Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2007, Article ID 93678, 20 pages
doi:10.1155/2007/93678

Research Article

Existence of Solutions and Convergence of a Multistep Iterative Algorithm for a System of Variational Inclusions with (H,η)-Accretive Operators

Jian-Wen Peng, Dao-Li Zhu, and Xiao-Ping Zheng

Received 5 April 2007; Accepted 6 July 2007

Recommended by Lech Gorniewicz

We introduce and study a new system of variational inclusions with (H,η)-accretive operators, which contains variational inequalities, variational inclusions, systems of variational inequalities, and systems of variational inclusions in the literature as special cases. By using the resolvent technique for the (H,η)-accretive operators, we prove the existence and uniqueness of solution and the convergence of a new multistep iterative algorithm for this system of variational inclusions in real q-uniformly smooth Banach spaces. The results in this paper unify, extend, and improve some known results in the literature.

1. Introduction

Variational inclusion problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics and transportation equilibrium, and engineering science. For the past years, many existence results and iterative algorithms for various variational inequality and variational inclusion problems have been studied. For details, please see [1–50] and the references therein.

Recently, some new and interesting problems, which are called to be system of variational inequality problems were introduced and studied. Pang [28], Cohen and Chaplais [29], Bianchi [30] and Ansari and Yao [16] considered a system of scalar variational inequalities and Pang showed that the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium, and the general equilibrium programming problem can be modeled as a system of variational inequalities. Ansari et al. [31] introduced and studied a system of vector equilibrium problems and a system of vector variational inequalities by a fixed point theorem. Allevi et al. [32] considered a system of generalized vector variational inequalities and established some existence results with relative pseudomonotonicity. Kassay and Kolumbán [17] introduced a system of variational inequalities and proved an existence theorem by the Ky Fan lemma. Kassay et al. [18] studied Minty and Stampacchia variational inequality systems with the help of the Kakutani-Fan-Glicksberg fixed point theorem. Peng [19, 20] introduced a system of quasivariational inequality problems and proved its existence theorem by maximal element theorems. Verma [21–25] introduced and studied some systems of variational inequalities and developed some iterative algorithms for approximating the solutions of system of variational inequalities in Hilbert spaces. K. Kim and S. Kim [26] introduced a new system of generalized nonlinear quasivariational inequalities and obtained some existence and uniqueness results of solution for this system of generalized nonlinear quasivariational inequalities in Hilbert spaces. Cho et al. [27] introduced and studied a new system of nonlinear variational inequalities in Hilbert spaces. They proved some existence and uniqueness theorems of solutions for the system of nonlinear variational inequalities.

As generalizations of the above systems of variational inequalities, Agarwal et al. [33] introduced a system of generalized nonlinear mixed quasivariational inclusions and investigated the sensitivity analysis of solutions for this system of generalized nonlinear mixed quasivariational inclusions in Hilbert spaces. Kazmi and Bhat [34] introduced a system of nonlinear variational-like inclusions and gave an iterative algorithm for finding its approximate solution. Fang and Huang [35] and Fang et al. [36] introduced and studied a new system of variational inclusions involving H-monotone operators and (H,η)-monotone, respectively. Peng and Huang [37] proved the existence and uniqueness of solutions and the convergence of some new three-step iterative algorithms for a new system of variational inclusions in Hilbert spaces.

On the other hand, Yu [10] introduced a new concept of (H,η)-accretive operators which provide unifying frameworks for H-monotone operators in [1], H-accretive operators in [9], (H,η)-monotone operators in [35], maximal η-monotone operators in [5], generalized m-accretive operators in [8], m-accretive operators in [12], and maximal monotone operators [13, 14].

Inspired and motivated by the above results, the purpose of this paper is to introduce a new mathematical model, which is called to be a system of variational inclusions with (H,η)-accretive operators, that is, a family of variational inclusions with (H,η)-accretive operators defined on a product set. This new mathematical model contains the system of inequalities in [16, 21–30] and the system of inclusions in [35–37], the variational inclusions in [1, 2, 9, 11], and some variational inequalities in the literature as special cases. By using the resolvent technique for the (H,η)-accretive operators, we prove the existence of solutions for this system of variational inclusions. We also prove the convergence of a multistep iterative algorithm approximating the solution for this system of variational inclusions. The result in this paper unifies, extends, and improves some results in [1, 2, 9, 11, 21–30, 35–37].

2. Preliminaries

We suppose that E is a real Banach space with dual space, norm, and the generalized dual pair denoted by E^*, $\|\cdot\|$, and $\langle\cdot,\cdot\rangle$, respectively, 2^E is the family of all the nonempty subsets of E, $CB(E)$ is the families of all nonempty closed bounded subsets of E, and the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|f^*\| \cdot \|x\|,\ \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in E, \tag{2.1}$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2}J_2(x)$, for all $x \neq 0$, and J_q is single valued if E^* is strictly convex.

The modulus of smoothness of E is the function $\varrho_E : [0,\infty) \to [0,\infty)$ defined by

$$\varrho_E(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1,\ \|y\| \leq t\right\}. \tag{2.2}$$

A Banach space E is called uniformly smooth if

$$\lim_{t\to 0}\frac{\varrho_E(t)}{t} = 0. \tag{2.3}$$

E is called q-uniformly smooth if there exists a constant $c > 0$, such that

$$\varrho_E(t) \leq ct^q, \quad q > 1. \tag{2.4}$$

Note that J_q is single valued if E is uniformly smooth. Xu and Roach [51] proved the following result.

Lemma 2.1. *Let E be a real uniformly smooth Banach space. Then, E is q-uniformly smooth if and only if there exists a constants $c_q > 0$, such that for all $x, y \in E$,*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q. \tag{2.5}$$

We recall some definitions needed later, for more details, please see [3, 4, 9, 10] and the references therein.

Definition 2.2. Let E be a real uniformly smooth Banach space, and let $T, H : E \to E$ be two single-valued operators. T is said to be

(i) accretive if

$$\langle T(x) - T(y), J_q(x-y)\rangle \geq 0, \quad \forall x, y \in E; \tag{2.6}$$

(ii) strictly accretive if T is accretive and

$$\langle T(x) - T(y), J_q(x-y)\rangle = 0 \quad \text{iff } x = y; \tag{2.7}$$

(iii) r-strongly accretive if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), J_q(x-y)\rangle \geq r\|x-y\|^q, \quad \forall x, y \in E; \tag{2.8}$$

(iv) r-strongly accretive with respect to H if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), J_q(H(x) - H(y)) \rangle \geq r\|x - y\|^q, \quad \forall x, y \in E; \tag{2.9}$$

(v) s-Lipschitz continuous if there exists a constant $s > 0$ such that

$$||T(x) - T(y)|| \leq s\|x - y\|, \quad \forall x, y \in E. \tag{2.10}$$

Definition 2.3. Let E be a real uniformly smooth Banach space, let $T : E \to E$ and $\eta : E \times E \to E$ be two single-valued operators. T is said to be

(i) η-accretive if

$$\langle T(x) - T(y), J_q(\eta(x, y)) \rangle \geq 0, \quad \forall x, y \in E; \tag{2.11}$$

(ii) strictly η-accretive if T is η-accretive and

$$\langle T(x) - T(y), J_q(\eta(x, y)) \rangle = 0 \quad \text{iff } x = y; \tag{2.12}$$

(iii) r-strongly η-accretive if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), J_q(\eta(x, y)) \rangle \geq r\|x - y\|^q, \quad \forall x, y \in E. \tag{2.13}$$

Definition 2.4. Let $\eta : E \times E \to E$, let $T, H : E \to E$ be single-valued operators and $M : E \to 2^E$ be a multivalued operator. M is said to be

(i) accretive if

$$\langle u - v, J_q(x - y) \rangle \geq 0, \quad \forall x, y \in E,\ u \in M(x),\ v \in M(y); \tag{2.14}$$

(ii) η-accretive if

$$\langle u - v, J_q(\eta(x, y)) \rangle \geq 0, \quad \forall x, y \in E,\ u \in M(x),\ v \in M(y); \tag{2.15}$$

(iii) strictly η-accretive if M is η-accretive, and equality holds if and only if $x = y$;

(iv) r-strongly η-accretive if there exists a constant $r > 0$ such that if

$$\langle u - v, J_q(\eta(x, y)) \rangle \geq r\|x - y\|^q, \quad \forall x, y \in E,\ u \in M(x),\ v \in M(y); \tag{2.16}$$

(v) m-accretive if M is accretive and $(I + \varrho M)(E) = E$ holds for all $\varrho > 0$, where I is the identity map on E;

(vi) generalized η-accretive if M is η-accretive and $(I + \varrho M)(E) = E$ holds for all $\varrho > 0$;

(vii) H-accretive if M is accretive and $(H + \varrho M)(E) = E$ holds for all $\varrho > 0$;

(viii) (H, η)-accretive if M is η-accretive and $(H + \varrho M)(E) = E$ holds for all $\varrho > 0$.

