NIELSEN NUMBER AND DIFFERENTIAL EQUATIONS

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In reply to a problem of Jean Leray (application of the Nielsen theory to differential equations), two main approaches are presented. The first is via Poincaré's translation operator, while the second one is based on the Hammerstein-type solution operator. The applicability of various Nielsen theories is discussed with respect to several sorts of differential equations and inclusions. Links with the Sharkovskii-like theorems (a finite number of periodic solutions imply infinitely many subharmonics) are indicated, jointly with some further consequences like the nontrivial R_δ-structure of solutions of initial value problems. Some illustrating examples are supplied and open problems are formulated.

1. Introduction: motivation for differential equations

Our main aim here is to show some applications of the Nielsen number to (multivalued) differential equations (whence the title). For this, applicable forms of various Nielsen theories will be formulated, and then applied—via Poincaré and Hammerstein operators—to associated initial and boundary value problems for differential equations and inclusions. Before, we, however, recall some Sharkovskii-like theorems in terms of differential equations which justify and partly stimulate our investigation.

Consider the system of ordinary differential equations

$$x' = f(t,x), \qquad f(t,x) \equiv f(t+\omega,x), \tag{1.1}$$

where $f : [0,\omega] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory mapping, that is,

(i) $f(\cdot,x) : [0,\omega] \to \mathbb{R}^n$ is measurable, for every $x \in \mathbb{R}^n$,

(ii) $f(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, for a.a. $t \in [0,\omega]$,

(iii) $|f(t,x)| \leq \alpha|x| + \beta$, for all $(t,x) \in [0,\omega] \times \mathbb{R}^n$, where α, β are suitable nonnegative constants.

By a *solution* to (1.1) on $J \subset \mathbb{R}$, we understand $x \in \mathrm{AC}_{\mathrm{loc}}(J,\mathbb{R}^n)$ which satisfies (1.1), for a.a. $t \in J$.

1.1. $n = 1$**.** For scalar equation (1.1), a version of the Sharkovskii cycle coexistence theorem (see [8, 14, 15, 17]) applies as follows.

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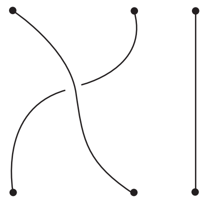

Figure 1.1. braid σ.

Theorem 1.1. *If (1.1) has an m-periodic solution, then it also admits a k-periodic solution, for every $k \lhd m$, with at most two exceptions, where $k \lhd m$ means that k is less than m in the celebrated Sharkovskii ordering of positive integers, namely* $3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \rhd \cdots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd 2^2\cdot 7 \rhd \cdots \rhd 2^m\cdot 3 \rhd 2^m\cdot 5 \rhd 2^m\cdot 7 \rhd \cdots \rhd 2^m \rhd \cdots \rhd 2^2 \rhd 2 \rhd 1$. *In particular, if $m \neq 2^k$, for all $k \in \mathbb{N}$, then infinitely many (subharmonic) periodic solutions of (1.1) coexist.*

Remark 1.2. Theorem 1.1 holds only in the lack of uniqueness; otherwise, it is empty. On the other hand, f on the right-hand side of (1.1) can be a (multivalued) upper-Carathéodory mapping with nonempty, convex, and compact values.

Remark 1.3. Although, for example, a 3ω-periodic solution of (1.1) implies, for every $k \in \mathbb{N}$ with a possible exception for $k = 2$ or $k = 4,6$, the existence of a $k\omega$-periodic solution of (1.1), it is very difficult to prove such a solution. Observe that a 3ω-periodic solution $x(\cdot,x_0)$ of (1.1) with $x(0,x_0) = x_0$ implies the existence of at least two more 3ω-periodic solutions of (1.1), namely $x(\cdot,x_1)$ with $x(0,x_1) = x(\omega,x_0) = x_1$ and $x(\cdot,x_2)$ with $x(0,x_2) = x(2\omega,x_0) = x(\omega,x_1) = x_2$.

1.2. $n = 2$. It follows from Boju Jiang's interpretation [43] of T. Matsuoka's results [47, 48, 49] that three (harmonic) ω-periodic solutions of the planar (i.e., in $\mathbb{R}^2$) system (1.1) imply "generically" the coexistence of infinitely many (subharmonic) $k\omega$-periodic solutions of (1.1), $k \in \mathbb{N}$. "Genericity" is understood here in terms of the Artin braid group theory, that is, with the exception of certain simplest braids, representing the three given harmonics.

