FIXED POINT THEOREMS IN LOCALLY CONVEX SPACES—THE SCHAUDER MAPPING METHOD

S. COBZAŞ

Received 22 March 2005; Revised 22 July 2005; Accepted 6 September 2005

In the appendix to the book by F. F. Bonsal, *Lectures on Some Fixed Point Theorems of Functional Analysis* (Tata Institute, Bombay, 1962) a proof by Singbal of the Schauder-Tychonoff fixed point theorem, based on a locally convex variant of Schauder mapping method, is included. The aim of this note is to show that this method can be adapted to yield a proof of Kakutani fixed point theorem in the locally convex case. For the sake of completeness we include also the proof of Schauder-Tychonoff theorem based on this method. As applications, one proves a theorem of von Neumann and a minimax result in game theory.

1. Introduction

Let B^n be the unit ball of the Euclidean space $\mathbb{R}^n$. Brouwer's fixed point theorem asserts that any continuous mapping $f : B^n \to B^n$ has a fixed point, that is, there exists $x \in B^n$ such that $f(x) = x$. The result holds for any nonempty convex bounded closed subset K of $\mathbb{R}^n$, or of any finite dimensional normed space (see [8, Theorems 18.9 and 18.9′]). Schauder [16] extended this result to the case when K is a convex compact subset of an arbitrary normed space X. Using some special functions, called Schauder mappings, the proof of Schauder's theorem can be reduced to Brouwer fixed point theorem (see. e.g. [8, page 197] or [12, page 180]). A further extension of this theorem was given by Tychonoff [18], who proved its validity when K is a compact convex subset of a Hausdorff locally convex space X. The proof given in the treatise of Dunford and Schwartz [4] is based on three lemmas and, with some minor modifications, the same proof appears in [5] and [9]. The extension of Schauder mapping method to locally convex case was given by Singbal who used it to prove the Schauder-Tychonoff theorem. This proof is included as an appendix to Bonsal's book [3] (see also [17, page 33]).

Kakutani [10] proved an extension of Brouwer's fixed point theorem to upper semi-continuous set-valued mappings defined on compact convex subsets of $\mathbb{R}^n$, which was

Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2006, Article ID 57950, Pages 1–13
DOI 10.1155/FPTA/2006/57950

extended to Banach spaces by Bohnenblust and Karlin [2], and to locally convex spaces by Glicksberg [7]. Nikaido [15] gave a new proof of Kakutani's theorem (in the case $\mathbb{R}^n$) based on the method of Schauder's mappings. This proof is extended to Banach spaces in [11].

The aim of this Note is to show that Schauder mapping method can be adapted to yield a proof of Kakutani fixed point theorem in locally convex spaces. For the sake of completeness we include also a proof of Schauder-Tychonoff theorem which is essentially Singbal's proof, with the difference that we use the fact that a net in a compact set admits a convergent subnet instead of the equivalent fact that it has a cluster point, as did Singbal. A similar proof appears also in [1, page 61], but it is based on the existence of a partition of unity instead of the Schauder mapping.

A *locally convex space* is a topological vector space (X,τ) admitting a neighborhood basis at 0 formed by convex sets. It follows that every point in X admits a neighborhood basis formed of convex sets and there is a neighborhood basis at 0 formed by open convex symmetric sets. Let P be a family of seminorms on a vector space X and let $\mathcal{F}(P) := \{F \subset P : F \text{ nonempty and finite}\}$. For $F \in \mathcal{F}(P)$ and $r > 0$, let

$$B'_F(x,r) = \{x' \in X : \forall p \in F,\ p(x'-x) < r\},$$
$$B_F(x,r) = \{x' \in X : \forall p \in F,\ p(x'-x) \le r\}. \tag{1.1}$$

If $F = \{p\}$, then we use the notation $B'_p(x,r)$ and $B_p(x,r)$ to designate the open, respectively closed, p-ball. The family of sets

$$\mathcal{B}'(x) = \{B'_F(x,r) : F \in \mathcal{F}(P) \text{ and } r > 0\} \tag{1.2}$$

forms a neighborhood basis of a locally convex topology τ_P on X.

The family of sets

$$\mathcal{B}(x) = \{B_F(x,r) : F \in \mathcal{F}(P) \text{ and } r > 0\} \tag{1.3}$$

is also a neighborhood basis at x for τ_P. If B is a convex symmetric absorbing subset of a vector space X, then the Minkowski functional $p_B : X \to [0,\infty)$ defined by

$$p_B(x) = \inf\{\lambda > 0 : x \in \lambda B\}, \quad x \in X, \tag{1.4}$$

is a seminorm on X and

$$\{x \in X : p_B(x) < 1\} \subset B \subset \{x \in X : p_B(x) \le 1\}. \tag{1.5}$$

If X is a topological vector space and B is an open convex symmetric neighborhood of 0, then the seminorm p_B is continuous,

$$B = \{x \in X : p_B(x) < 1\}, \qquad \operatorname{cl} B = \{x \in X : p_B(x) \le 1\}. \tag{1.6}$$

If $\mathcal{B}$ is a neighborhood basis at 0 of a locally convex space (X,τ), formed by open convex symmetric neighborhoods of 0, then $P = \{p_B : B \in \mathcal{B}\}$ is a directed family of seminorms generating the topology τ in the way described above. Therefore, there are two equivalent ways of defining a locally convex space—as a topological vector space (X,τ) such that 0 admits a neighborhood basis formed by convex sets, or as a pair (X,P) where P is a family of seminorms on X generating a locally convex topology on X. We consider only real vector spaces.

