A NOTE ON WELL-POSED NULL AND FIXED POINT PROBLEMS

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We establish generic well-posedness of certain null and fixed point problems for ordered Banach space-valued continuous mappings.

The notion of well-posedness is of great importance in many areas of mathematics and its applications. In this note, we consider two complete metric spaces of continuous mappings and establish generic well-posedness of certain null and fixed point problems (Theorems 1 and 2, resp.). Our results are a consequence of the variational principle established in [2]. For other recent results concerning the well-posedness of fixed point problems, see [1, 3].

Let $(X,\|\cdot\|,\geq)$ be a Banach space ordered by a closed convex cone $X_+=\{x\in X : x\geq 0\}$ such that $\|x\|\leq\|y\|$ for each pair of points $x,y\in X_+$ satisfying $x\leq y$. Let (K,ρ) be a complete metric space. Denote by $\mathfrak{M}$ the set of all continuous mappings $A: K\to X$. We equip the set $\mathfrak{M}$ with the uniformity determined by the following base:

$$E(\epsilon)=\{(A,B)\in\mathfrak{M}\times\mathfrak{M} : \|Ax-Bx\|\leq\epsilon\ \forall x\in K\}, \tag{1}$$

where $\epsilon>0$. It is not difficult to see that this uniform space is metrizable (by a metric d) and complete.

Denote by $\mathfrak{M}_p$ the set of all $A\in\mathfrak{M}$ such that

$$\begin{gathered} Ax\in X_+ \quad \forall x\in K, \\ \inf\{\|Ax\| : x\in K\}=0. \end{gathered} \tag{2}$$

It is not difficult to see that $\mathfrak{M}_p$ is a closed subset of $(\mathfrak{M},d)$.

We can now state and prove our first result.

Theorem 1. *There exists an everywhere dense G_δ subset $\mathcal{F}\subset\mathfrak{M}_p$ such that for each $A\in\mathcal{F}$, the following properties hold.*

(1) There is a unique $\bar{x}\in K$ such that $A\bar{x}=0$.

(2) For any $\epsilon>0$, there exist $\delta>0$ and a neighborhood U of A in $\mathfrak{M}_p$ such that if $B\in U$ and if $x\in K$ satisfies $\|Bx\|\leq\delta$, then $\rho(x,\bar{x})\leq\epsilon$.

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Proof. We obtain this theorem as a realization of the variational principle established in [2, Theorem 2.1] with $f_A(x) = \|Ax\|$, $x \in K$. In order to prove our theorem by using this variational principle, we need to prove the following assertion.

(A) For each $A \in \mathfrak{M}_p$ and each $\epsilon > 0$, there are $\bar{A} \in \mathfrak{M}_p$, $\delta > 0$, $\bar{x} \in K$, and a neighborhood W of $\bar{A}$ in $\mathfrak{M}_p$ such that

$$(A,\bar{A}) \in E(\epsilon), \tag{3}$$

and if $B \in W$ and $z \in K$ satisfy $\|Bz\| \le \delta$, then

$$\rho(z,\bar{x}) \le \epsilon. \tag{4}$$

Let $A \in \mathfrak{M}_p$ and $\epsilon > 0$. Choose $\bar{u} \in X_+$ such that

$$\|\bar{u}\| = \frac{\epsilon}{4}, \tag{5}$$

and $\bar{x} \in K$ such that

$$\|A\bar{x}\| \le \frac{\epsilon}{8}. \tag{6}$$

Since A is continuous, there is a positive number r such that

$$r < \min\left\{1, \frac{\epsilon}{16}\right\}, \tag{7}$$

$$\|Ax - A\bar{x}\| \le \frac{\epsilon}{8} \quad \text{for each } x \in K \text{ satisfying } \rho(x,\bar{x}) \le 4r. \tag{8}$$

By Urysohn's theorem, there is a continuous function $\phi : K \to [0,1]$ such that

$$\phi(x) = 1 \quad \text{for each } x \in K \text{ satisfying } \rho(x,\bar{x}) \le r, \tag{9}$$

$$\phi(x) = 0 \quad \text{for each } x \in K \text{ satisfying } \rho(x,\bar{x}) \ge 2r. \tag{10}$$

Define

$$\bar{A}x = \big(1 - \phi(x)\big)(Ax + \bar{u}), \quad x \in K. \tag{11}$$

It is clear that $\bar{A} : K \to X$ is continuous. Now (9), (10), and (11) imply that

$$\bar{A}x = 0 \quad \text{for each } x \in K \text{ satisfying } \rho(x,\bar{x}) \le r, \tag{12}$$

$$\bar{A}x \ge \bar{u} \quad \text{for each } x \in K \text{ satisfying } \rho(x,\bar{x}) \ge 2r. \tag{13}$$

It is not difficult to see that $\bar{A} \in \mathfrak{M}_p$. We claim that $(A,\bar{A}) \in E(\epsilon)$.