Theorem 1.4 (see [4, 43, 49]). *Assume a uniqueness condition is satisfied for (1.1). Let three (harmonic) ω-periodic solutions of (1.1) exist whose graphs are not conjugated to the braid σ^m in B_3/Z, for any integer $m \in \mathbb{N}$, where σ is shown in Figure 1.1, B_3/Z denotes the factor group of the Artin braid group B_3, and Z is its center (for definitions, see, e.g.,* [9, 43, 51]*). Then there exist infinitely many (subharmonic) $k\omega$-periodic solutions of (1.1), $k \in \mathbb{N}$.*

Remark 1.5. In the absence of uniqueness, there occur serious obstructions, but Theorem 1.4 still seems to hold in many situations; for more details, see [4].

Remark 1.6. The application of the Nielsen theory can determine the desired three harmonic solutions of (1.1). More precisely, it is more realistic to detect two harmonics by means of the related Nielsen number, and the third one by means of the related fixed-point index (see, e.g., [9]).

1.3. $n \geq 2$**.** For $n > 2$, statements like Theorem 1.1 or Theorem 1.4 appear only rarely. Nevertheless, if $f = (f_1, f_2, \dots, f_n)$ has a special triangular structure, that is,

$$f_i(x) = f_i(x_1, \dots, x_n) = f_i(x_1, \dots, x_i), \quad i = 1, \dots, n, \tag{1.2}$$

then Theorem 1.1 can be extended to hold in $\mathbb{R}^n$ (see [16, 18]).

THEOREM 1.7. *Under assumption (1.2), the conclusion of Theorem 1.1 remains valid in* $\mathbb{R}^n$.

Remark 1.8. Similarly to Theorem 1.1, Theorem 1.7 holds only in the lack of uniqueness. In other words, P. Kloeden's single-valued extension (cf. (1.2)) of the standard Sharkovskii theorem does not apply to differential equations (see [16]). On the other hand, the second parts of Remarks 1.2 and 1.3 are true here as well.

Remark 1.9. Without the special triangular structure (1.2), there is practically no chance to obtain an analogy to Theorem 1.1, for $n \geq 2$ (see the arguments in [6]).

Despite the mentioned difficulties, to satisfy the assumptions of Theorems 1.1, 1.4, and 1.7, it is often enough to show at least one subharmonic or several harmonic solutions, respectively. The multiplicity problem is sufficiently interesting in itself. Jean Leray posed at the first International Congress of Mathematicians, held after the World War II in Cambridge, Massachusetts, in 1950, the problem of adapting the Nielsen theory to the needs of nonlinear analysis and, in particular, of its application to differential systems for obtaining multiplicity results (cf. [9, 24, 25, 27]). Since then, only few papers have been devoted to this problem (see [2, 3, 4, 9, 10, 11, 12, 13, 22, 23, 24, 25, 26, 27, 28, 32, 33, 34, 35, 36, 37, 43, 44, 47, 48, 49, 50, 51, 52, 56]).

2. Nielsen theorems at our disposal

The following Nielsen numbers (defined in our papers [2, 7, 10, 11, 12, 13, 20]) are at our disposal for application to differential equations and inclusions:

(a) Nielsen number for compact maps $\varphi \in \mathbb{K}$ (see [2, 11]),
(b) Nielsen number for compact absorbing contractions $\varphi \in \mathbb{CAC}$ (see [10]),
(c) Nielsen number for condensing maps $\varphi \in \mathbb{C}$ (see [20]),
(d) relative Nielsen numbers (on the total space or on the complement) (see [12]),
(e) Nielsen number for periodic points (see [13]),
(f) Nielsen number for invariant and periodic sets (see [7]).

For the classical (single-valued) Nielsen theory, we recommend the monograph [42].

2.1. ad (a). Consider a multivalued map $\varphi : X \multimap X$, where

(i) X is a connected retract of an open subset of a convex set in a Fréchet space,
(ii) X has finitely generated abelian fundamental group,

(iii) φ is a compact (i.e., $\overline{\varphi(X)}$ is compact) composition of an R_δ-map $p^{-1}: X \multimap \Gamma$ and a continuous (single-valued) map $q: \Gamma \to X$, namely $\varphi = q \circ p^{-1}$, where Γ is a metric space.

Then a nonnegative integer $N(\varphi) = N(p,q)$ (we should write more correctly $N_H(\varphi) = N_H(p,q)$, because it is in fact a $(\operatorname{mod} H)$-Nielsen number; for the sake of simplicity, we omit the index H in the sequel), called the *Nielsen number* for $\varphi \in \mathbb{K}$, exists (for its definition, see [11]; cf. [9] or [7]) such that

$$N(\varphi) \leq \#C(\varphi), \tag{2.1}$$

where

$$\#C(\varphi) = \#C(p,q) := \operatorname{card}\{z \in \Gamma \mid p(z) = q(z)\}, \tag{2.2}$$

$$N(\varphi_0) = N(\varphi_1), \tag{2.3}$$

for compactly homotopic maps $\varphi_0 \sim \varphi_1$.