A directed set is a partially ordered set $(I,\leq)$ such that for every $i_1,i_2\in I$ there exists $i\in I$ with $i\geq i_1$, and $i\geq i_2$. A net in a set Z is a mapping $\psi:I\to Z$. If $(J,\leq)$ is another directed set and there exists a non-decreasing mapping $\gamma:J\to I$ such that for every $i\in I$ there exists $j\in J$ with $\gamma(j)\geq i$, then we say that $\psi\circ\gamma:J\to Z$ is a subnet of the net ψ. One uses also the notation $(z_i:i\in I)$, where $z_i=\psi(i)$, to designate the net ψ and $(z_{\gamma(j)}:j\in J)$ for a subnet. It is known that a subset K of a topological space T is compact if and only if every net in K admits a subnet converging to an element of K (see [6]).

If $\mathcal{B}(x)$ is a neighborhood basis of a point x of a topological space (X,τ), then it becomes a directed set with respect to the order $B_1\leq B_2\Leftrightarrow B_2\subset B_1$. If $x_B\in X$, $B\in\mathcal{B}$, then $(x_B:B\in\mathcal{B}(x))$ is a net in X. We denote by $\mathcal{V}(x)$ the family of all neighborhoods of a point $x\in X$, and by $\mathrm{cl}(Z)$ the closure of a subset Z of X.

We will use the following facts.

Proposition 1.1. *Let (X,τ) be a topological vector space and $\mathcal{B}$ a neighborhood basis of* 0.

(a) *The topology τ is Hausdorff separated if and only if*

$$\bigcap\{B:B\in\mathcal{B}\}=\{0\}. \tag{1.7}$$

(b) *The closure of any subset A of X can be calculated by the formula*

$$\mathrm{cl}A=\bigcap\{A+B:B\in\mathcal{B}\}. \tag{1.8}$$

(c) *Suppose that the topology of X is Hausdorff. Then for every finite subset $\{a_1,\dots,a_n\}$ of X there exists $m\in\mathbb{N}$, $m\leq n$, such that the set $\mathrm{co}\{a_1,\dots,a_n\}$ is linearly homeomorphic to a compact convex subset of $\mathbb{R}^m$.*

Proof. Properties (a) and (b) are well known (see, e.g. [13]). To prove (c), let $Y=\mathrm{sp}\{a_1,\dots,a_n\}$ and $m=\dim Y$. It follows that Y is linearly homeomorphic to $\mathbb{R}^m$, that is, there exists a linear homeomorphism $\Phi:Y\to\mathbb{R}^m$. Since $Z=\mathrm{co}\{a_1,\dots,a_n\}$ is a compact subset of Y, its image $\Phi(Z)$ will be a convex compact subset of $\mathbb{R}^m$. □

Based on this proposition one obtains the following extended form of Brouwer fixed point theorem.

Corollary 1.2. *If Z is a finite dimensional compact convex subset of a Hausdorff topological vector space X, then any continuous mapping $f:Z\to Z$ has a fixed point.*

Recall that a subset Z of a vector space X is called finite dimensional provided $\dim(\mathrm{sp}(Z))<\infty$.

2. The fixed point theorems

Before passing to the proofs of Schauder-Tychonoff and Kakutani fixed point theorems, we will present the construction of the Schauder projection mapping and its basic properties.

Let p be a seminorm on a vector space X and C a nonempty convex subset of X. For $\epsilon > 0$ suppose that there exists a (p,ϵ)-net $z^1,\dots,z^n \in C$ for C, that is, $C \subset \cup_{i=1}^n B'_p(z^i,\epsilon)$. For $i \in \{1,2,\dots,n\}$ define the real valued functions $g^i = g^i_{p,\epsilon}$, $w = w_{p,\epsilon}$ and $w^i = w^i_{p,\epsilon}$ by

$$g^i(x) = \max\{\epsilon - p(x-z^i),0\}, \qquad w(x) = \sum_{i=1}^n g^i(x),$$
$$w^i(x) = g^i(x)/w(x), \quad x \in C. \tag{2.1}$$

Let also $\varphi = \varphi_{p,\epsilon} : C \to C$ be defined by

$$\varphi(x) = \sum_{i=1}^n w^i(x)z^i, \quad x \in C. \tag{2.2}$$

The mapping $\varphi_{p,\epsilon}$ is called the *Schauder mapping*.