Let $x \in K$. There are two cases: either

$$\rho(x,\bar{x}) \geq 2r \tag{14}$$

or

$$\rho(x,\bar{x}) < 2r. \tag{15}$$

Assume first that (14) holds. Then it follows from (14), (10), (11), and (5) that

$$\|Ax - \bar{A}x\| = \|\bar{u}\| = \frac{\epsilon}{4}. \tag{16}$$

Now assume that (15) holds. Then by (15), (11), and (5),

$$\begin{aligned}\|\bar{A}x - Ax\| &= \big\|(1-\phi(x))(Ax+\bar{u}) - Ax\big\| \\ &\leq \|\bar{u}\| + \|Ax\| \leq \frac{\epsilon}{4} + \|Ax\|.\end{aligned} \tag{17}$$

It follows from this inequality, (15), (8), and (6) that

$$\|\bar{A}x - Ax\| \leq \frac{\epsilon}{4} + \|Ax\| < \frac{\epsilon}{2}. \tag{18}$$

Therefore, in both cases, $\|\bar{A}x - Ax\| \leq \epsilon/2$. Since this inequality holds for any $x \in K$, we conclude that

$$(A,\bar{A}) \in E(\epsilon). \tag{19}$$

Consider now an open neighborhood U of $\bar{A}$ in $\mathfrak{M}_p$ such that

$$U \subset \left\{B \in \mathfrak{M}_p : (\bar{A},B) \in E\left(\frac{\epsilon}{16}\right)\right\}. \tag{20}$$

Let

$$B \in U, \qquad z \in K, \tag{21}$$

$$\|Bz\| \leq \frac{\epsilon}{16}. \tag{22}$$

Relations (22), (21), (20), and (1) imply that

$$\|\bar{A}z\| \leq \|Bz\| + \|\bar{A}z - Bz\| \leq \frac{\epsilon}{16} + \frac{\epsilon}{16}. \tag{23}$$

We claim that

$$\rho(z,\bar{x}) \leq \epsilon. \tag{24}$$

We assume the converse. Then by (7),

$$\rho(z,\bar{x}) > \epsilon \geq 2r. \tag{25}$$

When combined with (13), this implies that

$$\bar{A}z \geq \bar{u}. \tag{26}$$

It follows from this inequality, the monotonicity of the norm, (21), (20), (1), and (5) that

$$\begin{aligned}\|Bz\| &\geq \|\bar{A}z\| - \frac{\epsilon}{16} \geq \|\bar{u}\| - \frac{\epsilon}{16}\\ &= \frac{\epsilon}{4} - \frac{\epsilon}{16} = \frac{3\epsilon}{16}.\end{aligned} \tag{27}$$

This, however, contradicts (22). The contradiction we have reached proves (24) and Theorem 1 itself. □

Now assume that the set K is a subset of X and

$$\rho(x,y) = \|x-y\|, \quad x,y \in K. \tag{28}$$

Denote by $\mathfrak{M}_n$ the set of all mappings $A \in \mathfrak{M}$ such that

$$\begin{gathered} Ax \geq x \quad \forall x \in K, \\ \inf\{\|Ax - x\| : x \in K\} = 0. \end{gathered} \tag{29}$$

Clearly, $\mathfrak{M}_n$ is a closed subset of $(\mathfrak{M}, d)$. Define a map $J : \mathfrak{M}_n \to \mathfrak{M}_p$ by

$$J(A)x = Ax - x \quad \forall x \in K \tag{30}$$

and all $A \in \mathfrak{M}_n$. Clearly, there exists $J^{-1} : \mathfrak{M}_p \to \mathfrak{M}_n$, and both J and its inverse J^{-1} are continuous. Therefore Theorem 1 implies the following result regarding the generic well-posedness of the fixed point problem for $A \in \mathfrak{M}_n$.

Theorem 2. *There exists an everywhere dense G_δ subset $\mathcal{F} \subset \mathfrak{M}_n$ such that for each $A \in \mathcal{F}$, the following properties hold.*

(1) There is a unique $\bar{x} \in K$ such that $A\bar{x} = \bar{x}$.

(2) For any $\epsilon > 0$, there exist $\delta > 0$ and a neighborhood U of A in $\mathfrak{M}_n$ such that if $B \in U$ and if $x \in K$ satisfies $\|Bx - x\| \leq \delta$, then $\|x - \bar{x}\| \leq \epsilon$.

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