Some remarks are in order. Condition (i) says that X is a particular case of a connected ANR-space and, in fact, X can be an arbitrary connected (metric) ANR-space (for the definition, see Part (f)). Condition (ii) can be avoided, provided X is the torus $\mathbb{T}^n$ (cf. [11]) or X is compact and $q = \mathrm{id}$ is the identity (cf. [2]).

By an *R_δ-map* $p^{-1}: X \multimap \Gamma$, we mean an upper semicontinuous (u.s.c.) one (i.e., for every open $U \subset \Gamma$, the set $\{x \in X \mid p^{-1}(x) \subset U\}$ is open in X) with R_δ-values (i.e., Y is an R_δ-set if $Y = \bigcap\{Y_n \mid n = 1,2,\ldots\}$, where $\{Y_n\}$ is a decreasing sequence of compact AR-spaces; for the definition of AR-spaces, see Part (f)).

Let $X \overset{p_0}{\Leftarrow} \Gamma_0 \overset{q_0}{\to} X$ and $X \overset{p_1}{\Leftarrow} \Gamma_1 \overset{q_1}{\to} X$ be two maps, namely $\varphi_0 = q_0 \circ p_0^{-1}$ and $\varphi_1 = q_1 \circ p_1^{-1}$. We say that φ_0 is *homotopic* to φ_1 (written $\varphi_0 \sim \varphi_1$ or $(p_0,q_0) \sim (p_1,q_1)$) if there exists a multivalued map $X \times [0,1] \overset{p}{\leftarrow} \overline{\Gamma} \overset{q}{\to} X$ such that the following diagram is commutative:

$$\begin{tikzcd}
X & \Gamma_i \arrow[l, Leftarrow, "p_i"'] \arrow[r, "q_i"] \arrow[d, "f_i"] & X \\
X \times [0,1] \arrow[from=1-1, "k_i"'] & \overline{\Gamma} \arrow[l, Leftarrow, "p"] \arrow[ur, dashed, "q"']
\end{tikzcd} \tag{2.4}$$

for $k_i(x) = (x,i)$, $i = 0,1$, and $f_i: \Gamma_i \to \overline{\Gamma}$ is a homeomorphism onto $p^{-1}(X \times i)$, $i = 0,1$, that is, $k_0 p_0 = p f_0$, $q_0 = q f_0$, $k_1 p_1 = p f_1$, and $q_1 = q f_1$.

Remark 2.1 (important). We have a counterexample in [11] that, under the above assumptions (i)–(iii), the Nielsen number $N(\varphi)$ is rather the topological invariant (see (2.3)) for the number of essential classes of coincidences (see (2.1)) than of fixed points. On the other hand, for a compact X and $q = \mathrm{id}$, $N(\varphi)$ gives even without (ii) a lower estimate of the number of fixed points of φ (see [2]), that is, $N(\varphi) \leq \#\mathrm{Fix}(\varphi)$, where $\#\mathrm{Fix}(\varphi) := \operatorname{card}\{x \in X \mid x \in \varphi(x)\}$. We have conjectured in [20] that if $\varphi = q \circ p^{-1}$ assumes only simply connected values, then also $N(\varphi) \leq \#\mathrm{Fix}(\varphi)$.

2.2. ad (b). Consider a multivalued map $\varphi : X \multimap X$, where X again satisfies the above conditions (i) and (ii), but this time

(iii′) φ is a $\mathbb{CAC}$-composition of an R_δ-map $p^{-1} : X \multimap \Gamma$ and a continuous (single-valued) map $q : \Gamma \to X$, namely $\varphi = q \circ p^{-1}$, where Γ is a metric space.

Let us recall (see, e.g., [9]) that the above composition $\varphi : X \multimap X$ is a *compact absorbing contraction* (written $\varphi \in \mathbb{CAC}$) if there exists an open set $U \subset X$ such that

(i) $\varphi|_U : U \multimap U$, where $\varphi|_U(x) = \varphi(x)$, for every $x \in U$, is compact,

(ii) for every $x \in X$, there exists $n = n_x$ such that $\varphi^n(x) \subset U$.

Then (i.e., under (i), (ii), (iii′)) a nonnegative integer $N(\varphi) = N(p,q)$, called the *Nielsen number* for $\varphi \in \mathbb{CAC}$, exists such that (2.1) and (2.3) hold. The homotopy invariance (2.3) is understood exactly in the same way as above.

Any compact map satisfying (iii) is obviously a compact absorbing contraction. In the class of *locally compact maps* φ (i.e., every $x \in X$ has an open neighborhood U_x of x in X such that $\varphi|_{U_x} : U_x \multimap X$ is a compact map), any *eventually compact* (written $\varphi \in \mathbb{EC}$), any *asymptotically compact* (written $\varphi \in \mathbb{ASC}$), or any *map with a compact attractor* (written $\varphi \in \mathbb{CA}$) becomes $\mathbb{CAC}$ (i.e., $\varphi \in \mathbb{CAC}$). More precisely, the following scheme takes place for the classes of locally compact compositions of R_δ-maps and continuous (single-valued) maps (cf. (iii′)):

$$\mathbb{K} \subset \mathbb{EC} \subset \mathbb{ASC} \subset \mathbb{CA} \subset \mathbb{CAC}, \tag{2.5}$$

where all the inclusions, but the last one, are proper (see [9]).