Lemma 2.1. *Let p be a continuous seminorm on a topological vector space (X,τ), C a convex subset of X and $\epsilon > 0$. The mappings defined by (2.1) and (2.2) have the following properties.*

(a) *The functions g^i are continuous and nonnegative on C.*
(b) *The function w is continuous and $\forall x \in C$, $w(x) > 0$.*
(c) *The functions w^i are well defined, continuous, nonnegative, and $\sum_{i=1}^n w^i(x) = 1$, $x \in C$.*
(d) *The mapping φ is continuous on C and*

$$\forall x \in C, \quad p(\varphi(x) - x) < \epsilon. \tag{2.3}$$

Proof. (a) The continuity of g^i follows from the continuity of p and the equality $g^i(x) = 2^{-1}(\epsilon - p(x-z^i) + |\epsilon - p(x-z^i)|)$.

(b) The continuity of w is obvious. Since for every $x \in C$ there exists $j \in \{1,2,\dots,n\}$ such that $p(x-z^j) < \epsilon$, it follows $w(x) \geq g^j(x) = \epsilon - p(x-z^j) > 0$.

(c) Follows from (a) and (b).

(d) By (b) and (c) the functions w^i are well defined and continuous, and $\varphi(x) \in C$ for every $x \in C$, as a convex combination of the elements $z^1,\dots,z^n \in C$. To prove inequality (2.3) observe that, for $x \in C$, $\varphi(x) - x = \sum_{i=1}^n w^i(x)(z^i - x)$, so that, by (c) and the fact that $p(z^i - x) < \epsilon$ whenever $w^i(x) > 0$, we have

$$p(\varphi(x) - x) \leq \sum_{i=1}^n w^i(x)p(z^i - x) < \epsilon. \tag{2.4}$$

□

Remark 2.2. It follows that for every $x \in C$, $\varphi(x)$ is a convex combination of the elements $z^1,\dots,z^n$, so that φ is a mapping from the set C to $\operatorname{co}\{z^1,\dots,z^n\}$.

Now we can state and prove Schauder-Tychonoff theorem.

Theorem 2.3. *If C is a convex compact subset of a Hausdorff locally convex space (X,τ), then any continuous mapping $f : C \to C$ has a fixed point in C.*

Proof. Let $\mathcal{B}$ be a basis of 0-neighborhoods formed by open convex symmetric subsets of X. The Minkowski functional p_B corresponding to a set $B \in \mathcal{B}$ is a continuous seminorm on X and

$$B = \{x \in X : p_B(x) < 1\}. \tag{2.5}$$

By the compactness of the set C there exist $z_B^1,\dots,z_B^{n(B)} \in C$ such that

$$C \subset \left\{z_B^1,\dots,z_B^{n(B)}\right\} + B. \tag{2.6}$$

Denote by φ_B the Schauder mapping corresponding to p_B, $\epsilon = 1$ and $z_B^1,\dots,z_B^{n(B)}$, and let $C_B = \text{co}\{z_B^1,\dots,z_B^{n(B)}\}$. It follows that $f_B = \varphi_B \circ f$ is a continuous mapping of the finite dimensional convex compact set C_B into itself, so that, by Brouwer's fixed point theorem (Corollary 1.2), it has a fixed point, that is, there exists $x_B \in C_B$ such that $f_B(x_B) = x_B$.

Using again the compactness of the set C, the net $(x_B : B \in \mathcal{B})$ admits a subnet $(x_{\gamma(\alpha)} : \alpha \in \Lambda)$ converging to an element $x \in C$. Here Λ is a directed set and $\gamma : \Lambda \to \mathcal{B}$ the non-decreasing mapping defining the subnet. We show that x is a fixed point of f, that is $f(x) = x$. Since the topology of the space X is separated Hausdorff this is equivalent to

$$\forall V \in \mathcal{V}(0), \quad x - f(x) \in V. \tag{2.7}$$

For $V \in \mathcal{V}(0)$ let $B \in \mathcal{B}$ be such that $B + B \subset V$. By the definition of the subnet there exists $\alpha_0 \in \Lambda$ such that $\gamma(\alpha_0) \subset B$. Then for all $\alpha \geq \alpha_0$, $\gamma(\alpha) \subset \gamma(\alpha_0) \subset B$, so that, by (2.3) (with $\epsilon = 1$), the fact that $\varphi_{\gamma(\alpha)}(f(x_{\gamma(\alpha)})) = x_{\gamma(\alpha)}$ and (2.5), we get

$$\begin{aligned} &p_{\gamma(\alpha)}\big(\varphi_{\gamma(\alpha)}(f(x_{\gamma(\alpha)})) - f(x_{\gamma(\alpha)})\big) < 1 \\ &\quad\Longrightarrow \varphi_{\gamma(\alpha)}(f(x_{\gamma(\alpha)})) - f(x_{\gamma(\alpha)}) \in \gamma(\alpha) \subset B \Longrightarrow x_{\gamma(\alpha)} - f(x_{\gamma(\alpha)}) \in B. \end{aligned} \tag{2.8}$$

Passing to limit for $\alpha \geq \alpha_0$ and taking into account the continuity of f, one obtains

$$x - f(x) \in \text{cl}\, B \subset B + B \subset V, \tag{2.9}$$

that is, (2.7) holds. □

Let (X,P) be a locally convex space. A subset Z of X is called bounded if $\sup p(Z) < \infty$ for every $p \in P$. The space X is called quasi-complete if every closed bounded subset of X is complete. In a quasi-complete locally convex space the closed convex hull of a compact set is compact (see [13, Section 20.6(3)]).

