leq \frac{1}{2}(d(F(x_{n-1}), F(x_n)) + d(F(x_n), F(x_{n+1}))). \tag{2.16}$$

Hence by (2.9),

$$\frac{1}{2}d(F(x_{n-1}), F(x_{n+1})) < d(F(x_{n-1}), F(x_n)). \tag{2.17}$$

Taking the upper limit as $n \to \infty$ we get

$$\limsup_{n\to\infty} \frac{1}{2} d(F(x_{n-1}), F(x_{n+1})) \leq \lim_{n\to\infty} d(F(x_{n-1}), F(x_n)). \tag{2.18}$$

If we set

$$\limsup_{n\to\infty} \frac{1}{2} d(F(x_{n-1}), F(x_{n+1})) = b, \tag{2.19}$$

then clearly $0 \leq b \leq \delta$. Now, taking the upper limit on the both sides of (2.12) and have in mind that $\varphi(t)$ is continuous, we get

$$\begin{aligned}\lim_{n\to\infty} d(F(x_n), F(x_{n+1})) \leq \max\Big\{&\varphi\Big(\lim_{n\to\infty} d(F(x_{n-1}), F(x_n))\Big), \varphi\Big(\lim_{n\to\infty} d(F(x_n), F(x_{n+1}))\Big),\\ &\varphi\Big(\limsup_{n\to\infty}\frac{1}{2}d(F(x_{n-1}), F(x_{n+1}))\Big)\Big\}.\end{aligned} \tag{2.20}$$

Hence by (2.15) and (2.19),

$$\delta \leq \max\{\varphi(\delta), \varphi(b)\}. \tag{2.21}$$

If we suppose that $\delta > 0$, then we have

$$\delta \leq \max\{\varphi(\delta), \varphi(b)\} < \max\{\delta, b\} = \delta, \tag{2.22}$$

a contradiction. Thus $\delta = 0$. Therefore, we proved that

$$\lim_{n\to\infty} d(F(x_n), F(x_{n+1})) = 0. \tag{2.23}$$

Now we prove that $\{F(x_n)\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{F(x_n)\}$ is not a Cauchy sequence. Then there exist an $\epsilon > 0$ and two sequences of integers $\{l(k)\}, \{m(k)\}$, $m(k) > l(k) \geq k$ with

$$r_k = d(F(x_{l(k)}), F(x_{m(k)})) \geq \epsilon \quad \text{for } k \in \{1, 2, \ldots\}. \tag{2.24}$$

We may also assume

$$d\left(F\left(x_{l(k)}\right), F\left(x_{m(k)-1}\right)\right) < \epsilon \tag{2.25}$$

by choosing $m(k)$ to be the smallest number exceeding $l(k)$ for which (2.24) holds. From (2.24), (2.25) and by the triangle inequality,

$$\epsilon \leq r_k \leq d\left(F\left(x_{l(k)}\right), F\left(x_{m(k)-1}\right) + d\left(F\left(x_{m(k)-1}\right), F\left(x_{m(k)}\right) < \epsilon + d\left(F\left(x_{m(k)-1}\right), F\left(x_{m(k)}\right).\right.\right.\right. \tag{2.26}$$

Hence by (2.23),

$$\lim_{k \to \infty} r_k = \epsilon. \tag{2.27}$$

Since from (2.7) and (2.4) we have $g(x_{l(k)+1}) = F(x_{l(k)}) \leq F(x_{m(k)}) = g(x_{m(k)+1})$, from (2.2) and (2.4) with $x = x_{m(k)+1}$ and $y = x_{l(k)+1}$ we get

$$\begin{aligned} d\left(F\left(x_{l(k)+1}\right), F\left(x_{m(k)+1}\right)\right) \leq \max \Bigg\{ & \varphi\left(d\left(F\left(x_{l(k)}\right), F\left(x_{m(k)}\right)\right)\right), \varphi\left(d\left(F\left(x_{l(k)}\right), F\left(x_{l(k)+1}\right)\right)\right), \\ & \varphi\left(d\left(F\left(x_{m(k)}\right), F\left(x_{m(k)+1}\right)\right)\right), \\ & \varphi\left(\frac{d(F(x_{l(k)}), F(x_{m(k)+1})) + d(F(x_{m(k)}), F(x_{l(k)+1}))}{2}\right) \Bigg\}. \end{aligned} \tag{2.28}$$

Denote $\delta_n = d(F(x_n), F(x_{n+1}))$. Then we have

$$\begin{aligned} d\left(F\left(x_{l(k)+1}\right), F\left(x_{m(k)+1}\right)\right) \leq \max \Bigg\{ & \varphi(r_k), \varphi(\delta_{l(k)}), \varphi(\delta_{m(k)}), \\ & \varphi\left(\frac{d(F(x_{l(k)}), F(x_{m(k)+1})) + d(F(x_{m(k)}), F(x_{l(k)+1}))}{2}\right) \Bigg\}. \end{aligned} \tag{2.29}$$

