

APPROXIMATION COMMON FIXED POINT OF ASYMPTOTICALLY QUASI-NONEXPANSIVE-TYPE MAPPINGS BY THE FINITE STEPS ITERATIVE SEQUENCES

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The purpose of this paper is to study sufficient and necessary conditions for finite-step iterative sequences with mean errors for a finite family of asymptotically quasi-nonexpansive and type mappings in Banach spaces to converge to a common fixed point. The results presented in this paper improve and extend the recent ones announced by Ghost-Debnath, Liu, Xu and Noor, Chang, Shahzad et al., Shahzad and Udomene, Chidume et al., and all the others.

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1. Introduction and preliminaries

Throughout this paper, we assume that E is a real Banach space, $F(T)$, $D(T)$, and N denote the set of fixed points of T , the domain of T , and the set of positive integers, respectively.

Definition 1.1. Let $T : D(T) = E \rightarrow E$ be a mapping.

- (1) T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$, for all $x \in E$ and $p \in F(T)$.
- (2) T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of positive real numbers with $k_n \geq 1$ and $\lim_{n \rightarrow +\infty} k_n = 1$, such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$, for all $x, y \in E$ and $n \in N$.
- (3) T is said to be *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ of positive real numbers with $k_n \geq 1$ and $\lim_{n \rightarrow +\infty} k_n = 1$ such that $\|T^n x - p\| \leq k_n \|x - p\|$, for all $x \in E$, $p \in F(T)$, and all $n \in N$.
- (4) T is said to be *asymptotically nonexpansive type* if

$$\limsup_{n \rightarrow \infty} \left\{ \sup_{x, y \in E} [\|T^n x - T^n y\|^2 - \|x - y\|^2] \right\} \leq 0. \quad (1.1)$$

2 Approximation of common fixed point

(5) T is said to be *asymptotically quasi-nonexpansive* type if

$$\limsup_{n \rightarrow \infty} \left\{ \sup_{x \in E, y \in F(T)} [\|T^n x - p\|^2 - \|x - p\|^2] \right\} \leq 0. \quad (1.2)$$

From the above definitions, it follows that if $F(T)$ is nonempty, quasi-nonexpansive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings, and asymptotically nonexpansive type-mappings are all special cases of asymptotically quasi-nonexpansive-type mappings.

Definition 1.2 (see [2]). Let $T_1, T_2, T_3 : E \rightarrow E$ be asymptotically quasi-nonexpansive-type mappings. Let $\{u_n\}, \{v_n\}, \{w_n\}$ be three given sequences in E and let x_1 be a given point. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\eta_n\}, \{\xi_n\}$ be sequences in $[0,1]$ satisfying the following conditions:

$$\begin{aligned} \alpha_n + \gamma_n &\leq 1, & \beta_n + \delta_n &\leq 1, & \eta_n + \xi_n &\leq 1, \\ \sum_{n=1}^{\infty} \gamma_n &< \infty, & \sum_{n=1}^{\infty} \delta_n &< \infty, & \sum_{n=1}^{\infty} \xi_n &< \infty. \end{aligned} \quad (1.3)$$

Then the sequence $\{x_n\} \subset E$ defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n - \gamma_n)x_n + \alpha_n T_1^n y_n + \gamma_n u_n, & n \geq 1, \\ y_n &= (1 - \beta_n - \delta_n)x_n + \beta_n T_2^n z_n + \delta_n v_n, & n \geq 1, \\ z_n &= (1 - \eta_n - \xi_n)x_n + \eta_n T_3^n x_n + \xi_n w_n, & n \geq 1, \end{aligned} \quad (1.4)$$

is called the *three-step iterative sequence with mean errors* of T_1, T_2, T_3 .

Let $T_1, T_2, \dots, T_N : E \rightarrow E$ be N asymptotically quasi-nonexpansive-type mappings. Let x_1 be a given point. Then the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_{n+1} &= (1 - a_{n1} - b_{n1})x_n + a_{n1} T_1^n y_{n1} + b_{n1} u_{n1}, \\ y_{n1} &= (1 - a_{n2} - b_{n2})x_n + a_{n2} T_2^n y_{n2} + b_{n2} u_{n2}, \\ &\vdots \\ y_{nN-2} &= (1 - a_{nN-1} - b_{nN-1})x_n + a_{nN-1} T_{N-1}^n y_{nN-1} + b_{nN-1} u_{nN-1}, \\ y_{nN-1} &= (1 - a_{nN} - b_{nN})x_n + a_{nN} T_N^n x_n + b_{nN} u_{nN}, \end{aligned} \quad (1.5)$$