The following result is a variant of the Schauder-Tychonoff fixed point theorem (see [8, Theorem 18.10′] for the Banach space case). In [9] and [14] one proves first this variant of Schauder's fixed point theorem in the Banach space case, by using uniform approximations of completely continuous nonlinear operators by operators with finite range. According to [14], an operator is called completely continuous if it is continuous and sends bounded sets onto relatively compact sets. Obviously that the operator f in the next theorem is completely continuous.

THEOREM 2.4. *Let (X,P) be a quasi-complete Hausdorff locally convex space and C a closed bounded convex subset of X. If $f : C \to C$ is a continuous mapping such that $\mathrm{cl}\, f(C)$ is a compact subset of C, then f has a fixed point in C.*

Proof. The closed convex hull $K = \text{cl-co}\, f(C)$ of the set $f(C)$ is a compact convex subset of C. Since $f(K) \subset f(C) \subset K$, then, by Theorem 2.3, the mapping f has a fixed point in K. □

The technique of Schauder mappings can be used to prove the Kakutani fixed point theorem for set-valued mappings in the locally convex case.

By a set-valued mapping between two sets X, Y we understand a mapping $F : X \to 2^Y$ such that $F(x) \neq \emptyset$ for all $x \in X$. We use the notation $F : X \rightrightarrows Y$. If X, Y are topological spaces, then a set-valued mapping $F : X \rightrightarrows Y$ is called upper semi-continuous (usc) provided for every $x \in X$ and every open set V in Y such that $F(x) \subset V$ there exists an open neighborhood U of x such that $F(U) \subset V$, where $F(U) = \bigcup\{F(x') : x' \in U\}$. The graph of F is the set $G_F = \{(x,y) \in X \times Y : y \in F(x)\}$. The set-valued mapping F is called closed if its graph G_F is a closed subset of $X \times Y$. Obviously that if F has closed graph, then $F(x)$ is closed in Y for every $x \in X$.

For proofs of the following proposition in the case $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ or in the case of normed spaces X, Y, see [15] and [11], respectively. In the case when X, Y are topological spaces, one can proceed similarly, by working with nets instead of sequences. For the sake of completeness we include the proof, but first recall some facts about separation properties in topological spaces (see [6, Chapter VI, Section 1]). A topological space X is called T_1 provided for every $x \in X$ the set $\{x\}$ is closed in X, and T_2, or Hausdorff, if any two distinct points in X have disjoint neighborhoods. If X, Y are topological spaces, Y is Hausdorff and $f,g : X \to Y$ are continuous, then the set $\{x \in X : f(x) = g(x)\}$ is closed in X. A topological space X is called regular if it is T_1 and for any $x \in X$ and any closed subset $A \subset X$ not containing x, there exist two disjoint open sets G_1, $G_2 \subset X$ such that $x \in G_1$ and $A \subset G_2$. This is equivalent to the fact that every point in X has a neighborhood basis formed of closed sets. It is obvious that a Hausdorff locally convex space is regular.

PROPOSITION 2.5. *Let X, Y be topological spaces and $F : X \rightrightarrows Y$ a set-valued mapping.*
(a) *If Y is regular, F is usc and for every $x \in X$ the set $F(x)$ is nonempty and closed, then F has closed graph.*
(b) *Conversely, if the space Y is compact Hausdorff and F is with closed graph, then F is usc.*

Proof. (a) Suppose that the nets $(x_i : i \in I)$ and $y_i \in F(x_i)$, $i \in I$, are such that $x_i \to x$ and $y_i \to y$, for some $x \in X$ and $y \in Y$ with $y \notin F(x)$. Since $F(x)$ is closed and Y is regular, there exists a closed neighborhood W of y such that $W \cap F(x) = \emptyset$. Then $V = Y \setminus W$ is an open set containing $F(x)$ so that, by the upper semi-continuity of F, there exists an open neighborhood U of x such that $F(U) \subset V$. If $i_0 \in I$ is such that for $i \geq i_0$, $x_i \in U$, then $y_i \in F(x_i) \subset V = X \setminus W$, for all $i \geq i_0$. It follows $y_i \notin W$, $\forall i \geq i_0$, in contradiction to $y_i \to y$.

(b) Let $x \in X$ and V an open subset of Y such that $F(x) \subset V$. Put

$$U := \{x' \in X : F(x') \subset V\}. \tag{2.10}$$

By the definition of U, $F(U) \subset V$, so it suffices to show that the set U is open or, equivalently, that the set $W := X \setminus U$ is closed.