We also recall that an *eventually compact* map $\varphi \in \mathbb{EC}$ is such that some of its iterates become compact; of course, so do all subsequent iterates, provided φ is u.s.c. with compact values as above.

Assuming, for the sake of simplicity, that φ is again a composition of an R_δ-map p^{-1} and a continuous map q, namely $\varphi = q \circ p^{-1}$, we can finally recall the definition of the classes $\mathbb{ASC}$ and $\mathbb{CA}$.

Definition 2.2. A map $\varphi : X \multimap X$ is called *asymptotically compact* (written $\varphi \in \mathbb{ASC}$) if

(i) for every $x \in X$, the orbit $\bigcup_{n=1}^{\infty} \varphi^n(x)$ is contained in a compact subset of X,

(ii) the *center* (sometimes also called the *core*) $\bigcap_{n=1}^{\infty} \varphi^n(X)$ of φ is nonempty, contained in a compact subset of X.

Definition 2.3. A map $\varphi : X \multimap X$ is said to have a *compact attractor* (written $\varphi \in \mathbb{CA}$) if there exists a compact $K \subset X$ such that, for every open neighborhood U of K in X and for every $x \in X$, there exists $n = n_x$ such that $\varphi^m(x) \subset U$, for every $m \geq n$. K is then called the *attractor* of φ.

Remark 2.4. Obviously, if X is locally compact, then so is φ. If φ is not locally compact, then the following scheme takes place for the composition of an R_δ-map and a continuous map:

$$\begin{array}{ccccc} \mathbb{EC} & \subset & \mathbb{ASC} & \subset & \mathbb{CA} \\ \cup & & & & \cup \\ \mathbb{K} & & \subset & & \mathbb{CAC}, \end{array} \tag{2.6}$$

where all the inclusions are again proper (see [9]).

Remark 2.5. Although the $\mathbb{CA}$-class is very important for applications, it is (even in the single-valued case) an open problem whether local compactness of φ can be avoided or, at least, replaced by some weaker assumption.

2.3. ad (c). For single-valued continuous self-maps in metric (e.g., Fréchet) spaces, including condensing maps, the Nielsen theory was developed in [55], provided only that (i) the set of fixed points is compact, (ii) the space is a (metric) ANR, and (iii) the related generalized Lefschetz number is well defined. However, to define the Lefschetz number for condensing maps on non-simply connected sets is a difficult task (see [9, 19]). Roughly speaking, once we have defined the generalized Lefschetz number, the Nielsen number can be defined as well.

In the multivalued case, the situation becomes still more delicate, but the main difficulty related to the definition of the generalized Lefschetz number remains actual. Before going into more detail, let us recall the notion of a *condensing* map which is based on the concept of the measure of noncompactness (MNC).

Let (X,d) be a metric (e.g., Fréchet) space and let $\mathcal{B}(X)$ be the set of nonempty bounded subsets of X. The function $\alpha : \mathcal{B} \to [0,\infty)$, where $\alpha(B) := \inf\{\delta > 0 \mid B \in \mathcal{B}$ admits a finite covering by sets of diameter less than or equal to $\delta\}$, is called the *Kuratowski MNC* and the function $\gamma : \mathcal{B} \to [0,\infty)$, where $\gamma(B) := \inf\{\varepsilon > 0 \mid B \in \mathcal{B}$ has a finite ε-net$\}$, is called the *Hausdorff MNC*. These MNC are related by the inequality $\gamma(B) \le \alpha(B) \le 2\gamma(B)$. Moreover, they satisfy the following properties, where μ denotes either α or γ:

(i) $\mu(B) = 0 \Leftrightarrow \overline{B}$ is compact,
(ii) $B_1 \subset B_2 \Rightarrow \mu(B_1) \le \mu(B_2)$,
(iii) $\mu(\overline{B}) = \mu(B)$,
(iv) if $\{B_n\}$ is a decreasing sequence of nonempty, closed sets $B_n \in \mathcal{B}$ with $\lim_{n\to\infty} \mu(B_n) = 0$, then $\bigcap\{B_n \mid n = 1,2,\dots\} \neq \emptyset$,
(v) $\mu(B_1 \cup B_2) = \max\{\mu(B_1),\mu(B_2)\}$,
(vi) $\mu(B_1 \cap B_2) = \min\{\mu(B_1),\mu(B_2)\}$.