Therefore, since

$$\begin{aligned} r_k &\leq d\left(F\left(x_{l(k)}\right), F\left(x_{l(k)+1}\right)\right) + d\left(F\left(x_{l(k)+1}\right), F\left(x_{m(k)+1}\right)\right) + d\left(F\left(x_{m(k)}\right), F\left(x_{m(k)+1}\right)\right) \\ &= \delta_{l(k)} + \delta_{m(k)} + d\left(F\left(x_{l(k)+1}\right), F\left(x_{m(k)+1}\right)\right), \end{aligned} \tag{2.30}$$

we have

$$\epsilon \leq r_k \leq \delta_{l(k)} + \delta_{m(k)} + \max\left\{\varphi(r_k), \varphi(\delta_{l(k)}), \varphi(\delta_{m(k)}), \varphi\left(\frac{d(F(x_{l(k)}), F(x_{m(k)+1})) + d(F(x_{m(k)}), F(x_{l(k)+1}))}{2}\right)\right\}. \tag{2.31}$$

By the triangle inequality, (2.24) and (2.25),

$$\begin{gathered}\epsilon \leq r_k \leq d(F(x_{l(k)}), F(x_{m(k)+1})) + \delta_{m(k)}, \\ d(F(x_{l(k)}), F(x_{m(k)+1})) \leq d(F(x_{l(k)}), F(x_{m(k)-1})) + \delta_{m(k)-1} + \delta_{m(k)} \leq \epsilon + \delta_{m(k)-1} + \delta_{m(k)}.\end{gathered} \tag{2.32}$$

From (2.32),

$$\epsilon - \delta_{m(k)} \leq d(F(x_{l(k)}), F(x_{m(k)+1})) \leq \epsilon + \delta_{m(k)-1} + \delta_{m(k)}. \tag{2.33}$$

Similarly,

$$\begin{gathered}\epsilon \leq r_k \leq \delta_{l(k)} + d(F(x_{l(k)+1}), F(x_{m(k)})), \\ d(F(x_{l(k)+1}), F(x_{m(k)})) \leq \delta_{l(k)} + d(F(x_{l(k)}), F(x_{m(k)-1})) + \delta_{m(k)-1} \leq \epsilon + \delta_{m(k)-1} + \delta_{m(k)}.\end{gathered} \tag{2.34}$$

Hence

$$\epsilon - \delta_{l(k)} \leq d(F(x_{m(k)}), F(x_{l(k)+1})) \leq \epsilon + \delta_{m(k)-1} + \delta_{l(k)}. \tag{2.35}$$

From (2.33) and (2.35),

$$\begin{aligned}\epsilon - \frac{\delta_{l(k)} + \delta_{m(k)}}{2} &\leq \frac{d(F(x_{l(k)}), F(x_{m(k)+1})) + d(F(x_{m(k)}), F(x_{l(k)+1}))}{2} \\ &\leq \epsilon + \delta_{m(k)-1} + \frac{\delta_{l(k)} + \delta_{m(k)}}{2}.\end{aligned} \tag{2.36}$$

Thus from (2.36) and (2.23) we get

$$\lim_{k\to\infty} \frac{d(F(x_{l(k)}), F(x_{m(k)+1})) + d(F(x_{m(k)}), F(x_{l(k)+1}))}{2} = \epsilon. \tag{2.37}$$

Letting $n \to \infty$ in (2.31), then by (2.23), (2.27) and (2.37) we get, as φ is continuous,

$$\epsilon \leq \max\{\varphi(\epsilon), 0, 0, \varphi(\epsilon)\} < \epsilon, \tag{2.38}$$

a contradiction. Thus our assumption (2.24) is wrong. Therefore, $\{F(x_n)\}$ is a Cauchy sequence. Since by (2.4) we have $\{F(x_n)\} = \{g(x_{n+1})\} \subseteq g(X)$ and $g(X)$ is closed, there exists $z \in X$ such that

$$\lim_{n\to\infty} g(x_n) = g(z). \tag{2.39}$$

Now we show that z is a coincidence point of F and g. Since from (2.3) and (2.39) we have $g(x_n) \leq g(z)$ for all n, then by the triangle inequality and (2.2) we get

$$\begin{aligned} d(g(z), F(z)) &\leq d(g(z), F(x_n)) + d(F(x_n), F(z)) \\ &\leq d(g(z), F(x_n)) \\ &\quad + \max\Big\{\varphi(d(g(x_n), g(z))), \varphi(d(g(x_n), F(x_n))), \\ &\qquad \varphi(d(g(z), F(z))), \varphi\Big(\frac{d(g(x_n), F(z)) + d(g(z), F(x_n))}{2}\Big)\Big\}. \end{aligned} \tag{2.40}$$

So letting $n \to \infty$ yields $d(g(z), F(z)) \leq \max\{\varphi(d(g(z), F(z))), \varphi(d(g(z), F(z))/2\}$. Hence $d(g(z), F(z)) = 0$, hence $F(z) = g(z)$. Thus we proved that F and g have a coincidence.