Suppose that there exists a net $x_i \in W$, $i \in I$, that converges to an element $x \in U$. By the definition (2.10) of the set U, for every $i \in I$ there exists $y_i \in F(x_i) \setminus V$. By the compactness of the space Y, the net (y_i) contains a subnet $(y_{\gamma(j)} : j \in J)$ converging to an element $y \in Y$. We have $x_{\gamma(j)} \to x$, $y_{\gamma(j)} \in F(x_{\gamma(j)})$ and $y_{\gamma(j)} \to y$, so that, by the closedness of F, $y \in F(x)$. By the choice of the elements y_i, the elements $y_{\gamma(j)}$ belong to the closed set $Y \setminus V$, as well as their limit y, implying $y \in F(x) \setminus V$, in contradiction to $F(x) \subset V$. □

We can state and prove the Kakutani theorem in the locally convex case. An element $x \in X$ is called a fixed point of a set-valued mapping $F : X \rightrightarrows Y$ if $x \in F(x)$. If F is single-valued then we get the usual notion of fixed point.

Theorem 2.6. *Let C be a nonempty compact convex subset of a Hausdorff locally convex space (X, τ). Then any upper semi-continuous mapping $F : C \rightrightarrows C$, such that $F(x)$ is nonempty closed and convex for every $x \in C$, has a fixed point in C.*

Proof. Let $\mathcal{B}$ be a basis of 0-neighborhoods formed by open convex symmetric subsets of X. For $B \in \mathcal{B}$ choose $z_B^1, \dots, z_B^{n(B)} \in C$ such that

$$C \subset \left\{z_B^1, \dots, z_B^{n(B)}\right\} + B, \tag{2.11}$$

and let $y_B^i \in F(z_B^i)$, $i = 1, \dots, n(B)$. Denote by w_B^i, $i = 1, \dots, n(B)$, the functions from (2.1) corresponding to the Minkowski functional p_B of the set B, $\epsilon = 1$, and to the points $z_B^1, \dots, z_B^{n(B)}$, and let

$$f_B(x) = \sum_{i=1}^{n(B)} w_B^i(x) y_B^i, \quad x \in C. \tag{2.12}$$

By Schauder-Tychonoff theorem (Theorem 2.3) the continuous mapping $f_B : C \to C$ has a fixed point, that is, there exists $x_B \in C$ such that $f_B(x_B) = x_B$. The net $(x_B : B \in \mathcal{B})$ admits a subnet $(x_{\gamma(\alpha)} : \alpha \in \Lambda)$, $\gamma : \Lambda \to \mathcal{B}$, converging to an element $x \in C$. We show that x is a fixed point for F, that is, $x \in F(x)$. Since $F(x)$ is closed this is equivalent to

$$\forall\, V \in \mathcal{V}(0), \quad x \in F(x) + V. \tag{2.13}$$

Let $V \in \mathcal{V}(0)$ and let $B \in \mathcal{B}$ such that $B + B \subset V$. Since the set $F(x) + B$ is open and contains $F(x)$, by the upper semi-continuity of the mapping F there exists $U \in \mathcal{B}$ such that

$$F(C \cap (x + U)) \subset F(x) + B \tag{2.14}$$

Let $D \in \mathcal{B}$ such that $D + D \subset U$ and let $\alpha_0 \in \Lambda$ be such that

$$\gamma(\alpha_0) \subset D, \qquad \forall \alpha \geq \alpha_0, \quad x_{\gamma(\alpha)} \in x + D. \tag{2.15}$$

Then, for all $\alpha \geq \alpha_0$, $\gamma(\alpha) \subset \gamma(\alpha_0) \subset D$ and

$$\begin{aligned} x_{\gamma(\alpha)} &= f_{\gamma(\alpha)}(x_{\gamma(\alpha)}) \\ &= \sum \{ w^i_{\gamma(\alpha)}(x_{\gamma(\alpha)}) y^i_{\gamma(\alpha)} : 1 \leq i \leq n(\gamma(\alpha)),\ w^i_{\gamma(\alpha)}(x_{\gamma(\alpha)}) > 0 \}. \end{aligned} \tag{2.16}$$

But

$$\begin{aligned} w^i_{\gamma(\alpha)}(x_{\gamma(\alpha)}) > 0 &\Longleftrightarrow p_{\gamma(\alpha)}(z^i_{\gamma(\alpha)} - x_{\gamma(\alpha)}) < 1 \\ &\Longleftrightarrow z^i_{\gamma(\alpha)} - x_{\gamma(\alpha)} \in \gamma(\alpha) \subset D, \end{aligned} \tag{2.17}$$

so that

$$z^i_{\gamma(\alpha)} \in x_{\gamma(\alpha)} + D \subset x + D + D \subset x + U, \tag{2.18}$$

for every $\alpha \geq \alpha_0$. Taking into account (2.14) it follows

$$y^i_{\gamma(\alpha)} \in F(z^i_{\gamma(\alpha)}) \subset F(x) + B, \quad i = 1, \ldots, n(\gamma(\alpha)). \tag{2.19}$$

By (2.16), $x_{\gamma(\alpha)}$ is a convex combination of the elements $y^i_{\gamma(\alpha)}$, $i = 1, \ldots, n(\gamma(\alpha))$, so that it belongs to the convex set $F(x) + B$ for all $\alpha \geq \alpha_0$. Consequently

$$x \in \operatorname{cl}(F(x) + B) \subset F(x) + B + B \subset F(x) + V, \tag{2.20}$$

showing that (2.13) holds. □

3. Applications

In this section we will give some applications of Kakutani's fixed point theorem to game theory. First we show that Kakutani's theorem has as consequence a result of J. von Neumann [19] (see also [15]).