Remark 2.5. (i) If $\eta(x, y) = x - y$, for all $x, y \in E$, then the definition of (H, η)-accretive operators becomes that of H-accretive operators in [9]. If $E = \mathcal{H}$ is a Hilbert space, the definition of (H, η)-accretive operator becomes that of (H, η)-monotone operators in [36], the definition of H-accretive operators becomes that of H-monotone operators in [1, 35]. Hence, the definition of (H, η)-accretive operators provides unifying frameworks for classes of H-accretive operators, generalized η-accretive operators, m-accretive operators, maximal monotone operators, maximal η-monotone operators, H-monotone operators, and (H,η)-monotone operators.

Definition 2.6 [5]. Let $\eta : E\times E \to E$ be a single-valued operator, then $\eta(\cdot,\cdot)$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that

$$||\eta(u,v)|| \le \tau\|u-v\|, \quad \forall u,v\in E. \tag{2.17}$$

Definition 2.7 [10]. Let $\eta : E\times E \to E$ be a single-valued operator, let $H : E\to E$ be a strictly η-accretive single-valued operator, and let $M : E \to 2^E$ be an (H,η)-accretive operator, and let $\lambda > 0$ be a constant. The resolvent operator $R^{H,\eta}_{M,\lambda} : E\to E$ associated with H, η, M, λ is defined by

$$R^{H,\eta}_{M,\lambda}(u) = (H+\lambda M)^{-1}(u), \quad \forall u\in E. \tag{2.18}$$

Lemma 2.8 [10]. *Let $\eta : E\times E\to E$ be a τ-Lipschitz continuous operator, $H : E\to E$ be a γ-strongly η-accretive operator, and let $M : E \to 2^E$ be an (H,η)-accretive operator. Then, the resolvent operator $R^{H,\eta}_{M,\lambda} : E\to E$ is τ^{q-1}/γ-Lipschitz continuous, that is,*

$$\left\|R^{H,\eta}_{M,\lambda}(x) - R^{H,\eta}_{M,\lambda}(y)\right\| \le \frac{\tau^{q-1}}{\gamma}\|x-y\|, \quad \forall x,y\in E. \tag{2.19}$$

We extend some definitions in [6, 37, 46] to more general cases as follows.

Definition 2.9. Let $E_1,E_2,\dots,E_p$ be Banach spaces, let $g_1 : E_1\to E_1$ and $N_1 : \prod_{j=1}^{p} E_j \to E_1$ be two single-valued mappings.

(i) N_1 is said to be ξ-Lipschitz continuous in the first argument if there exists a constant $\xi > 0$ such that

$$\begin{gathered} ||N_1(x_1,x_2,\dots,x_p) - N_1(y_1,x_2,\dots,x_p)|| \le \xi||x_1-y_1||, \\ \forall x_1,y_1\in E_1,\ x_j\in E_j\ (j=2,3,\dots,p). \end{gathered} \tag{2.20}$$

(ii) N_1 is said to be accretive in the first argument if

$$\begin{gathered} \langle N_1(x_1,x_2,\dots,x_p) - N_1(y_1,x_2,\dots,x_p), J_q(x_1-y_1)\rangle \ge 0, \\ \forall x_1,y_1\in E_1,\ x_j\in E_j\ (j=2,3,\dots,p). \end{gathered} \tag{2.21}$$

(iii) N_1 is said to be α-strongly accretive in the first argument if there exists a constant $\alpha > 0$ such that

$$\begin{gathered} \langle N_1(x_1,x_2,\dots,x_p) - N_1(y_1,x_2,\dots,x_p), J_q(x_1-y_1)\rangle \ge \alpha||x_1-y_1||^q, \\ \forall x_1,y_1\in E_1,\ x_j\in E_j\ (j=2,3,\dots,p). \end{gathered} \tag{2.22}$$

(iv) N_1 is said to be accretive with respect to g in the first argument if

$$\begin{gathered} \langle N_1(x_1,x_2,\dots,x_p) - N_1(y_1,x_2,\dots,x_p), J_q(g(x_1)-g(y_1))\rangle \ge 0, \\ \forall x_1,y_1\in E_1,\ x_j\in E_j\ (j=2,3,\dots,p). \end{gathered} \tag{2.23}$$

(v) N_1 is said to be β-strongly accretive with respect to g in the first argument if there exists a constant $\beta > 0$ such that

$$\begin{aligned}&\langle N_1(x_1,x_2,\dots,x_p) - N_1(y_1,x_2,\dots,x_p), J_q(g(x_1)-g(y_1))\rangle \geq \beta\|x_1-y_1\|^q,\\ &\qquad\qquad\qquad\qquad\qquad \forall x_1,y_1 \in E_1,\ x_j \in E_j\ (j=2,3,\dots,p).\end{aligned} \tag{2.24}$$

In a similar way, we can define the Lipschitz continuity and the strong accretivity (accretivity) of $N_i : \prod_{j=1}^{p} E_j \to E_i$ (with respect to $g_i : E_i \to E_i$) in the ith argument $(i = 2,3,\dots,p)$.

3. A system of variational inclusions

In this section, we will introduce a new system of variational inclusions with (H,η)-accretive operators. In what follows, unless other specified, for each $i = 1,2,\dots,p$, we always suppose that E_i is a real q-uniformly smooth Banach space, $H_i, g_i : E_i \to E_i$, $\eta_i : E_i \times E_i \to E_i$, $F_i, G_i : \prod_{j=1}^{p} E_j \to E_i$ are single-valued mappings, and that $M_i : E_i \to 2^{E_i}$ is an (H_i,η_i)-accretive operator. We consider the following problem of finding $(x_1,x_2,\dots,x_p) \in \prod_{i=1}^{p} E_i$ such that for each $i = 1,2,\dots,p$,

$$0 \in F_i(x_1,x_2,\dots,x_p) + G_i(x_1,x_2,\dots,x_p) + M_i(g_i(x_i)). \tag{3.1}$$

The problem (3.1) is called a system of variational inclusions with (H,η)-accretive operators.

Below are some special cases of problem (3.1).

(i) For each $j = 1,2,\dots,p$, if $E_j = \mathscr{H}_j$ is a Hilbert space, then problem (3.1) becomes the following system of variational inclusions with (H,η)-monotone operators, which is to find $(x_1,x_2,\dots,x_p) \in \prod_{i=1}^{p} E_i$ such that for each $i = 1,2,\dots,p$,

$$0 \in F_i(x_1,x_2,\dots,x_p) + G_i(x_1,x_2,\dots,x_p) + M_i(g_i(x_i)). \tag{3.2}$$

(ii) For each $j = 1,2,\dots,p$, if $g_j \equiv I_j$ (the identity map on E_j) and $G_j \equiv 0$, then problem (3.1) reduces to the system of variational inclusions with (H,η)-accretive operators, which is to find $(x_1,x_2,\dots,x_p) \in \prod_{j=1}^{p} E_j$ such that for each $i = 1,2,\dots,p$,

$$0 \in F_i(x_1,x_2,\dots,x_p) + M_i(x_i). \tag{3.3}$$

(iii) If $p = 1$, then problem (3.2) becomes the following variational inclusion with an (H_1,η_1)-monotone operator, which is to find $x_1 \in \mathscr{H}_1$ such that

$$0 \in F_1(x_1) + G_1(x_1) + M_1(g_1(x_1)). \tag{3.4}$$

Moreover, if $\eta_1(x_1,y_1) = x_1 - y_1$ for all $x_1,y_1 \in \mathscr{H}_1$ and $H_1 = I_1$ (the identity map on $\mathscr{H}_1$), then problem (3.4) becomes the variational inclusion introduced and researched by Adly [11] which contains the variational inequality in [2] as a special case.

If $p = 1$, then problem (3.3) becomes the following variational inclusion with an (H_1, η_1)-accretive operator, which is to find $x_1 \in E_1$ such that

$$0 \in F_1(x_1) + M_1(x_1). \tag{3.5}$$

Problem (3.5) was introduced and studied by Yu [10] and contains the variational inclusions in [1, 9] as special cases.

If $p = 2$, then problem (3.3) becomes the following system of variational inclusions with (H,η)-accretive operators, which is to find $(x_1,x_2) \in E_1 \times E_2$ such that

$$
\begin{aligned}
&0 \in F_1(x_1,x_2) + M_1(x_1),\\
&0 \in F_2(x_1,x_2) + M_2(x_2).
\end{aligned}
\tag{3.6}
$$

Problem (3.6) contains the system of variational inclusions with H-monotone operators in [35], the system of variational inclusions with (H,η)-monotone operators in [36] as special cases.