is called the *N -step iterative sequence with mean errors* of T_1, T_2, \dots, T_N , where $\{u_{ni}\}_{n=1}^{\infty}$, $i = 1, 2, \dots, N$, are N sequences in E , $\{a_{ni}\}_{n=1}^{\infty}$, $\{b_{ni}\}_{n=1}^{\infty}$, $i = 1, 2, \dots, N$, are N sequences in $[0,1]$ satisfying the following conditions:

$$\begin{aligned} a_{ni} + b_{ni} &\leq 1, & n &\leq 1, & i &= 1, 2, \dots, N, \\ \sum_{n=1}^{\infty} b_{ni} &< \infty, & i &= 1, 2, \dots, N. \end{aligned} \quad (1.6)$$

Petryshyn and Williamson [9] proved a sufficient and necessary condition for the Mann iterative sequences to converge to a fixed point for quasi-nonexpansive mappings. Ghosh and Debnath [5] extended the result of [9] and gave a sufficient and necessary condition for the Ishikawa iterative sequence to converge to a fixed point for quasi-nonexpansive mappings. Liu [6–8] extended the above results and proved some sufficient and necessary conditions for the Ishikawa iterative sequence or the Ishikawa iterative sequences with errors for asymptotically quasi-nonexpansive mappings to converge to a fixed point. Chidume et al. [4] obtained a strong convergence theorem to a fixed point of a family of nonself nonexpansive mappings in Banach spaces by an algorithm for nonself-mappings. Shahzad and Udomene [10] established necessary and sufficient conditions for the convergence of the Ishikawa-type iterative sequences involving two asymptotically quasi-nonexpansive mappings to a common fixed point of the mappings defined on a nonempty closed convex subset of a Banach space and a sufficient condition for the convergence of the Ishikawa-type iterative sequences involving two uniformly continuous asymptotically quasi-nonexpansive mappings to a common fixed point of the mappings defined on a nonempty closed convex subset of a uniformly convex Banach space. Alber [1] studied the approximating methods for finding the fixed points of asymptotically nonexpansive mappings.

Recently, Chang et al. [2] complement, improve, and perfect all the above results and obtained some necessary and sufficient conditions for the Ishikawa iterative sequence with mixed errors of asymptotically quasi-nonexpansive-type mappings in Banach spaces to converge to a fixed point in Banach spaces. And also using the N -step iterative sequences (1.5), Chang et al. [3] proved the weak and strong convergence of finite steps iterative sequences with mean errors to a common fixed point for a finite family of asymptotically nonexpansive mappings.

The purpose of this paper is to study sufficient and necessary conditions for finite-step iterative sequences with mean errors for a finite family of asymptotically quasi-nonexpansive-type mappings in Banach spaces to converge to a common fixed point. Our result shows that [2, Condition (2.1) in Theorem 2.1] can be removed. The results present in this paper improve, extend, and perfect the recent ones announced by Petryshyn and Williamson [9], Ghost and Debnath [5], Liu [6, 7], Xu and Noor [12], Chang [2, 3], Shahzad et al. [4], Shahzad and Udomene [10], Chidume et al. [1], and all the others.

In order to prove our main results, we will need the following lemma.

LEMMA 1.3 (see [11]). *Let $\{a_n\}, \{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \quad n \geq 1. \quad (1.7)$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists.

2. Main results

THEOREM 2.1. *Let E be a Banach space and $T_i: E \rightarrow E$ ($i = 1, 2, \dots, N$) be N asymptotically quasi-nonexpansive-type mappings with a nonempty fixed-point set $F(T) = \bigcap_{i=1}^N F(T_i)$, that*

4 Approximation of common fixed point

is,

$$\limsup_{n \rightarrow \infty} \left\{ \sup_{x \in E, p \in F(T)} [\|T_i^n x - p\|^2 - \|x - p\|^2] \right\} \leq 0, \quad i = 1, 2, \dots, N. \quad (2.1)$$

Let $\{u_n\}$ be a bounded sequence in E . For any given point x_1 in E , generate the sequence $\{x_n\}$ defined by (1.5). If $\sum_{n=1}^{\infty} \alpha_n < \infty$, then sequence $\{x_n\}$ strongly converges to a common fixed point of T_i ($i = 1, 2, \dots, N$) if and only if $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$, where $d(y, S)$ denotes the distance of y to set S ; that is, $d(y, S) = \inf_{s \in S} \|y - s\|$.