THEOREM 3.1. *Let (X,P) and (Y,Q) be Hausdorff locally convex spaces and $A \subset X$, $B \subset Y$ nonempty compact convex sets. For $M,N \subset A \times B$ let $M_x = \{y \in B : (x,y) \in M\}$, $x \in A$, and $N_y = \{x \in A : (x,y) \in N\}$, $y \in B$.*

If the sets M, N are closed and for every $(x,y) \in A \times B$ the sets M_x and N_y are nonempty closed and convex, then $M \cap N \neq \emptyset$.

Proof. Define the set-valued mapping $F : A \times B \rightrightarrows A \times B$ by $F(x,y) = N_y \times M_x$, $(x,y) \in A \times B$. If we show that F satisfies the hypotheses of Kakutani fixed point theorem, then there exists $(x_0,y_0) \in A \times B$ such that $(x_0,y_0) \in F(x_0,y_0) = N_{y_0} \times M_{x_0}$. It follows $x_0 \in N_{y_0} \Leftrightarrow (x_0,y_0) \in N$ and $y_0 \in M_{x_0} \Leftrightarrow (x_0,y_0) \in M$, so that $(x_0,y_0) \in M \cap N$.

Consider the locally convex space $(X \times Y, P \times Q)$, where $(p,q)(x,y) = p(x) + q(y)$, for $(p,q) \in P \times Q$ and $(x,y) \in X \times Y$. The set $C = A \times B$ is a compact convex subset of $X \times Y$ and, by hypothesis, $F(x,y) = N_y \times M_x$ is nonempty and convex for every $(x,y) \in A \times B$.

By Proposition 2.5, if we show that F is with closed graph, then it will be usc and with closed image sets $F(x,y)$. Define the mappings $\varphi,\psi : (A\times B)^2 \to A\times B$ by

$$\varphi(x,y,u,v) = (u,y), \qquad \psi(x,y,u,v) = (x,v), \tag{3.1}$$

for $(x,y,u,v) \in (A\times B)^2$. Then φ and ψ are continuous and the sets

$$\begin{aligned} \varphi^{-1}(N) &= \{(x,y,u,v) \in (A\times B)^2 : (u,y) \in N\}, \\ \psi^{-1}(M) &= \{(x,y,u,v) \in (A\times B)^2 : (x,v) \in N\} \end{aligned} \tag{3.2}$$

are closed. The equivalences

$$\begin{aligned} (u,v) \in F(x,y) \Longleftrightarrow u \in N_y &\Longleftrightarrow (u,y) \in N \\ v \in M_x &\Longleftrightarrow (x,v) \in M, \end{aligned} \tag{3.3}$$

imply

$$\begin{aligned} G_F &= \{(x,y,u,v) \in (A\times B)^2 : (u,v) \in F(x,y)\} \\ &= \{(x,y,u,v) \in (A\times B)^2 : (u,y) \in N,\ (x,v) \in M\} \\ &= \varphi^{-1}(N) \cap \psi^{-1}(M), \end{aligned} \tag{3.4}$$

showing that G_F is closed. □

Remark 3.2. Note that Kakutani's fixed point theorem is a particular case of von Neumann's theorem. Indeed, taking $A = B = C$, $M = G_F$ and $N = \{(x,x) : x \in C\}$, then $(x,y) \in M \cap N$ is equivalent to $y = x \in F(x)$, that is, x is a fixed point of F.

Another application of the Kakutani fixed point theorem is to game theory.

A game is a triple (A,B,K), where A, B are nonempty sets, whose elements are called strategies, and $K : A\times B \to \mathbb{R}$ is the gain function. There are two players, α and β, and $K(x,y)$ represents the gain of the player α when he chooses the strategy $x \in A$ and the player β chooses the strategy $y \in B$. The quantity $-K(x,y)$ represents the gain of the player β in the same situation. The target of the player α is to maximize his gain when the player β chooses a strategy that is the worst for α, that is, to choose $x_0 \in A$ such that

$$\inf_{y\in B} K(x_0,y) = \max_{x\in A}\inf_{y\in B} K(x,y). \tag{3.5}$$

Similarly, the player β chooses $y_0 \in B$ such that

$$\sup_{x\in A} K(x,y_0) = \min_{y\in B}\sup_{x\in A} K(x,y). \tag{3.6}$$

It follows

$$\sup_{x\in A}\inf_{y\in B} K(x,y) = \inf_{y\in B} K(x_0,y) \le K(x_0,y_0) \le \sup_{x\in A} K(x,y_0) \le \inf_{y\in B}\sup_{x\in A} K(x,y). \tag{3.7}$$

Note that in general

$$\sup_{x\in A}\inf_{y\in B} K(x,y) \leq \inf_{y\in B}\sup_{x\in A} K(x,y). \tag{3.8}$$

If the equality holds in (3.8), then, by (3.7),

$$\sup_{x\in A}\inf_{y\in B} K(x,y) = K(x_0,y_0) = \inf_{y\in B}\sup_{x\in A} K(x,y). \tag{3.9}$$

The common value in (3.9) is called the *value of the game*, $(x_0,y_0) \in A\times B$ a *solution of the game* and x_0 and y_0 *winning strategies*. It follows that to prove the existence of a solution of a game we have to prove equality (3.8), that is, to prove a *minimax theorem*.