If $p = 3$ and for each $j = 1,2,3$, $E_j = \mathcal{H}_j$ is a Hilbert space and $G_j = 0$, then problem (3.1) becomes the system of variational inclusions with (H,η)-monotone operators in [37] with $f_j = 0$ and $\zeta_j = 1$.

(iv) For each $j = 1,2,\dots,p$, if $E_j = \mathcal{H}_j$ is a Hilbert space, and $M_j(x_j) = \Delta_{\eta_j}\varphi_j$ for all $x_j \in \mathcal{H}_j$, where $\varphi_j : \mathcal{H}_j \to R \cup \{+\infty\}$ is a proper, η_j-subdifferentiable functional and $\Delta_{\eta_j}\varphi_j$ denotes the η_j-subdifferential operator of φ_j, then problem (3.3) reduces to the following system of variational-like inequalities, which is to find $(x_1,x_2,\dots,x_p) \in \prod_{i=1}^{p}\mathcal{H}_i$ such that for each $i = 1,2,\dots,p$,

$$
\langle F_i(x_1,x_2,\dots,x_p),\eta_i(z_i,x_i)\rangle + \varphi_i(z_i) - \varphi_i(x_i) \geq 0, \quad \forall z_i \in \mathcal{H}_i. \tag{3.7}
$$

(v) For each $j = 1,2,\dots,p$, if $E_j = \mathcal{H}_j$ is a Hilbert space, and $M_j(x_j) = \partial\varphi_j(x_j)$, for all $x_j \in \mathcal{H}_j$, where $\varphi_j : \mathcal{H}_j \to R \cup \{+\infty\}$ is a proper, convex, lower semicontinuous functional and $\partial\varphi_j$ denotes the subdifferential operator of φ_j, then problem (3.3) reduces to the following system of variational inequalities, which is to find $(x_1,x_2,\dots,x_p) \in \prod_{i=1}^{p}\mathcal{H}_i$ such that for each $i = 1,2,\dots,p$,

$$
\langle F_i(x_1,x_2,\dots,x_p),z_i - x_i\rangle + \varphi_i(z_i) - \varphi_i(x_i) \geq 0, \quad \forall z_i \in \mathcal{H}_i. \tag{3.8}
$$

(vi) For each $j = 1,2,\dots,p$, if $M_j(x_j) = \partial\delta_{K_j}(x_j)$ for all $x_j \in \mathcal{H}_j$, where $K_j \subset \mathcal{H}_j$ is a nonempty, closed, and convex subsets and δ_{K_j} denotes the indicator of K_j, then problem (3.8) reduces to the following system of variational inequalities, which is to find $(x_1,x_2,\dots,x_p) \in \prod_{i=1}^{p}\mathcal{H}_i$ such that for each $i = 1,2,\dots,p$,

$$
\langle F_i(x_1,x_2,\dots,x_p),z_i - x_i\rangle \geq 0, \quad \forall z_i \in K_i. \tag{3.9}
$$

Problem (3.9) was introduced and researched in [16, 28–30]. If $p = 2$, then problems (3.7), (3.8), and (3.9), respectively, become the problems (3.2), (3.3) and (3.4) in [36]. It is easy to see that problem (3.4) in [36] contains the models of system of variational inequalities in [21–25] as special cases.

It is worthy noting that problem (3.1)–(3.8) are all new problems.

4. Existence and uniqueness of the solution

In this section, we will prove existence and uniqueness for solutions of problem (3.1). For our main results, we give a characterization of the solution of problem (3.1) as follows.

Lemma 4.1. *For $i = 1,2,\dots,p$, let $\eta_i : E_i \times E_i \to E_i$ be a single-valued operator, let $H_i : E_i \to E_i$ be a strictly η_i-accretive operator, and let $M_i : E_i \to 2^{E_i}$ be an (H_i,η_i)-accretive operator. Then $(x_1,x_2,\dots,x_p) \in \prod_{i=1}^{p} E_i$ is a solution of the problem (3.1) if and only if for each $i = 1,2,\dots,p$,*

$$g_i(x_i) = R_{M_i,\lambda_i}^{H_i,\eta_i}\left(H_i(g_i(x_i)) - \lambda_i F_i(x_1,x_2,\dots,x_p) - \lambda_i G_i(x_1,x_2,\dots,x_p)\right), \tag{4.1}$$

where $R_{M_i,\lambda_i}^{H_i,\eta_i} = (H_i + \lambda_i M_i)^{-1}$ and $\lambda_i > 0$ are constants.

Proof. The fact directly follows from Definition 2.9. □

Let $\Gamma = \{1,2,\dots,p\}$.

Theorem 4.2. *For $i = 1,2,\dots,p$, let $\eta_i : E_i \times E_i \to E_i$ be σ_i-Lipschitz continuous, let $H_i : E_i \to E_i$ be γ_i-strongly η_i-accretive and τ_i-Lipschitz continuous, let $g_i : E_i \to E_i$ be β_i-strongly accretive and θ_i-Lipschitz continuous, let $M_i : E_i \to 2^{E_i}$ be an (H_i,η_i)-accretive operator, let $F_i : \prod_{j=1}^{p} E_j \to E_i$ be a single-valued mapping such that F_i is r_i-strongly accretive with respect to $\hat{g}_i$ and s_i-Lipschitz continuous in the ith argument, where $\hat{g}_i : E_i \to E_i$ is defined by $\hat{g}_i(x_i) = H_i \circ g_i(x_i) = H_i(g_i(x_i))$, for all $x_i \in E_i$, F_i is t_{ij}-Lipschitz continuous in the jth arguments for each $j \in \Gamma$, $j \neq i$, $G_i : \prod_{j=1}^{p} E_j \to E_i$ be a single-valued mapping such that G_i is l_{ij}-Lipschitz continuous in the jth argument for each $j \in \Gamma$. If there exist constants $\lambda_i > 0$ $(i = 1,2,\dots,p)$ such that*

$$\begin{gathered}
\sqrt[q]{1 - q\beta_1 + c_q\theta_1^q} + \frac{\sigma_1^{q-1}}{\gamma_1}\sqrt[q]{\tau_1^q\theta_1^q - q\lambda_1 r_1 + c_q{\lambda_1}^q s_1^q} + \frac{l_{11}\lambda_1\sigma_1^{q-1}}{\gamma_1} + \sum_{k=2}^{p}\frac{\lambda_k\sigma_k^{q-1}}{\gamma_k}(t_{k1} + l_{k1}) < 1, \\
\sqrt[q]{1 - q\beta_2 + c_q\theta_2^q} + \frac{\sigma_2^{q-1}}{\gamma_2}\sqrt[q]{\tau_2^q\theta_2^q - q\lambda_2 r_2 + c_q{\lambda_2}^q s_2^q} + \frac{l_{22}\lambda_2\sigma_2^{q-1}}{\gamma_2} + \sum_{k\in\Gamma,k\neq 2}\frac{\lambda_k\sigma_k^{q-1}}{\gamma_k}(t_{k2} + l_{k2}) < 1, \\
\cdots \\
\sqrt[q]{1 - q\beta_p + c_q\theta_p^q} + \frac{\sigma_p^{q-1}}{\gamma_p}\sqrt[q]{\tau_p^q\theta_p^q - q\lambda_p r_p + c_q{\lambda_p}^q s_p^q} + \frac{l_{pp}\lambda_p\sigma_p^{q-1}}{\gamma_p} + \sum_{k=1}^{p-1}\frac{\sigma_k^{q-1}\lambda_k}{\gamma_k}(t_{k,p} + l_{k,p}) < 1.
\end{gathered} \tag{4.2}$$

Then, problem (3.1) admits a unique solution.

Proof. For $i = 1,2,\dots,p$ and for any given $\lambda_i > 0$, define a single-valued mapping $T_{i,\lambda_i} : \prod_{j=1}^{p} E_j \to E_i$ by

$$\begin{aligned}
&T_{i,\lambda_i}(x_1,x_2,\dots,x_p) \\
&\quad = x_i - g_i(x_i) + R_{M_i,\lambda_i}^{H_i,\eta_i}\left(H_i g_i(x_i) - \lambda_i F_i(x_1,x_2,\dots,x_p) - \lambda_i G_i(x_1,x_2,\dots,x_p)\right),
\end{aligned} \tag{4.3}$$

for any $(x_1,x_2,\dots,x_p) \in \prod_{i=1}^{p} E_i$.