Suppose now that F and g commute at z. Set $w = g(z) = F(z)$. Then

$$F(w) = F(g(z)) = g(F(z)) = g(w). \tag{2.41}$$

Since from (2.3) we have $g(z) \leq g(g(z)) = g(w)$ and as $g(z) = F(z)$ and $g(w) = F(w)$, from (2.2) we get

$$\begin{aligned} d(F(z), F(w)) &\leq \max\Big\{\varphi(d(g(z), g(w))), \varphi(d(g(z), F(z))), \\ &\qquad \varphi(d(g(w), F(w))), \varphi\Big(\frac{d(g(z), F(w)) + d(g(w), F(z))}{2}\Big)\Big\} \\ &= \varphi(d(F(z), F(w))). \end{aligned} \tag{2.42}$$

Hence $d(F(z), F(w)) = 0$, that is, $d(w, F(w)) = 0$. Therefore,

$$F(w) = g(w) = w. \tag{2.43}$$

Thus we proved that F and g have a common fixed point. □

Remark 2.3. Note F is g-non-decreasing can be replaced by F is g-non-increasing in Theorem 2.2 provided $g(x_0) \leq F(x_0)$ is replaced by $F(x_0) \geq g(x_0)$ in Theorem 2.2.

Corollary 2.4. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(t) < t$ for each $t > 0$ and also suppose $F : X \to X$ is a non-decreasing mapping and*

$$d(F(x), F(y)) \leq \max\left\{\varphi(d(x,y)), \varphi(d(x, F(x))), \varphi(d(y, F(y))), \varphi\left(\frac{d(x, F(y)) + d(y, F(x))}{2}\right)\right\} \tag{2.44}$$

for all $x, y \in X$ for which $x \leq y$. Also suppose either

(i) *if $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to z$ in X then $x_n \leq z$ for all n hold or*

(ii) *F is continuous.*

If there exists an $x_0 \in X$ with $x_0 \leq F(x_0)$ then F has a fixed point.

Proof. If (i) holds, then taking $g = I$ (I = the identity mapping) in Theorem 2.2 we obtain Corollary 2.4. If (ii) holds, then from (2.39) with $g = I$ we get

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n) = F\left(\lim_{n\to\infty} x_n\right) = F(z). \tag{2.45}$$

□

Corollary 2.5. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(t) < t$ for each $t > 0$ and also suppose $F : X \to X$ is a non-decreasing mapping and*

$$d(F(x), F(y)) \leq \max\{\varphi(d(x,y)), \varphi(d(x, F(x))), \varphi(d(y, F(y)))\} \tag{2.46}$$

for all $x, y \in X$ for which $x \leq y$. Also suppose either

(i) *if $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to z$ in X then $x_n \leq z$ for all n hold or*

(ii) *F is continuous.*

If there exists an $x_0 \in X$ with $x_0 \leq F(x_0)$ then F has a fixed point.

Remark 2.6. Since (1.2) implies (2.46) with $\psi = \varphi$, Corollary 2.5 is a generalization of Theorem 1.2. If in addition ψ and φ are non-decreasing, then Theorem 1.2 and Corollary 2.5 are equivalent.

Taking $\varphi(t) = kt$, $0 < k < 1$, in Corollary 2.4 we obtain the following generalization of the results in [20, 26].

Corollary 2.7. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X \to X$ is a non-decreasing mapping and*

$$d(F(x), F(y)) \leq k \max\left\{d(x,y), d(x, F(x)), d(y, F(y)), \frac{d(x, F(y)) + d(y, F(x))}{2}\right\} \tag{2.47}$$

for all $x, y \in X$ *for which* $x \leq y$, *where* $0 < k < 1$. *Also suppose either*

(i) *if* $\{x_n\} \subset X$ *is a non-decreasing sequence with* $x_n \to z$ *in* X *then* $x_n \leq z$ *for all* n *hold or*

(ii) F *is continuous.*

If there exists an $x_0 \in X$ *with* $x_0 \leq F(x_0)$ *then* F *has a fixed point.*

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