We will prove first a lemma.

LEMMA 3.3. *If A, B are compact Hausdorff topological spaces and $K : A\times B \to \mathbb{R}$ is continuous, then the functions*

$$\begin{aligned} \varphi(x) &:= \min_{y\in B} K(x,y) = \min K(x\times B), \quad x\in A, \\ \psi(y) &:= \max_{x\in A} K(x,y) = \max K(A\times y), \quad y\in B, \end{aligned} \tag{3.10}$$

are continuous too.

Proof. We will prove that ψ is continuous. The continuity of φ can be proved in a similar way.

Let $(y_i : i\in I)$ be a net in B converging to $y\in B$. By the compactness of A there exists $x_i\in A$ such that $\psi(y_i) = K(x_i,y_i)$, $i\in I$. Using again the compactness of A, the net (x_i) contains a subnet $(x_{\gamma(j)} : j\in J)$, $\gamma : J\to I$, converging to an element $x\in A$. Then, by the continuity of K,

$$\lim_j \psi(y_{\gamma(j)}) = \lim_j K(x_{\gamma(j)}, y_{\gamma(j)}) = K(x,y). \tag{3.11}$$

But, for every $u\in A$ and $j\in J$, $K(u,y_{\gamma(j)}) \leq K(x_{\gamma(j)},y_{\gamma(j)})$, implying $K(u,y)\leq K(x,y)$, $u\in A$, that is, $K(x,y) = \max K(A\times y) = \psi(y)$, which is equivalent to the continuity of ψ at y. Indeed, if ψ would not be continuous at y, then it would exists $\epsilon > 0$ such that for every $V\in\mathcal{V}(y)$ there exists $y_V\in V$ with $|\psi(y_V)-\psi(y)|\geq\epsilon$. Ordering $\mathcal{V}(y)$ by $V_1\leq V_2 \Leftrightarrow V_2\subset V_1$, it follows that the net $(y_V : V\in\mathcal{V}(y))$ converges to y and no subnet of $(\psi(y_V) : V\in\mathcal{V}(y))$ converges to $\psi(y)$. □

The minimax result we will prove is the following.

THEOREM 3.4. *Let (X,P) and (Y,Q) be Hausdorff locally convex spaces and $A\subset X$, $B\subset Y$ nonempty compact convex sets.*

Suppose that $K : A\times B\to\mathbb{R}$ is continuous and

(i) *for every $x\in A$ the function $K(x,\cdot)$ is convex, and*

(ii) *for every $y\in B$ the function $K(\cdot,y)$ is concave.*

Then

$$\min_{y\in B}\max_{x\in A}K(x,y)=\max_{x\in A}\min_{y\in B}K(x,y), \tag{3.12}$$

and the game (A,B,K) has a solution.

Proof. Let the functions $\varphi(x)=\min K(x\times B)$ and $\psi(y)=\min K(A\times y)$ be as in Lemma 3.3, and let

$$M_x=\{y\in B:K(x,y)=\varphi(x)\},\qquad N_y=\{x\in A:K(x,y)=\psi(y)\}, \tag{3.13}$$

for $x\in A$ and $y\in B$. Since A, B are Hausdorff compact spaces and the functions K,φ,ψ are continuous, the sets M_x and N_y are nonempty and closed, for every $(x,y)\in A\times B$.

We will show that they are convex too. Let $y_1,y_2\in M_x$, $t\in(0,1)$, and $y=(1-t)y_1+ty_2$. Then, by (i),

$$\varphi(x)\le K(x,y)\le(1-t)K(x,y_1)+tK(x,y_2)=(1-t)\varphi(x)+t\varphi(x)=\varphi(x), \tag{3.14}$$

showing that $K(x,y)=\varphi(x)$, that is, $y\in M_x$. Similarly, if $x_1,x_2\in N_y$ and $t\in(0,1)$, we have by (ii),

$$\psi(y)\ge K(x,y)\ge(1-t)K(x_1,y)+tK(x_2,y)=(1-t)\psi(y)+t\psi(y)=\psi(y), \tag{3.15}$$

showing that $K(x,y)=\psi(y)$, that is, $x\in N_y$.