For any $(x_1,x_2,\dots,x_p),(y_1,y_2,\dots,y_p)\in\prod_{i=1}^{p}E_i$, it follows from (4.3) that for $i=1,2,\dots,p$,

$$\begin{aligned}
&\|T_{i,\lambda_i}(x_1,x_2,\dots,x_p)-T_{i,\lambda_i}(y_1,y_2,\dots,y_p)\|_i\\
&\quad=\|x_i-g_i(x_i)+R^{H_i,\eta_i}_{M_i,\lambda_i}(H_i(g_i(x_i))-\lambda_iF_i(x_1,x_2,\dots,x_p)-\lambda_iG_i(x_1,x_2,\dots,x_p))\\
&\qquad\quad-[y_i-g_i(y_i)+R^{H_i,\eta_i}_{M_i,\lambda_i}(H_i(g_i(y_i))-\lambda_iF_i(y_1,y_2,\dots,y_p)-\lambda_iG_i(y_1,y_2,\dots,y_p))]\|_i\\
&\quad\le\|x_i-y_i-(g_i(x_i)-g_i(y_i))\|_i\\
&\qquad+\|R^{H_i,\eta_i}_{M_i,\lambda_i}(H_i(g_i(x_i))-\lambda_iF_i(x_1,x_2,\dots,x_p)-\lambda_iG_i(x_1,x_2,\dots,x_p))\\
&\qquad\quad-R^{H_i,\eta_i}_{M_i,\lambda_i,m_i}(H_i(g_i(y_i))-\lambda_iF_i(y_1,y_2,\dots,y_p)-\lambda_iG_i(y_1,y_2,\dots,y_p))\|_i.
\end{aligned}\tag{4.4}$$

For $i=1,2,\dots,p$, since g_i is β_i-strongly accretive and θ_i-Lipschitz continuous, we have

$$\begin{aligned}
&\|x_i-y_i-(g_i(x_i)-g_i(y_i))\|_i^q\\
&\quad=\|x_i-y_i\|_i^q-q\langle g_i(x_i)-g_i(y_i),J_q(x_i-y_i)\rangle+c_q\|g_i(x_i)-g_i(y_i)\|_i^q\\
&\quad\le(1-q\beta_i+c_q\theta_i^q)\|x_i-y_i\|_i^q.
\end{aligned}\tag{4.5}$$

It follows from Lemma 2.1 that for $i=1,2,\dots,p$,

$$\begin{aligned}
&\Big\|R^{H_i,\eta_i}_{M_i,\lambda_i}(H_i(g_i(x_i))-\lambda_iF_i(x_1,x_2,\dots,x_p)-\lambda_iG_i(x_1,x_2,\dots,x_p))\\
&\quad-R^{H_i,\eta_i}_{M_i,\lambda_i}(H_i(g_i(y_i))-\lambda_iF_i(y_1,y_2,\dots,y_p)-\lambda_iG_i(y_1,y_2,\dots,y_p))\Big\|_i\\
&\qquad\le\frac{\sigma_i^{q-1}}{\gamma_i}\|(H_i(g_i(x_i))-H_i(g_i(y_i)))-\lambda_i(F_i(x_1,x_2,\dots,x_p)-F_i(y_1,y_2,\dots,y_p))\|_i\\
&\qquad\quad+\frac{\sigma_i^{q-1}\lambda_i}{\gamma_i}\|G_i(x_1,x_2,\dots,x_p)-G_i(y_1,y_2,\dots,y_p)\|_i\\
&\qquad\le\frac{\sigma_i^{q-1}}{\gamma_i}\|H_i(g_i(x_i))-H_i(g_i(y_i))-\lambda_i(F_i(x_1,x_2,\dots,x_{i-1},x_i,x_{i+1},\dots,x_p)\\
&\qquad\qquad\quad-F_i(x_1,x_2,\dots,x_{i-1},y_i,x_{i+1},\dots,x_p))\|_i\\
&\qquad\quad+\frac{\sigma_i^{q-1}\lambda_i}{\gamma_i}\Bigg(\sum_{j\in\Gamma,j\ne i}\|F_i(x_1,x_2,\dots,x_{j-1},x_j,x_{j+1},\dots,x_p)\\
&\qquad\qquad\qquad\qquad-F_i(x_1,x_2,\dots,x_{j-1},y_j,x_{j+1},\dots,x_p)\|_i\Bigg)\\
&\qquad\quad+\frac{\sigma_i^{q-1}\lambda_i}{\gamma_i}\Bigg(\sum_{j=1}^{p}\|G_i(x_1,x_2,\dots,x_{j-1},x_j,x_{j+1},\dots,x_p)\\
&\qquad\qquad\qquad\qquad-G_i(x_1,x_2,\dots,x_{j-1},y_j,x_{j+1},\dots,x_p)\|_i\Bigg).
\end{aligned}\tag{4.6}$$

For $i = 1,2,\dots,p$, since H_i is τ_i-Lipschitz continuous, and g_i is θ_i-Lipschitz continuous and F_i is r_i-$\widehat{g}_i$-strongly accretive and s_i-Lipschitz continuous in the ith argument, we have

$$\begin{aligned}
&||H_i(g_i(x_i)) - H_i(g_i(y_i)) - \lambda_i(F_i(x_1,x_2,\dots,x_{i-1},x_i,x_{i+1},\dots,x_p) \\
&\qquad\qquad - F_i(x_1,x_2,\dots,x_{i-1},y_i,x_{i+1},\dots,x_p))||_i^q \\
&\quad \le ||(H_i(g_i(x_i)) - H_i(g_i(y_i)))||_i^q - q\lambda_i\langle F_i(x_1,x_2,\dots,x_{i-1},x_i,x_{i+1},\dots,x_p) \\
&\qquad - F_i(x_1,x_2,\dots,x_{i-1},y_i,x_{i+1},\dots,x_p), H_i(g_i(x_i)) - H_i(g_i(y_i))\rangle \\
&\qquad + c_q{\lambda_i}^q||F_i(x_1,x_2,\dots,x_{i-1},x_i,x_{i+1},\dots,x_p) - F_i(x_1,x_2,\dots,x_{i-1},y_i,x_{i+1},\dots,x_p)||_i^q \\
&\quad \le \tau_i^q||g_i(x_i) - g_i(y_i)||_i^q - q\lambda_i r_i||x_i - y_i||_i^q + c_q{\lambda_i}^q s_i^q||x_i - y_i||_i^q \\
&\quad \le (\tau_i^q\theta_i^q - q\lambda_i r_i + c_q{\lambda_i}^q s_i^q)||x_i - y_i||_i^q.
\end{aligned} \tag{4.7}$$

For $i = 1,2,\dots,p$, since F_i is t_{ij}-Lipschitz continuous in the jth arguments ($j \in \Gamma$, $j \ne i$), we have

$$||F_i(x_1,x_2,\dots,x_{j-1},x_j,x_{j+1},\dots,x_p) - F_i(x_1,x_2,\dots,x_{j-1},y_j,x_{j+1},\dots,x_p)||_i \le t_{ij}||x_j - y_j||_j. \tag{4.8}$$

For $i = 1,2,\dots,p$, since G_i is l_{ij}-Lipschitz continuous in the jth arguments ($j = 1, 2,\dots,p$), we have

$$||G_i(x_1,x_2,\dots,x_{j-1},x_j,x_{j+1},\dots,x_p) - G_i(x_1,x_2,\dots,x_{j-1},y_j,x_{j+1},\dots,x_p)||_i \le l_{ij}||x_j - y_j||_j. \tag{4.9}$$

It follows from (4.4)–(4.9) that for each $i = 1,2,\dots,p$

$$\begin{aligned}
&||T_{i,\lambda_i}(x_1,x_2,\dots,x_p) - T_{i,\lambda_i}(y_1,y_2,\dots,y_p)||_i \\
&\quad \le \left(\sqrt[q]{1 - q\beta_i + c_q\theta_i^q} + \frac{\sigma_i^{q-1}}{\gamma_i}\sqrt[q]{\tau_i^q\theta_i^q - q\lambda_i r_i + c_q{\lambda_i}^q s_i^q} + \frac{l_{ii}\lambda_i\sigma_i^{q-1}}{\gamma_i}\right)||x_i - y_i||_i \\
&\qquad + \frac{\lambda_i\sigma_i^{q-1}}{\gamma_i}\left[\sum_{j\in\Gamma,\, j\ne i}(t_{ij} + l_{ij})||x_j - y_j||_j\right].
\end{aligned} \tag{4.10}$$