In Fréchet spaces, MNC μ can be shown to have further properties like the essential requirement that

(vii) $\mu(\overline{\operatorname{conv}}B) = \mu(B)$
and the seminorm property, that is,
(viii) $\mu(\lambda B) = |\lambda|\mu(B)$ and $\mu(B_1 \cup B_2) \le \mu(B_1) + \mu(B_2)$, for every $\lambda \in \mathbb{R}$ and $B, B_1, B_2 \in \mathcal{B}$.

It is, however, more convenient to take $\mu = \{\mu_s\}_{s\in S}$ as a countable family of MNC μ_s, $s \in S$ (S is the index set), related to the generating seminorms of the locally convex topology in this case.

Letting $\mu := \alpha$ or $\mu := \gamma$, a bounded mapping $\varphi : X \multimap X$ (i.e., $\varphi(B) \in \mathcal{B}$, for any $B \in \mathcal{B}$) is said to be μ*-condensing* (shortly, *condensing*) if $\mu(\varphi(B)) < \mu(B)$, whenever $B \in \mathcal{B}$ and $\mu(B) > 0$, or, equivalently, if $\mu(\varphi(B)) \ge \mu(B)$ implies $\mu(B) = 0$, whenever $B \in \mathcal{B}$.

Because of the mentioned difficulties with defining the generalized Lefschetz number for condensing maps on non-simply connected sets, we have actually two possibilities: either to define the Lefschetz number on special neighborhood retracts (see, e.g., [7, 9]) or to define the essential Nielsen classes recursively without explicit usage of the Lefschetz number (see [7, 20]). Of course, once the generalized Lefschetz number is well defined, the essentiality of classes can immediately be distinguished.

For the first possibility, by a *special neighborhood retract* (written SNR), we mean a closed bounded subset X of a Fréchet space with the following property: there exists an open subset U of (a convex set in) a Fréchet space such that $X \subset U$ and a continuous retraction $r : U \to X$ with $\mu(r(A)) \leq \mu(A)$, for every $A \subset U$, where μ is an MNC.

Hence, if $X \in$ SNR and $\varphi : X \multimap X$ is a condensing composition of an R_δ-map and continuous map, then the generalized Lefschetz number $\Lambda(\varphi)$ of φ is well defined (cf. [9]) as required, and subsequently if $X \in$ SNR is additionally connected with a finitely generated abelian fundamental group (cf. (i), (ii)), then we can define the Nielsen number $N(\varphi)$, for $\varphi \in \mathbb{C}$, as in the previous cases (a) and (b). The best candidate for a non-simply connected X to be an SNR seems to be that it is a suitable subset of a Hilbert manifold. Nevertheless, so far it is an open problem.

For the second possibility of a recursive definition of essential Nielsen classes, let us only mention that every Nielsen class $C \neq \emptyset$ is called 0-*essential* and, for $n = 1,2,\ldots,$ class C is further called *n-essential*, if for each $(p_1,q_1) \sim (p,q)$ and each corresponding lifting $(\widetilde{q},\widetilde{q}_1)$, there is a natural transformation α of the covering $p_{X_H} : \widetilde{X}_H \Rightarrow X$ with $C = C_\alpha(p,q,\widetilde{q}) := p_{\Gamma_H}(C(\widetilde{p}_H,\alpha\widetilde{q}_H))$ (the symbol H refers to the case modulo a subgroup $H \subset \pi_1(X)$ with a finite index) such that the Nielsen class $C_\alpha(p_1,q_1,\widetilde{q}_1)$ is $(n-1)$-essential (for the definitions and more details, see [20]). Class C is finally called *essential* if it is n-essential, for each $n \in \mathbb{N}$. For the lower estimate of the number of coincidence points of $\varphi = (p,q)$, it is sufficient to use the number of 1-essential Nielsen classes. The related Nielsen number is therefore a lower bound for the cardinality of $C(p_1,q_1)$. For more details, see [20] (cf. [7]).

2.4. ad (d). Consider a multivalued map $\varphi : X \multimap X$ and assume that conditions (i), (ii), and (iii′) are satisfied. Let $A \subset X$ be a closed and connected subset. Using the above notation $\varphi = (p,q)$, namely $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\rightarrow} X$, denote still $\Gamma_A = p^{-1}(A)$ and consider the restriction $A \overset{p|}{\Leftarrow} \Gamma_A \overset{q|}{\rightarrow} A$, where $p|$ and $q|$ denote the natural restrictions. It can be checked (see [12]) that the map $A \overset{p|}{\Leftarrow} \Gamma_A \overset{q|}{\rightarrow} A$ also satisfies sufficient conditions for the definition of essential Nielsen classes.

Hence, let $S(\varphi;A) = S(p,q;A)$ denote the set of essential Nielsen classes for $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\rightarrow} X$ which contain no essential Nielsen classes for $A \overset{p|}{\Leftarrow} \Gamma_A \overset{q|}{\rightarrow} A$.