Let $C=A\times B$ and define $F:C\rightrightarrows C$ by $F(x,y)=N_y\times M_x$, $(x,y)\in C$. It follows that $F(x,y)$ is a nonempty closed convex subset of C for every $(x,y)\in C$. If we show that F has closed graph, then by Proposition 2.5, it is usc, so that, by Theorem 2.6, F has a fixed point (x_0,y_0). We have

$$(x_0,y_0)\in F(x_0,y_0)\Longleftrightarrow x_0\in N_{y_0},\quad y_0\in M_{x_0}. \tag{3.16}$$

But

$$\begin{aligned} x_0\in N_{y_0}&\Longleftrightarrow K(x_0,y_0)=\max_{x\in A}K(x,y_0)\ge\inf_{y\in B}\max_{x\in A}K(x,y),\\ y_0\in M_{x_0}&\Longleftrightarrow K(x_0,y_0)=\min_{y\in B}K(x_0,y)\le\sup_{x\in A}\min_{y\in B}K(x,y). \end{aligned}\tag{3.17}$$

Taking into account these last two inequalities and (3.8), we get

$$K(x_0,y_0)\le\sup_{y\in B}\min_{x\in A}K(x,y)\le\inf_{x\in A}\max_{y\in B}K(x,y)\le K(x_0,y_0), \tag{3.18}$$

implying

$$\max_{x\in A}\min_{y\in B}K(x,y)=K(x_0,y_0)=\min_{y\in B}\max_{x\in A}K(x,y). \tag{3.19}$$

It remained to show that the graph G_F of F, given by

$$G_F=\{(x,y,u,v)\in C^2:(u,v)\in F(x,y)\}, \tag{3.20}$$

is closed in C^2. Suppose that $((x_i, y_i) : i \in I)$ is a net in C converging to $(x, y) \in C$, and $(u_i, v_i) \in F(x_i, y_i)$, $i \in I$, are such that the net $((u_i, v_i) : i \in I)$ converges to $(u, v) \in C$. We have to show that $(u, v) \in F(x, y)$. We have

$$(u_i, v_i) \in F(x_i, y_i) \iff K(u_i, y_i) = \psi(y_i), \qquad K(x_i, v_i) = \varphi(x_i). \tag{3.21}$$

Passing to limits for $i \in I$, and taking into account the continuity of the functions K, φ and ψ, we get $K(u, y) = \psi(y)$ and $K(x, v) = \varphi(x)$, that is, $(u, v) \in N_y \times M_x = F(x, y)$.

The proof is complete. □

Acknowledgment

The author thanks one of the referees for mentioning reference [3] that leads to an improvement of the presentation.

References

[1] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis. Vol. 1*, American Mathematical Society Colloquium Publications, vol. 48, American Mathematical Society, Rhode Island, 2000.

[2] H. F. Bohnenblust and S. Karlin, *On a theorem of Ville*, Contributions to the Theory of Games, Annals of Mathematics Studies, no. 24, Princeton University Press, New Jersey, 1950, pp. 155–160.

[3] F. F. Bonsal, *Lectures on some Fixed Point Theorems and Functional Analysis*, Notes by K. B. Vedak, Tata Institute of Fundamental Research, Bombay, 1962.

[4] N. Dunford and J. T. Schwartz, *Linear Operators. I. General Theory*, Pure and Applied Mathematics, vol. 7, Interscience, New York, 1958.

[5] R. E. Edwards, *Functional Analysis. Theory and Applications*, Corrected reprint of the 1965 original, Dover, New York, 1995.

[6] R. Engelking, *General Topology*, 2nd ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann, Berlin, 1989.

[7] I. L. Glicksberg, *A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points*, Proceedings of the American Mathematical Society **3** (1952), no. 1, 170–174.

[8] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Studies in Advanced Mathematics, vol. 28, Cambridge University Press, Cambridge, 1990.

[9] V. I. Istrăţescu, *Fixed Point Theory. An Introduction*, Mathematics and Its Applications, vol. 7, D. Reidel, Dordrecht, 1981.

[10] S. Kakutani, *A generalization of Brouwer's fixed point theorem*, Duke Mathematical Journal **8** (1941), 457–459.

[11] L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, 3rd ed., Nauka, Moscow, 1984, English translation of the 1959 edition: Macmillan, New York 1964.

[12] M. A. Khamsi and W. A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, Pure and Applied Mathematics (New York), Wiley-Interscience, New York, 2001.

[13] G. Köthe, *Topological Vector Spaces. I.*, Translated from the German by D. J. H. Garling, Die Grundlehren der mathematischen Wissenschaften, vol. 159, Springer, New York, 1969.

[14] L. A. Lusternik and V. J. Sobolev, *Elements of Functional Analysis*, International Monographs on Advanced Mathematics Physics, Hindustan, Delhi; Halsted Press [John Wiley & Sons], New York, 1974.

[15] H. Nikaidô, *Convex Structures and Economic Theory*, Mathematics in Science and Engineering, vol. 51, Academic Press, New York, 1968.
[16] J. Schauder, *Der Fixpunktsatz in Funktionalräume*, Studia Mathematica **2** (1930), 171–180 (German).
[17] D. R. Smart, *Fixed Point Theorems*, Cambridge Tracts in Mathematics, no. 66, Cambridge University Press, London, 1974.
[18] A. Tychonoff, *Ein Fixpunktsatz*, Mathematische Annalen **111** (1935), no. 1, 767–776 (German).
[19] J. von Neumann, *Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes*, Ergebnisse eines Mathematischen Kolloquiums **8** (1937), 73–83 (German).

S. Cobzaş: Faculty of Mathematics and Computer Science, Babeş-Bolyai University,
400084 Cluj-Napoca, Romania
E-mail address: scobzas@math.ubbcluj.ro