Hence,

$$\begin{aligned}
&\sum_{i=1}^{p}\|T_{i,\lambda_i}(x_1,x_2,\ldots,x_p)-T_{i,\lambda_i}(y_1,y_2,\ldots,y_p)\|_i\\
&\quad\le\sum_{i=1}^{p}\Bigg\{\Bigg(\sqrt[q]{1-q\beta_i+c_q\theta_i^q}+\frac{\sigma_i^{q-1}}{\gamma_i}\sqrt[q]{\tau_i^q\theta_i^q-q\lambda_i r_i+c_q\lambda_i{}^q s_i^q}+\frac{l_{ii}\lambda_i\sigma_i^{q-1}}{\gamma_i}\Bigg)\|x_i-y_i\|_i\\
&\qquad\qquad+\frac{\lambda_i\sigma_i^{q-1}}{\gamma_i}\Bigg[\sum_{j\in\Gamma,\,j\neq i}(t_{ij}+l_{ij})\|x_j-y_j\|_j\Bigg]\Bigg\}\\
&\quad=\Bigg(\sqrt[q]{1-q\beta_1+c_q\theta_1^q}+\frac{\sigma_1^{q-1}}{\gamma_1}\sqrt[q]{\tau_1^q\theta_1^q-q\lambda_1 r_1+c_q\lambda_1{}^q s_1^q}\\
&\qquad\quad+\frac{l_{11}\lambda_1\sigma_1^{q-1}}{\gamma_1}+\sum_{k=2}^{p}\frac{\lambda_k\sigma_k^{q-1}}{\gamma_k}(t_{k1}+l_{k1})\Bigg)\|x_1-y_1\|_1\\
&\qquad+\Bigg(\sqrt[q]{1-q\beta_2+c_q\theta_2^q}+\frac{\sigma_2^{q-1}}{\gamma_2}\sqrt[q]{\tau_2^q\theta_2^q-q\lambda_2 r_2+c_q\lambda_2{}^q s_2^q}\\
&\qquad\quad+\frac{l_{22}\lambda_2\sigma_2^{q-1}}{\gamma_2}+\sum_{k\in\Gamma,k\neq 2}\frac{\lambda_k\sigma_k^{q-1}}{\gamma_k}(t_{k2}+l_{k2})\Bigg)\|x_2-y_2\|_2\\
&\qquad+\cdots+\Bigg(\sqrt[q]{1-q\beta_p+c_q\theta_p^q}+\frac{\sigma_p^{q-1}}{\gamma_p}\sqrt[q]{\tau_p^q\theta_p^q-q\lambda_p r_p+c_q\lambda_p{}^q s_p^q}\\
&\qquad\quad+\frac{l_{pp}\lambda_p\sigma_p^{q-1}}{\gamma_p}+\sum_{k=1}^{p-1}\frac{\sigma_k^{q-1}\lambda_k}{\gamma_k}(t_{k,p}+l_{k,p})\Bigg)\|x_p-y_p\|_p\\
&\quad\le\xi\Bigg(\sum_{k=1}^{p}\|x_k-y_k\|_k\Bigg),
\end{aligned}\tag{4.11}$$

where

$$\begin{aligned}
\xi=\max\Bigg\{&\sqrt[q]{1-q\beta_1+c_q\theta_1^q}+\frac{\sigma_1^{q-1}}{\gamma_1}\sqrt[q]{\tau_1^q\theta_1^q-q\lambda_1 r_1+c_q\lambda_1{}^q s_1^q}+\frac{l_{11}\lambda_1\sigma_1^{q-1}}{\gamma_1}\\
&+\sum_{k=2}^{p}\frac{\lambda_k\sigma_k^{q-1}}{\gamma_k}(t_{k1}+l_{k1}),\sqrt[q]{1-q\beta_2+c_q\theta_2^q}+\frac{\sigma_2^{q-1}}{\gamma_2}\sqrt[q]{\tau_2^q\theta_2^q-q\lambda_2 r_2+c_q\lambda_2{}^q s_2^q}\\
&+\frac{l_{22}\lambda_2\sigma_2^{q-1}}{\gamma_2}+\sum_{k\in\Gamma,k\neq 2}\frac{\lambda_k\sigma_k^{q-1}}{\gamma_k}(t_{k2}+l_{k2}),\ldots,\sqrt[q]{1-q\beta_p+c_q\theta_p^q}\\
&+\frac{\sigma_p^{q-1}}{\gamma_p}\sqrt[q]{\tau_p^q\theta_p^q-q\lambda_p r_p+c_q\lambda_p{}^q s_p^q}+\frac{l_{pp}\lambda_p\sigma_p^{q-1}}{\gamma_p}+\sum_{k=1}^{p-1}\frac{\sigma_k^{q-1}\lambda_k}{\gamma_k}(t_{k,p}+l_{k,p})\Bigg\}.
\end{aligned}\tag{4.12}$$

Define $\|\cdot\|_\Gamma$ on $\prod_{i=1}^{p} E_i$ by $\|(x_1,x_2,\dots,x_p)\|_\Gamma = \|x_1\|_1 + \|x_2\|_2 + \cdots + \|x_p\|_p$, for all $(x_1,x_2,\dots,x_p) \in \prod_{i=1}^{p} E_i$. It is easy to see that $\prod_{i=1}^{p} E_i$ is a Banach space. For any given $\lambda_i > 0$ ($i \in \Gamma$), define $W_{\Gamma,\lambda_1,\lambda_2,\dots,\lambda_p} : \prod_{i=1}^{p} E_i \to \prod_{i=1}^{p} E_i$ by

$$\begin{aligned} &W_{\Gamma,\lambda_1,\lambda_2,\dots,\lambda_p}(x_1,x_2,\dots,x_p) \\ &\quad = (T_{1,\lambda_1}(x_1,x_2,\dots,x_p), T_{2,\lambda_2}(x_1,x_2,\dots,x_p)),\dots, T_{p,\lambda_p}(x_1,x_2,\dots,x_p)), \end{aligned} \tag{4.13}$$

for all $(x_1,x_2,\dots,x_p) \in \prod_{i=1}^{p} E_i$.

By (4.2), we know that $0 < \xi < 1$, it follows from (4.11) that

$$\begin{aligned} &||W_{\Gamma,\lambda_1,\lambda_2,\dots,\lambda_p}(x_1,x_2,\dots,x_p) - W_{\Gamma,\lambda_1,\lambda_2,\dots,\lambda_p}(x_1,x_2,\dots,x_p)||_\Gamma \\ &\qquad \le \xi||(x_1,x_2,\dots,x_p) - (y_1,y_2,\dots,y_p)||_\Gamma. \end{aligned} \tag{4.14}$$

This shows that $W_{\Gamma,\lambda_1,\lambda_2,\dots,\lambda_p}$ is a contraction operator. Hence, there exists a unique $(x_1,x_2,\dots,x_p) \in \prod_{i=1}^{p} E_i$, such that

$$W_{\Gamma,\lambda_1,\lambda_2,\dots,\lambda_p}(x_1,x_2,\dots,x_p) = (x_1,x_2,\dots,x_p), \tag{4.15}$$

that is, for $i = 1,2,\dots,p$,

$$g_i(x_i) = R_{M_i,\lambda_i}^{H_i,\eta_i}(H_i(g_i(x_i)) - \lambda_i F_i(x_1,x_2,\dots,x_p) - \lambda_i G_i(x_1,x_2,\dots,x_p)). \tag{4.16}$$

By Lemma 4.1, $(x_1,x_2,\dots,x_p)$ is the unique solution of problem (3.1). This completes this proof. □

5. Iterative algorithm and convergence

In this section, we will construct a new multistep iterative algorithm for approximating the unique solution of problem (3.1) and discuss the convergence analysis of this algorithm.

Lemma 5.1 [36]. *Let $\{c_n\}$ and $\{k_n\}$ be two real sequences of nonnegative numbers that satisfy the following conditions:*

(1) $0 \le k_n < 1$, $n = 0,1,2,\dots$ *and* $\limsup_n k_n < 1$;

(2) $c_{n+1} \le k_n c_n$, $n = 0,1,2,\dots$;

then c_n converges to 0 as $n \to \infty$.

Algorithm 5.2. For $i = 1,2,\dots,p$, let H_i, M_i, F_i, g_i, η_i be the same as in Theorem 4.2. For any given $(x_1^0, x_2^0,\dots,x_p^0) \in \prod_{j=1}^{p} E_j$, define a multistep iterative sequence $\{(x_1^n, x_2^n,\dots,x_p^n)\}$ by

$$\begin{aligned} x_i^{n+1} = \alpha_n x_i^n + (1-\alpha_n)\Big[x_i^n - g_i(x_i^n) + R_{M_i,\lambda_i}^{H_i,\eta_i}\big(H_i(g_i(x_i^n)) \\ - \lambda_i F_i(x_1^n, x_2^n,\dots,x_p^n) - \lambda_i G_i(x_1^n, x_2^n,\dots,x_p^n)\big)\Big], \end{aligned} \tag{5.1}$$

where

$$0 \le \alpha_n < 1, \quad \limsup_n \alpha_n < 1. \tag{5.2}$$

Theorem 5.3. *For $i = 1,2,\dots,p$, let H_i, M_i, F_i, g_i, η_i be the same as in Theorem 4.2. Assume that all the conditions of Theorem 4.2 hold. Then $\{(x_1^n, x_2^n,\dots,x_p^n)\}$ generated by Algorithm 5.2 converges strongly to the unique solution $(x_1, x_2,\dots,x_p)$ of problem (3.1).*