The following theorem considers the relative Nielsen number for $\mathbb{CAC}$-maps on the total space.

Theorem 2.6 (see [12]). *Let X be a set satisfying conditions (i), (ii), and let $A \subset X$ be its closed connected subset. A $\mathbb{CAC}$-composition φ satisfying (iii′) has at least $N(\varphi) + \#S(\varphi;A)$ coincidences, that is,*

$$N(\varphi) + \#S(\varphi;A) \leq \#C(\varphi), \tag{2.7}$$

$$N(\varphi_0) + \#S(\varphi_0;A) = N(\varphi_1) + \#S(\varphi_1;A), \tag{2.8}$$

for homotopic maps $\varphi_0 \sim \varphi_1$.

Similarly, the following theorem relates to a relative Nielsen number for $\mathbb{CAC}$-maps on the complement.

THEOREM 2.7 [12]. *Let X be a set satisfying conditions (i), (ii), and let $A \subset X$ be its closed connected subset. A $\mathbb{CAC}$-composition φ satisfying (iii′) has at least* $\mathrm{SN}(\varphi;A)$ *coincidences on $\Gamma\backslash\Gamma_A$, that is,*

$$SN(\varphi;A) \leq \#C(\varphi), \tag{2.9}$$

$$SN(\varphi_0;A) = SN(\varphi_1;A), \tag{2.10}$$

for homotopic maps $\varphi_0 \sim \varphi_1$.

Remark 2.8. The relative Nielsen number $SN(\varphi;A)$ is defined by means of essential Reidemeister classes. More precisely, it is the number of essential classes in $\mathcal{R}_H(\varphi)\backslash \mathrm{Im}\mathcal{R}(i)$, where the meaning of $\mathcal{R}(i)$ can be seen from the commutative diagram

$$\begin{tikzcd} \mathcal{N}_{H_0}(p|,q|) \arrow[r, "\mathcal{N}(i)"] \arrow[d, "\eta"'] & \mathcal{N}_H(p,q) \arrow[d, "\eta"'] \\ \mathcal{R}_{H_0}(p|,q|) \arrow[r, "\mathcal{R}(i)"] & \mathcal{R}_H(p,q) \end{tikzcd} \tag{2.11}$$

concerning the Nielsen classes $\mathcal{N}_H(p,q)$, $\mathcal{N}_{H_0}(p|,q|)$ and the Reidemeister classes $\mathcal{R}_H(p,q)$, $\mathcal{R}_{H_0}(p|,q|)$; η is a natural injection, $H \subset \pi_1(X)$ and $H_0 \subset \pi_1(A)$ are fixed normal subgroups of finite order. For more details, see [12].

Remark 2.9. Theorem 2.6 generalizes in an obvious way the results presented in parts (a) and (b) (cf. (2.7), (2.8) with (2.1), (2.3)); Theorem 2.7 can be regarded as their improvement as the localization of the coincidences concerns (cf. (2.9), (2.10) with (2.1), (2.3)).

2.5. ad (e). Consider a map $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} X$, that is, $\varphi = q \circ p^{-1}$. A sequence of points $(z_1,\dots,z_k)$ satisfying $z_i \in \Gamma$, $i = 1,\dots,k$, such that $q(z_i) = p(z_{i+1})$, $i = 1,\dots,k-1$, and $q(z_k) = p(z_1)$ will be called a *k-periodic orbit of coincidences*, for $\varphi = (p,q)$. Observe that, for $(p,q) = (\mathrm{id}_X, f)$, a k-periodic orbit of coincidences equals the orbit of periodic points for f.

We will consider periodic orbits of coincidences with the fixed first element $(z_1,\dots,z_k)$. Thus, $(z_2,z_3,\dots,z_k,z_1)$ is another periodic orbit. Orbits $(z_1,\dots,z_k)$ and $(z_1',\dots,z_k')$ are said to be *cyclically equal* if $(z_1',\dots,z_k') = (z_l,\dots,z_k;z_1,\dots,z_{l-1})$, for some $l \in \{1,\dots,k\}$. Otherwise, they are said to be *cyclically different*. Let us note that, unlike in the single-valued case, there can exist distinct orbits starting from a given point z_1 (the second element z_2 satisfying only $z_2 \in q^{-1}(p(z_1))$ need not be uniquely determined).

Denoting $\Gamma_k := \{(z_1,\dots,z_k) \mid z_i \in \Gamma,\ q(z_i) = p(z_{i+1}), i = 1,\dots,k-1\}$, we define maps $p_k, q_k : \Gamma_k \to X$ by $p_k(z_1,\dots,z_k) = p(z_1)$ and $q_k(z_1,\dots,z_k) = q(z_k)$. Since a sequence of points $(z_1,\dots,z_k) \in \Gamma_k$ is an orbit of coincidences if and only if $(z_1,\dots,z_k) \in C(p_k,q_k)$, the study of k-periodic orbits of coincidences reduces to the one for the coincidences of the pair $X \overset{p_k}{\leftarrow} \Gamma_k \overset{q_k}{\rightarrow} X$.