Proof. (1) For the sake of convenience, we prove the conclusion only for the case of $N = 3$ and then the other cases can be proved by the same way. For the purpose, let $\alpha_n = a_n$, $\beta_n = a_n^2$, $\eta_n = a_n^3$, $\gamma_n = b_n$, $\delta_n = b_n^2$, $\xi_n = b_n^3$. Then we can consider the sequence $\{x_n\}$ defined by (1.4) and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded. For all $p \in F(T)$, let

$$\begin{aligned} M_1 &= \sup \{ \|u_n - p\| : n \geq 1 \}, & M_2 &= \sup \{ \|v_n - p\| : n \geq 1 \}, \\ M_3 &= \sup \{ \|w_n - p\| : n \geq 1 \}, & M &= \max \{ M_i : i = 1, 2, 3 \}. \end{aligned} \quad (2.2)$$

It follows from (2.1) that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left\{ \sup_{x \in E, p \in F(T)} [(\|T_i^n x - p\| - \|x - p\|) (\|T_i^n x + p\| - \|x - p\|)] \right\} \\ &= \limsup_{n \rightarrow \infty} \left\{ \sup_{x \in E, p \in F(T)} [\|T_i^n x - p\|^2 - \|x - p\|^2] \right\} \leq 0, \quad i = 1, 2, 3. \end{aligned} \quad (2.3)$$

Therefore we have

$$\limsup_{n \rightarrow \infty} \left\{ \sup_{x \in E, p \in F(T)} [\|T_i^n x - p\| - \|x - p\|] \right\} \leq 0, \quad i = 1, 2, 3. \quad (2.4)$$

This implies that for any given $\epsilon > 0$, there exists a positive integer n_0 such that for $n \geq n_0$, we have

$$\sup_{x \in E, p \in F(T)} \left\{ \|T_i^n x - p\| - \|x - p\| \right\} < \epsilon, \quad i = 1, 2, 3. \quad (2.5)$$

Since $\{x_n\}, \{y_n\}, \{z_n\} \subset E$, we have

$$\|T_1^n y_n - p\| - \|y_n - p\| < \epsilon, \quad \forall p \in F(T), \forall n \geq n_0, \quad (2.6)$$

$$\|T_2^n z_n - p\| - \|z_n - p\| < \epsilon, \quad \forall p \in F(T), \forall n \geq n_0, \quad (2.7)$$

$$\|T_3^n x_n - p\| - \|x_n - p\| < \epsilon, \quad \forall p \in F(T), \forall n \geq n_0. \quad (2.8)$$

Thus for any $p \in F(T)$, using (1.4) and (2.6), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \alpha_n - \gamma_n)(x_n - p) + \alpha_n(T_1^n y_n - p) + \gamma_n(u_n - p)\| \\
&\leq (1 - \alpha_n - \lambda_n)\|x_n - p\| + \alpha_n(\|T_1^n y_n - p\| - \|y_n - p\|) \\
&\quad + \alpha_n\|y_n - p\| + \gamma_n\|u_n - p\| \\
&\leq (1 - \alpha_n - \lambda_n)\|x_n - p\| + \alpha_n\epsilon + \alpha_n\|y_n - p\| + \gamma_n M.
\end{aligned} \tag{2.9}$$

Consider the third term in the right-hand side of (2.9), using (1.4) and (2.7), we have that

$$\begin{aligned}
\|y_n - p\| &= \|(1 - \beta_n - \delta_n)(x_n - p) + \beta_n(T_2^n z_n - p) + \delta_n(v_n - p)\| \\
&\leq (1 - \beta_n - \delta_n)\|x_n - p\| + \beta_n(\|T_2^n z_n - p\| - \|z_n - p\|) \\
&\quad + \beta_n\|z_n - p\| + \delta_n\|v_n - p\| \\
&\leq (1 - \beta_n - \delta_n)\|x_n - p\| + \beta_n\epsilon + \beta_n\|z_n - p\| + \delta_n M.
\end{aligned} \tag{2.10}$$