Proof. By Theorem 4.2, problem (3.1) admits a unique solution $(x_1, x_2,\dots,x_p)$, it follows from Lemma 4.1 that for each $i = 1,2,\dots,p$,

$$g_i(x_i) = R_{M_i,\lambda_i}^{H_i,\eta_i}\big(H_i(g_i(x_i)) - \lambda_i F_i(x_1, x_2,\dots,x_p) - \lambda_i G_i(x_1, x_2,\dots,x_p)\big). \tag{5.3}$$

It follows from (5.1) and (5.3) that for each $i = 1,2,\dots,p$,

$$\begin{aligned} \|x_i^{n+1} - x_i\|_i = \Big\|\alpha_n(x_i^n - x_i) + (1-\alpha_n)\Big[x_i^n - g_i(x_i^n) - (x_i - g_i(x_i)) \\ + R_{M_i,\lambda_i}^{H_i,\eta_i}\big(H_i(g_i(x_i^n)) - \lambda_i F_i(x_1^n, x_2^n,\dots,x_p^n) - \lambda_i G_i(x_1^n, x_2^n,\dots,x_p^n)\big) \\ - R_{M_i,\lambda_i}^{H_i,\eta_i}\big(H_i(g_i(x_i)) - \lambda_i F_i(x_1, x_2,\dots,x_p) - \lambda_i G_i(x_1, x_2,\dots,x_p)\big)\Big]\Big\|_i \\ \le \alpha_n\|x_i^n - x_i\|_i + (1-\alpha_n)\|x_i^n - g_i(x_i^n) - (x_i - g_i(x_i))\|_i \\ + (1-\alpha_n)\Big\|R_{M_i,\lambda_i}^{H_i,\eta_i}\big(H_i(g_i(x_i^n)) - \lambda_i F_i(x_1^n, x_2^n,\dots,x_p^n) - \lambda_i G_i(x_1^n, x_2^n,\dots,x_p^n)\big) \\ - R_{M_i,\lambda_i}^{H_i,\eta_i}\big(H_i(g_i(x_i)) - \lambda_i F_i(x_1, x_2,\dots,x_p) - \lambda_i G_i(x_1, x_2,\dots,x_p)\big)\Big\|_i. \end{aligned} \tag{5.4}$$

For $i = 1,2,\dots,p$, since g_i is β_i-strongly accretive and θ_i-Lipschitz continuous, we have

$$\|x_i^n - g_i(x_i^n) - (x_i - g_i(x_i))\|_i^q \le (1 - q\beta_i + c_q\theta_i^q)\|x_i^n - x_i\|_i^q. \tag{5.5}$$

It follows from Lemma 2.1 that for $i = 1, 2, \ldots, p$,

$$\begin{aligned}
&\Big\|R_{M_i,\lambda_i}^{H_i,\eta_i}\big(H_i(g_i(x_i^n)) - \lambda_i F_i(x_1^n, x_2^n, \ldots, x_p^n) - \lambda_i G_i(x_1^n, x_2^n, \ldots, x_p^n)\big) \\
&\quad - R_{M_i,\lambda_i}^{H_i,\eta_i}\big(H_i(g_i(x_i)) - \lambda_i F_i(x_1, x_2, \ldots, x_p) - \lambda_i G_i(x_1, x_2, \ldots, x_p)\big)\Big\|_i \\
&\quad\le \frac{\sigma_i^{q-1}}{\gamma_i}\big\|H_i(g_i(x_i^n)) - H_i(g_i(x_i)) \\
&\qquad\qquad - \lambda_i\big(F_i(x_1^n, x_2^n, \ldots, x_{i-1}^n, x_i^n, x_{i+1}^n, \ldots, x_p^n) \\
&\qquad\qquad\qquad - F_i(x_1^n, x_2^n, \ldots, x_{i-1}^n, x_i, x_{i+1}^n, \ldots, x_p^n)\big)\big\|_i \\
&\qquad + \frac{\lambda_i\sigma_i^{q-1}}{\gamma_i}\Bigg(\sum_{j\in\Gamma, j\neq i}\big\|F_i(x_1^n, x_2^n, \ldots, x_{j-1}^n, x_j^n, x_{j+1}^n, \ldots, x_p^n) \\
&\qquad\qquad\qquad - F_i(x_1^n, x_2^n, \ldots, x_{j-1}^n, x_j, x_{j+1}^n, \ldots, x_p^n)\big\|_i\Bigg) \\
&\qquad + \frac{\lambda_i\sigma_i^{q-1}}{\gamma_i}\Bigg(\sum_{j=1}^{p}\big\|G_i(x_1^n, x_2^n, \ldots, x_{j-1}^n, x_j^n, x_{j+1}^n, \ldots, x_p^n) \\
&\qquad\qquad\qquad - G_i(x_1^n, x_2^n, \ldots, x_{j-1}^n, x_j, x_{j+1}^n, \ldots, x_p^n)\big\|_i\Bigg).
\end{aligned} \tag{5.6}$$

For $i = 1, 2, \ldots, p$, since H_i is τ_i-Lipschitz continuous, and g_i is θ_i-Lipschitz continuous and F_i is r_i-$\hat{g}_i$-strongly accretive and s_i-Lipschitz continuous in the ith argument, we have

$$\begin{aligned}
&\big\|H_i(g_i(x_i^n)) - H_i(g_i(x_i)) - \lambda_i\big(F_i(x_1^n, x_2^n, \ldots, x_{i-1}^n, x_i^n, x_{i+1}^n, \ldots, x_p^n) \\
&\qquad\qquad - F_i(x_1^n, x_2^n, \ldots, x_{i-1}^n, x_i, x_{i+1}^n, \ldots, x_p^n)\big)\big\|_i^q \\
&\qquad \le \big(\tau_i^q\theta_i^q - q\lambda_i r_i + c_q\lambda_i^q s_i^q\big)\|x_i^n - x_i\|^q.
\end{aligned} \tag{5.7}$$

For $i = 1, 2, \ldots, p$, since F_i is t_{ij}-Lipschitz continuous in the jth arguments ($j \in \Gamma$, $j \neq i$), we have

$$\big\|F_i(x_1^n, x_2^n, \ldots, x_{j-1}^n, x_j^n, x_{j+1}^n, \ldots, x_p^n) - F_i(x_1^n, x_2^n, \ldots, x_{j-1}^n, x_j, x_{j+1}^n, \ldots, x_p^n)\big\|_i \le t_{ij}\|x_j^n - x_j\|_j. \tag{5.8}$$

For $i = 1, 2, \ldots, p$, since G_i is l_{ij}-Lipschitz continuous in the jth arguments ($j = 1, 2, \ldots, p$), we have

$$\big\|G_i(x_1^n, x_2^n, \ldots, x_{j-1}^n, x_j^n, x_{j+1}^n, \ldots, x_p^n) - G_i(x_1^n, x_2^n, \ldots, x_{j-1}^n, x_j, x_{j+1}^n, \ldots, x_p^n)\big\|_i \le l_{ij}\|x_j^n - x_j\|_j. \tag{5.9}$$

It follows from (5.4)–(5.9) that for $i = 1, 2, \dots, p$,

$$\begin{aligned}
&\|x_i^{n+1} - x_i\|_i \\
&\quad \le \alpha_n \|x_i^n - x_i\|_i + (1-\alpha_n)\sqrt[q]{1 - q\beta_i + c_q\theta_i^q}\,\|x_i^n - x_i\|_i \\
&\qquad + (1-\alpha_n)\frac{\sigma_i^{q-1}}{\gamma_i}\sqrt[q]{\tau_i^q\theta_i^q - q\lambda_i r_i + c_q\lambda_i^q s_i^q}\,\|x_i^n - x_i\|_i \\
&\qquad + (1-\alpha_n)\frac{\lambda_i\sigma_i^{q-1}}{\gamma_i}\left(\sum_{j\in\Gamma,\, j\neq i} t_{ij}\|x_j^n - x_j\|_j\right) + (1-\alpha_n)\frac{\lambda_i\sigma_i^{q-1}}{\gamma_i}\left(\sum_{j=1}^{p} l_{ij}\|x_j^n - x_j\|_j\right) \\
&\quad = \alpha_n\|x_i^n - x_i\|_i + (1-\alpha_n)\Bigg(\sqrt[q]{1 - q\beta_i + c_q\theta_i^q} \\
&\qquad\qquad + \frac{\sigma_i^{q-1}}{\gamma_i}\sqrt[q]{\tau_i^q\theta_i^q - q\lambda_i r_i + c_q\lambda_i^q s_i^q} + \frac{l_{ii}\lambda_i\sigma_i^{q-1}}{\gamma_i}\Bigg)\|x_i^n - x_i\|_i \\
&\qquad + (1-\alpha_n)\frac{\sigma_i^{q-1}}{\gamma_i}\left(\sum_{j\in\Gamma,\, j\neq i}(t_{ij} + l_{ij})\|x_j^n - x_j\|_j\right).
\end{aligned} \tag{5.10}$$