Hence, in order to make an estimation of the number of k-orbits of coincidences of the pair (p,q), we will need the following assumptions:

(i′) X is a compact, connected retract of an open subset of (a convex set in) a Fréchet space,

(ii) X has a finitely generated abelian fundamental group,

(iii) φ is a (compact) composition of an R_δ-map $p^{-1}: X \multimap \Gamma$ and a continuous (single-valued) map $q: \Gamma \to X$, namely $\varphi = q \circ p^{-1}$, where Γ is a metric space.

We can again define, under (i′), (ii), and (iii), Nielsen and Reidemeister classes $\mathcal{N}(p_k,q_k)$ and $\mathcal{R}(p_k,q_k)$ and speak about orbits of Nielsen and Reidemeister classes.

Definition 2.10. A k-orbit of coincidences $(z_1,\dots,z_k)$ is called *reducible* if $(z_1,\dots,z_k) = j_{kl}(z_1,\dots,z_l)$, for some $l<k$ dividing k, where $j_{kl}: C(p_l,q_l) \to C(p_k,q_k)$ sends the Nielsen class corresponding to $[\alpha] \in \mathcal{R}(\widetilde{p}_l,\widetilde{q}_l)$ to the Nielsen class corresponding to $[i_{kl}(\alpha)] \in \mathcal{R}(\widetilde{p}_k,\widetilde{q}_k)$, that is, for which the following diagram commutes:

$$\begin{tikzcd}
\mathcal{N}(p_l,q_l) \arrow[r, "j_{kl}"] \arrow[d] & \mathcal{N}(p_k,q_k) \arrow[d] \\
\mathcal{R}(\overline{p}_l,\overline{q}_l) \arrow[r, "i_{kl}"] & \mathcal{R}(\overline{p}_k,\overline{q}_k)
\end{tikzcd} \tag{2.12}$$

(for more details, see [13]). Otherwise, $(z_1,\dots,z_k)$ is called *irreducible*.

Denoting by $S_k(\widetilde{p},\widetilde{q})$ the number of irreducible and essential orbits in $\mathcal{R}(\overline{p_l},\overline{q_l})$, we can state the following theorem.

Theorem 2.11 (see [13]). *Let X be a set satisfying conditions (i′), (ii). A (compact) composition $\varphi = (p,q)$ satisfying (iii) has at least $S_k(\widetilde{p},\widetilde{q})$ irreducible cyclically different k-orbits of coincidences.*

Remark 2.12. Since the essentiality is a homotopy invariant and irreducibility is defined in terms of Reidemeister classes, $S_k(\widetilde{p},\widetilde{q})$ is a homotopy invariant.

Remark 2.13. It seems to be only a technical (but rather cumbersome) problem to generalize Theorem 2.11 for $\varphi \in \mathbb{K}$, provided (i)–(iii) hold, or even for $\varphi \in \mathbb{CAC}$, provided (i), (ii), and (iii′) hold. One can also develop multivalued versions of relative Nielsen theorems for periodic coincidences (on the total space, on the complement, etc.). For single-valued versions of relative Nielsen theorems for periodic points (including those on the closure of the complement), see [57] and cf. the survey paper [40].

2.6. ad (f). One can easily check that, in the single-valued case, condition (ii) can be avoided and X in condition (i) or (i′) (for cases (a)–(e)) can be very often a (compact) ANR-space.

Definition 2.14. *ANR* (or *AR*) denotes the class of *absolute neighborhood retracts* (or *absolute retracts*), namely, X is an ANR-space (or an AR-space) if each embedding $h: X \hookrightarrow Y$ of X into a metrizable space Y (an embedding $h: X \hookrightarrow Y$ is a homeomorphism which takes

X to a closed subset $h(X) \subset Y$) satisfies that $h(X)$ is a neighborhood retract (or a retract) of Y.

In this subsection, we will employ the *hyperspace* $(\mathcal{K}(X), d_H)$, where $\mathcal{K}(X) := \{K \subset X \mid K \text{ is compact }\}$ and d_H stands for the Hausdorff metric; for its definition and properties, see, for example, [9]. According to the results in [31], if X is locally continuum connected (or connected and locally continuum connected), then $\mathcal{K}(X)$ is ANR (or AR).

Remark 2.15. Obviously, condition (i) implies $X \in \text{ANR}$ which makes X locally continuum connected. Hence, in order to deal with hyperspaces $(\mathcal{K}(X), d_H)$ which are ANR, it is sufficient to take $X \in \text{ANR}$. On the other hand, to have hyperspaces which are ANR, but not AR, X has to be disconnected.