Consider the third term in the right-hand side of (2.10), using (1.4) and (2.8), we have that

$$\begin{aligned}
\|z_n - p\| &= \|(1 - \eta_n - \xi_n)(x_n - p) + \eta_n(T_3^n x_n - p) + \xi_n(w_n - p)\| \\
&\leq (1 - \eta_n - \xi_n)\|x_n - p\| + \eta_n(\|T_3^n x_n - p\| - \|x_n - p\|) \\
&\quad + \eta_n\|x_n - p\| + \xi_n\|w_n - p\| \\
&\leq (1 - \xi_n)\|x_n - p\| + \eta_n\epsilon + \xi_n M.
\end{aligned} \tag{2.11}$$

Substituting (2.11) into (2.10) and simplifying, we have

$$\|y_n - p\| \leq (1 - \beta_n \xi_n - \delta_n)\|x_n - p\| + \beta_n\epsilon(1 + \eta_n) + \beta_n \xi_n M + \delta_n M. \tag{2.12}$$

Substituting (2.12) into (2.9) and simplifying, we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq (1 - \gamma_n - \alpha_n \beta_n \xi_n - \alpha_n \delta_n)\|x_n - p\| + \alpha_n\epsilon + \alpha_n \beta_n \epsilon(1 + \eta_n) \\
&\quad + \alpha_n \delta_n M + \alpha_n \beta_n \xi_n M + \gamma_n M \\
&\leq \|x_n - p\| + \alpha_n(1 + \beta_n + \beta_n \eta_n)\epsilon + (\gamma_n + \delta_n + \xi_n)M \\
&\leq \|x_n - p\| + 3\alpha_n\epsilon + (\gamma_n + \delta_n + \xi_n)M.
\end{aligned} \tag{2.13}$$

Let $A_n = 3\alpha_n\epsilon + (\gamma_n + \delta_n + \xi_n)M$. Then $A_n \geq 0$. It follows from (1.3) and $\sum_{n=1}^{\infty} \alpha_n^i < \infty$ that $\sum_{n=1}^{\infty} A_n < \infty$. Then by (2.13), we have

$$\|x_{n+1} - p\| \leq \|x_n - p\| + A_n. \tag{2.14}$$

It follows from (2.14) and $\sum_{n=1}^{\infty} A_n < \infty$ that

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + A_n. \tag{2.15}$$

6 Approximation of common fixed point

By Lemma 1.3, we know that $\lim_{n \rightarrow \infty} d(x_n, F(T))$ exists. Because $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$, then we have

$$\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0. \quad (2.16)$$

Next we prove that $\{x_n\}$ is a Cauchy sequence in E .

It follows from (2.14) that for any $m \geq 1$, for all $n \geq n_0$, for all $p \in F(T)$,

$$\begin{aligned} \|x_{n+m} - p\| &\leq \|x_{n+m-1} - p\| + A_{n+m-1} \\ &\leq \|x_{n+m-2} - p\| + (A_{n+m-1} + A_{n+m-2}) \\ &\leq \dots \leq \|x_n - p\| + \sum_{k=n}^{n+m-1} A_k. \end{aligned} \quad (2.17)$$

So by (2.17), we have

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p\| + \|x_n - p\| \leq 2\|x_n - p\| + \sum_{k=n}^{\infty} A_k. \quad (2.18)$$

By the arbitrariness of $p \in F(T)$ and (2.18), we know that

$$\|x_{n+m} - x_n\| \leq 2d(x_n, F(T)) + \sum_{k=n}^{\infty} A_k, \quad \forall n \geq n_0. \quad (2.19)$$

For any given $\bar{\epsilon} > 0$, there exists a positive integer $n_1 \geq n_0$ such that for any $n \geq n_1$, $d(x_n, F(T)) < \bar{\epsilon}/4$ and $\sum_{k=n}^{\infty} A_k < \bar{\epsilon}/2$. Thus when $n \geq n_1$, $\|x_{n+m} - x_n\| < \bar{\epsilon}$. So we have that

$$\lim_{n \rightarrow \infty} \|x_{n+m} - x_n\| = 0. \quad (2.20)$$

This implies that $\{x_n\}$ is a Cauchy sequence in E . Since E is complete, there exists a $p^* \in E$ such that $x_n \rightarrow p^*$ as $n \rightarrow \infty$.

Now we have to prove that p^* is a common fixed point of T_i , $i = 1, 2, \dots, N$, that is, $p^* \in F(T)$.