It follows from (5.10) that

$$\begin{aligned}
&\sum_{i=1}^{p}\|x_i^{n+1} - x_i\|_i \\
&\quad \le \sum_{i=1}^{p}\Bigg[\alpha_n\|x_i^n - x_i\|_i \\
&\qquad + (1-\alpha_n)\left(\sqrt[q]{1 - q\beta_i + c_q\theta_i^q} + \frac{\sigma_i^{q-1}}{\gamma_i}\sqrt[q]{\tau_i^q\theta_i^q - q\lambda_i r_i + c_q\lambda_i^q s_i^q} + \frac{l_{ii}\lambda_i\sigma_i^{q-1}}{\gamma_i}\right)\|x_i^n - x_i\|_i \\
&\qquad + (1-\alpha_n)\frac{\sigma_i^{q-1}}{\gamma_i}\left(\sum_{j\in\Gamma,\, j\neq i}(t_{ij} + l_{ij})\|x_j^n - x_j\|_j\right)\Bigg] \\
&\quad \le \alpha_n\left(\sum_{i=1}^{p}\|x_i^n - x_i\|_i\right) + (1-\alpha_n)\xi\left(\sum_{i=1}^{p}\|x_i^n - x_i\|_i\right) \\
&\quad = (\xi + (1-\xi)\alpha_n)\left(\sum_{i=1}^{p}\|x_i^n - x_i\|_i\right),
\end{aligned} \tag{5.11}$$

where ξ is defined by

$$\begin{aligned}
\xi = \max\Bigg\{ & \sqrt[q]{1-q\beta_1+c_q\theta_1^q}+\frac{\sigma_1^{q-1}}{\gamma_1}\sqrt[q]{\tau_1^q\theta_1^q-q\lambda_1 r_1+c_q{\lambda_1}^q s_1^q} \\
& +\frac{l_{11}\lambda_1\sigma_1^{q-1}}{\gamma_1}+\sum_{k=2}^{p}\frac{\lambda_k\sigma_k^{q-1}}{\gamma_k}(t_{k1}+l_{k1}),\sqrt[q]{1-q\beta_2+c_q\theta_2^q} \\
& +\frac{\sigma_2^{q-1}}{\gamma_2}\sqrt[q]{\tau_2^q\theta_2^q-q\lambda_2 r_2+c_q{\lambda_2}^q s_2^q}+\frac{l_{22}\lambda_2\sigma_2^{q-1}}{\gamma_2} \\
& +\sum_{k\in\Gamma,k\neq 2}\frac{\lambda_k\sigma_k^{q-1}}{\gamma_k}(t_{k2}+l_{k2}),\ldots,\sqrt[q]{1-q\beta_p+c_q\theta_p^q}+\frac{\sigma_p^{q-1}}{\gamma_p}\sqrt[q]{\tau_p^q\theta_p^q-q\lambda_p r_p+c_q{\lambda_p}^q s_p^q} \\
& +\frac{l_{pp}\lambda_p\sigma_p^{q-1}}{\gamma_p}+\sum_{k=1}^{p-1}\frac{\sigma_k^{q-1}\lambda_k}{\gamma_k}(t_{k,p}+l_{k,p})\Bigg\}.
\end{aligned} \tag{5.12}$$

It follows from hypothesis (4.2) that $0<\xi<1$.

Let $a_n=\sum_{i=1}^{p}\|x_i^n-x_i\|_i,\xi_n=\xi+(1-\xi)\alpha_n$. Then, (5.11) can be rewritten as $a_{n+1}\leq \xi_n a_n$, $n=0,1,2,\ldots$. By (5.2), we know that $\limsup_n \xi_n<1$, it follows from Lemma 5.1 that

$$a_n=\sum_{i=1}^{p}\left\|x_i^n-x_i\right\|_i \quad \text{converges to 0 as } n\longrightarrow\infty. \tag{5.13}$$

Therefore, $\{(x_1^n,x_2^n,\ldots,x_p^n)\}$ converges to the unique solution $(x_1,x_2,\ldots,x_p)$ of problem (3.1). This completes the proof. □

Remark 5.4. If E is a 2-uniformly smooth Banach space and there exist constants $\lambda_i>0$ $(i=1,2,\ldots,p)$ such that

$$\begin{gathered}
\sqrt{1-2\beta_1+c_2\theta_1^2}+\frac{\sigma_1}{\gamma_1}\sqrt{\tau_1^2\theta_1^2-2\lambda_1 r_1+c_2{\lambda_1}^2 s_1^2}+\frac{l_{11}\lambda_1\sigma_1}{\gamma_1}+\sum_{k=2}^{p}\frac{\lambda_k\sigma_k}{\gamma_k}(t_{k1}+l_{k1})<1, \\
\sqrt{1-2\beta_2+c_2\theta_2^2}+\frac{\sigma_2}{\gamma_2}\sqrt{\tau_2^2\theta_2^2-2\lambda_2 r_2+c_2{\lambda_2}^2 s_2^2}+\frac{l_{22}\lambda_2\sigma_2}{\gamma_2}+\sum_{k\in\Gamma,k\neq 2}\frac{\lambda_k\sigma_k}{\gamma_k}(t_{k2}+l_{k2})<1, \\
\cdots \\
\sqrt{1-2\beta_2+c_2\theta_p^2}+\frac{\sigma_p}{\gamma_p}\sqrt{\tau_p^2\theta_p^2-2\lambda_p r_p+c_2{\lambda_p}^2 s_p^2}+\frac{l_{pp}\lambda_p\sigma_p}{\gamma_p}+\sum_{k=1}^{p-1}\frac{\sigma_k\lambda_k}{\gamma_k}(t_{k,p}+l_{k,p})<1,
\end{gathered} \tag{5.14}$$

then (4.2) holds. It is worth noting that the Hilbert space and L_P (or l_p) spaces ($2 \le q \le \infty$) are 2 unifomly smooth Banach spaces.

Remark 5.5. Theorems 4.2 and 5.3 unify, improve, and extend those results in [1, 2, 9, 11, 21–30, 35–37] in several aspects.

Remark 5.6. By the results in Sections 4 and 5, it is easy to obtain the existence of solutions and the convergence results of iterative algorithms for the special cases of problem (3.1). And we omit them here.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (Grant no. 70432001 and Grant no. 70502006), the Science and Technology Research Project of Chinese Ministry of Education (Grant no. 206123), the Education Committee project Research Foundation of Chongqing (Grant no. KJ070816), and the Postdoctoral Science Foundation of China (Grant no. 2005038133).

References

[1] Y.-P. Fang and N.-J. Huang, “H-monotone operator and resolvent operator technique for variational inclusions,” *Applied Mathematics and Computation*, vol. 145, no. 2-3, pp. 795–803, 2003.

[2] A. Hassouni and A. Moudafi, “A perturbed algorithm for variational inclusions,” *Journal of Mathematical Analysis and Applications*, vol. 185, no. 3, pp. 706–712, 1994.

[3] H.-Y. Lan, Y. J. Cho, and R. U. Verma, “Nonlinear relaxed cocoercive variational inclusions involving (A,η)-accretive mappings in Banach spaces,” *Computers & Mathematics with Applications*, vol. 51, no. 9-10, pp. 1529–1538, 2006.

[4] H.-Y. Lan, “On multivalued nonlinear variational inclusion problems with (A,η)-accretive mappings in Banach spaces,” *Journal of Inequalities and Applications*, vol. 2006, Article ID 59836, 12 pages, 2006.

[5] N.-J. Huang and Y.-P. Fang, “A new class of general variational inclusions involving maximal η-monotone mappings,” *Publicationes Mathematicae Debrecen*, vol. 62, no. 1-2, pp. 83–98, 2003.

[6] M. A. Noor, “Generalized set-valued variational inclusions and resolvent equations,” *Journal of Mathematical Analysis and Applications*, vol. 228, no. 1, pp. 206–220, 1998.

[7] R. U. Verma, “Sensitivity analysis for generalized strongly monotone variational inclusions based on the (A,η)-resolvent operator technique,” *Applied Mathematics Letters*, vol. 19, no. 12, pp. 1409–1413, 2006.

[8] N.-J. Huang, “Nonlinear implicit quasi-variational inclusions involving generalized m-accretive mappings,” *Archives of Inequalities and Applications*, vol. 2, no. 4, pp. 413–425, 2004.

[9] Y.-P. Fang and N.-J. Huang, “H-accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces,” *Applied Mathematics Letters*, vol. 17, no. 6, pp. 647–653, 2004.

[10] X. Z. Yu, “Ishikawa iterative process variational inclusions with (H,η)-accretive operators in Banach spaces,” to appear in *Journal of Inequalities and Applications*.