By contradiction, we assume that p^* is not in $F(T)$. Since $F(T)$ is closed in Banach spaces, $d(p^*, F(T)) > 0$. So for all $p \in F(T)$, we have

$$\|p^* - p\| \leq \|p^* - x_n\| + \|x_n - p\|. \quad (2.21)$$

By the arbitrary of $p \in F(T)$, we know that

$$d(p^*, F(T)) \leq \|p^* - x_n\| + d(x_n, F(T)). \quad (2.22)$$

By (2.16), above inequality and $x_n \rightarrow p^*$ as $n \rightarrow \infty$, we have

$$d(p^*, F(T)) = 0, \quad (2.23)$$

which contracts $d(p^*, F(T)) > 0$. This completes the proof of Theorem 2.1. \square

COROLLARY 2.2. *Suppose the conditions in Theorem 2.1 are satisfied. Then the N -step iterative sequence $\{x_n\}$ generated by (1.5) converges to a common fixed point $p \in E$ if and only if there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges to p .*

THEOREM 2.3. *Let E be a Banach space and let $T_i : E \rightarrow E$ ($i = 1, 2, \dots, N$) be N asymptotically quasi-nonexpansive mappings with a nonempty fixed-point set $F(T) = \bigcap_{i=1}^N F(T_i)$. Let $\{u_{n_i}\}$ be a bounded sequence in E . For any given point x_1 in E , generate the sequence $\{x_n\}$ by (1.5). If $\sum_{n=1}^{\infty} \alpha_{n_i} < \infty$, then sequence $\{x_n\}$ strongly converges to a common fixed point of T_i ($i = 1, 2, \dots, N$) if and only if $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$, where $d(y, S)$ denotes the distance of y to set S .*

Proof. Since T_i are asymptotically quasi-nonexpansive mappings with a nonempty fixed-point set $F(T) = \bigcap_{i=1}^N F(T_i)$, by [3, Proposition 1] or [13], we know that there must exist a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that

$$\|T_i^n x - p\| \leq k_n \|x - p\|, \quad \forall p \in F(T), \forall x \in E, n \geq 1. \tag{2.24}$$

This implies that

$$\|T_i^n x - p\|^2 - (k_n)^2 \|x - p\|^2 \leq 0, \quad \forall p \in F(T), \forall x \in E, n \geq 1. \tag{2.25}$$

Therefore we have

$$\limsup_{n \rightarrow \infty} \left\{ \sup_{x \in D, p \in F(T)} [\|T_i^n x - p\|^2 - \|x - p\|^2] \right\} \leq 0, \quad i = 1, 2, \dots, N. \tag{2.26}$$

This implies that T_i , $i = 1, 2, \dots, N$, are N asymptotically quasi-nonexpansive-type mappings with a nonempty fixed-point set $F(T) = \bigcap_{i=1}^N F(T_i)$. Theorem 2.3 can be proved by Theorem 2.1 immediately. \square

THEOREM 2.4. *Let E be a Banach space and let $T_i : E \rightarrow E$ ($i = 1, 2, \dots, N$) be N asymptotically nonexpansive mappings with a nonempty fixed-point set $F(T) = \bigcap_{i=1}^N F(T_i)$. Let $\{u_{n_i}\}$ be a bounded sequence in E . For any given point x_1 in E , generate the sequence $\{x_n\}$ by (1.5). If $\sum_{n=1}^{\infty} \alpha_{n_i} < \infty$, then sequence $\{x_n\}$ strongly converges to a common fixed point of T_i ($i = 1, 2, \dots, N$) if and only if $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$.*

Remarks 2.5. We would like to point out that Theorems 2.1, 2.3, and 2.4 generalize and improve the corresponding results of Petryshyn and Williamson [9], Ghost and Debnath [5], Liu [6, 7], and Xu and Noor [12]. These theorems especially improve Chang’s results [2] in the following aspects.

- (1) We removed the condition (2.1) “there exists constant $L > 0$ and $\alpha > 0$ such that $\|Tx - p\| \leq L\|x - p\|^\alpha, \forall x \in E, \forall p \in F(T)$ ” in [2].
- (2) “The Ishikawa iterative sequence with mixed errors” is extended to N -step iterative sequence with mean errors, and so we obtain the common fixed point of N asymptotically nonexpansive-type mappings.

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