[11] S. Adly, “Perturbed algorithms and sensitivity analysis for a general class of variational inclusions,” *Journal of Mathematical Analysis and Applications*, vol. 201, no. 2, pp. 609–630, 1996.

[12] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, Leyden, The Netherlands, 1996.

[13] E. Zeidler, *Nonlinear Functional Analysis and Its Applications. II/A. Linear Monotone Operators*, Springer, New York, NY, USA, 1990.

[14] E. Zeidler, *Nonlinear Functional Analysis and Its Applications. II/B. Nonlinear Monotone Operators*, Springer, New York, NY, USA, 1990.

[15] P. T. Harker and J.-S. Pang, "Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications," *Mathematical Programming*, vol. 48, no. 2, pp. 161–220, 1990.

[16] Q. H. Ansari and J.-C. Yao, "A fixed point theorem and its applications to a system of variational inequalities," *Bulletin of the Australian Mathematical Society*, vol. 59, no. 3, pp. 433–442, 1999.

[17] G. Kassay and J. Kolumbán, "System of multi-valued variational variational inequalities," *Publicationes Mathematicae*, vol. 56, no. 1-2, pp. 185–195, 2000.

[18] G. Kassay, J. Kolumbán, and Z. Páles, "Factorization of Minty and Stampacchia variational inequality systems," *European Journal of Operational Research*, vol. 143, no. 2, pp. 377–389, 2002.

[19] J.-W. Peng, "System of generalised set-valued quasi-variational-like inequalities," *Bulletin of the Australian Mathematical Society*, vol. 68, no. 3, pp. 501–515, 2003.

[20] J.-W. Peng and X. Yang, "On existence of a solution for the system of generalized vector quasi-equilibrium problems with upper semicontinuous set-valued maps," *International Journal of Mathematics and Mathematical Sciences*, vol. 2005, no. 15, pp. 2409–2420, 2005.

[21] R. U. Verma, "Projection methods, algorithms, and a new system of nonlinear variational inequalities," *Computers & Mathematics with Applications*, vol. 41, no. 7-8, pp. 1025–1031, 2001.

[22] R. U. Verma, "Iterative algorithms and a new system of nonlinear quasivariational inequalities," *Advances in Nonlinear Variational Inequalities*, vol. 4, no. 1, pp. 117–124, 2001.

[23] R. U. Verma, "General convergence analysis for two-step projection methods and applications to variational problems," *Applied Mathematics Letters*, vol. 18, no. 11, pp. 1286–1292, 2005.

[24] R. U. Verma, "On a new system of nonlinear variational inequalities and associated iterative algorithms," *Mathematical Sciences Research Hot-Line*, vol. 3, no. 8, pp. 65–68, 1999.

[25] R. U. Verma, "Generalized system for relaxed cocoercive variational inequalities and projection methods," *Journal of Optimization Theory and Applications*, vol. 121, no. 1, pp. 203–210, 2004.

[26] J. K. Kim and D. S. Kim, "A new system of generalized nonlinear mixed variational inequalities in Hilbert spaces," *Journal of Convex Analysis*, vol. 11, no. 1, pp. 235–243, 2004.

[27] Y. J. Cho, Y. P. Fang, N. J. Huang, and H. J. Hwang, "Algorithms for systems of nonlinear variational inequalities," *Journal of the Korean Mathematical Society*, vol. 41, no. 3, pp. 489–499, 2004.

[28] J.-S. Pang, "Asymmetric variational inequality problems over product sets: applications and iterative methods," *Mathematical Programming*, vol. 31, no. 2, pp. 206–219, 1985.

[29] G. Cohen and F. Chaplais, "Nested monotony for variational inequalities over product of spaces and convergence of iterative algorithms," *Journal of Optimization Theory and Applications*, vol. 59, no. 3, pp. 369–390, 1988.

[30] M. Bianchi, "Pseudo P-monotone operators and variational inequalities," Report 6, Istituto di econometria e Matematica per le Decisioni Economiche, Universita Cattolica del Sacro Cuore, Milan, Italy, 1993.

[31] Q. H. Ansari, S. Schaible, and J. C. Yao, "System of vector equilibrium problems and its applications," *Journal of Optimization Theory and Applications*, vol. 107, no. 3, pp. 547–557, 2000.

[32] E. Allevi, A. Gnudi, and I. V. Konnov, "Generalized vector variational inequalities over product sets," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 47, no. 1, pp. 573–582, 2001.

[33] R. P. Agarwal, N.-J. Huang, and M.-Y. Tan, "Sensitivity analysis for a new system of generalized nonlinear mixed quasi-variational inclusions," *Applied Mathematics Letters*, vol. 17, no. 3, pp. 345–352, 2004.

[34] K. R. Kazmi and M. I. Bhat, "Iterative algorithm for a system of nonlinear variational-like inclusions," *Computers & Mathematics with Applications*, vol. 48, no. 12, pp. 1929–1935, 2004.

[35] Y. P. Fang and N. J. Huang, "H-monotone operators and system of variational inclusions," *Communications on Applied Nonlinear Analysis*, vol. 11, no. 1, pp. 93–101, 2004.

[36] Y.-P. Fang, N.-J. Huang, and H. B. Thompson, "A new system of variational inclusions with (H,η)-monotone operators in Hilbert spaces," *Computers & Mathematics with Applications*, vol. 49, no. 2-3, pp. 365–374, 2005.

[37] J.-W. Peng and J. Huang, "A new system of variational inclusions with (H,η)-monotone operators," *Bulletin of the Australian Mathematical Society*, vol. 74, no. 2, pp. 301–319, 2006.

[38] X. Q. Yang and J. C. Yao, "Gap functions and existence of solutions to set-valued vector variational inequalities," *Journal of Optimization Theory and Applications*, vol. 115, no. 2, pp. 407–417, 2002.

[39] X. Q. Yang, "Vector variational inequality and its duality," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 21, no. 11, pp. 869–877, 1993.

[40] X. Q. Yang, "Generalized convex functions and vector variational inequalities," *Journal of Optimization Theory and Applications*, vol. 79, no. 3, pp. 563–580, 1993.

[41] M. A. Noor, K. I. Noor, and Th. M. Rassias, "Set-valued resolvent equations and mixed variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 220, no. 2, pp. 741–759, 1998.

[42] E. Al-Shemas and S. C. Billups, "An iterative method for generalized set-valued nonlinear mixed quasi-variational inequalities," *Journal of Computational and Applied Mathematics*, vol. 170, no. 2, pp. 423–432, 2004.

[43] G. X.-Z. Yuan, "The study of minimax inequalities and applications to economies and variational inequalities," *Memoirs of the American Mathematical Society*, vol. 132, no. 625, p. 140, 1998.

[44] D.-L. Zhu and P. Marcotte, "Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities," *SIAM Journal on Optimization*, vol. 6, no. 3, pp. 714–726, 1996.

[45] P. Marcotte and D.-L. Zhu, "Weak sharp solutions of variational inequalities," *SIAM Journal on Optimization*, vol. 9, no. 1, pp. 179–189, 1999.

[46] M. A. Noor, "Three-step iterative algorithms for multivalued quasi variational inclusions," *Journal of Mathematical Analysis and Applications*, vol. 255, no. 2, pp. 589–604, 2001.

[47] R. P. Agarwal, N. J. Huang, and Y. J. Cho, "Generalized nonlinear mixed implicit quasi-variational inclusions with set-valued mappings," *Journal of Inequalities and Applications*, vol. 7, no. 6, pp. 807–828, 2002.

[48] J.-W. Peng and X.-M. Yang, "Generalized vector quasi-variational-like inequalities," *Journal of Inequalities and Applications*, vol. 2006, Article ID 59387, 11 pages, 2006.

[49] J.-W. Peng, "Set-valued variational inclusions with T-accretive operators in Banach spaces," *Applied Mathematics Letters*, vol. 19, no. 3, pp. 273–282, 2006.

[50] F. Giannessi, "Theorems of alternative, quadratic programs and complementarity problems," in *Variational Inequalities and Complementarity Problems (Proceedings of an International School of Mathematics, Erice, 1978)*, R. W. Cottle, F. Giannessi, and J.-L. Lions, Eds., pp. 151–186, John Wiley & Sons, Chichester, UK, 1980.

[51] Z. B. Xu and G. F. Roach, "Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 157, no. 1, pp. 189–210, 1991.

Jian-Wen Peng: College of Mathematics and Computer Science, Chongqing Normal University, Chongqing 400047, China; Department of Management Science, School of Management, Fudan University, Shanghai 200433, China
Email address: jwpeng6@yahoo.com.cn

Dao-Li Zhu: Department of Management Science, School of Management, Fudan University, Shanghai 200433, China
Email address: dlzhu@fudan.edu.cn

Xiao-Ping Zheng: The Institute of Safety Management, Beijing University of Chemical Technology, Beijing 100029, China
Email address: asean@vip.163.com