Furthermore, if $\varphi : X \multimap X$ is a Hausdorff-continuous map with compact values (or, equivalently, an upper semicontinuous and lower semicontinuous map with compact values), then the induced (single-valued) map $\varphi^* : \mathcal{K}(X) \to \mathcal{K}(X)$ can be proved to be continuous (see, e.g., [9]). If φ is still compact (i.e., $\varphi \in \mathbb{K}$), then φ^* becomes compact, too. It is a question whether similar implications hold for $\varphi \in \mathbb{CAC}$ or $\varphi \in \mathbb{C}$.

Applying the Nielsen theory (cf. [55]) in the hyperspace $(\mathcal{K}(X), d_H)$ which is ANR, we can immediately state the following corollary.

Corollary 2.16 (see [7]). *Let X be a locally continuum connected metric space and let $\varphi : X \multimap X$ be a Hausdorff-continuous compact map (with compact values). Then there exist at least $N(\varphi^*)$ compact invariant subsets $K \subset X$, that is,*

$$N(\varphi^*) \leq \#\{K \subset X \mid K \text{ is compact with } \varphi(K) = K\}, \tag{2.13}$$

where $N(\varphi^*)$ is the Nielsen number for fixed points of the induced (single-valued) map $\varphi^* : \mathcal{K}(X) \to \mathcal{K}(X)$ in the hyperspace $(\mathcal{K}(X), d_H)$.

If X is compact, so is $\mathcal{K}(X)$ (see, e.g., [9]). Applying, therefore, the Nielsen theory for periodic points in $(\mathcal{K}(X), d_H) \in \text{ANR}$, we obtain the following corollary.

Corollary 2.17 (see [7]). *Let X be a compact, locally connected metric space and let $\varphi : X \multimap X$ be a Hausdorff-continuous compact map (with compact values). Then there exist at least $S_k(\varphi^*)$ compact periodic subsets $K \subset X$, that is,*

$$\begin{aligned} S_k(\varphi^*) \leq \#\{K \subset X \mid K &\text{ is compact with } \varphi^k(K) = K, \\ &\varphi^j(K) \neq K, \text{ for } j < k\}, \end{aligned} \tag{2.14}$$

where $S_k(\varphi^*)$ is the Nielsen number for k-periodic points of the induced (single-valued) map $\varphi^* : \mathcal{K}(X) \to \mathcal{K}(X)$ in the hyperspace $(\mathcal{K}(X), d_H)$.

Remark 2.18. Similar corollaries can be obtained by means of relative Nielsen numbers in hyperspaces, for the estimates of the number of compact invariant (or periodic) sets on the total space X or of those with $\varphi(K) = K \not\subset A$ (or with $\varphi^k(K) = K$ and $\varphi^j(K) \neq K$, for $j < k$), where $A \subset X$ is a closed subset. For more details, see [7].

3. Poincaré translation operator approach

In [5] (cf. [9]), the following types of Poincaré operators are considered separately:

(a) translation operator for ordinary systems,
(b) translation operator for functional systems,
(c) translation operator for systems with constraints,
(d) translation operator for systems in Banach spaces,
(e) translation operator for random systems,
(f) translation operator for directionally u.s.c. systems.

For all the types (a)–(f), it can be proved that, under natural assumptions, the Poincaré operators related to given systems are the desired compositions of R_δ-maps with continuous (single-valued) maps. On the other hand, these operators can be easily checked to be admissibly homotopic to identity which signalizes that they are useless as far as they are considered on some nontrivial ANR-subsets (e.g., on an annulus or on a torus). Thus, the only chance to overcome this handicap seems to be the composition with a suitable homeomorphism, because the associated Nielsen number can be reduced to the Nielsen number of this homeomorphism.

3.1. ad (a). Consider the upper-Carathéodory system

$$x' \in F(t,x), \quad x \in \mathbb{R}^n, \tag{3.1}$$

where

(i) the values of $F(t,x)$ are nonempty, compact, and convex, for all $(t,x) \in [0,\tau] \times \mathbb{R}^n$,
(ii) $F(t,\cdot)$ is u.s.c., for a.a. $t \in [0,\tau]$,
(iii) $F(\cdot,x)$ is measurable, for every $x \in \mathbb{R}^n$, that is, for any closed $U \subset \mathbb{R}^n$ and every $x \in \mathbb{R}^n$, the set $\{t \in [0,\tau] \mid F(t,x) \cap U \neq \emptyset\}$ is measurable,
(iv) $|F(t,x)| \leq \alpha + \beta|x|$, for every $x \in \mathbb{R}^n$ and a.a. $t \in [0,\tau]$, where α and β are suitable nonnegative constants.

By a *solution* to (3.1), we mean an absolutely continuous function $x \in \mathrm{AC}([0,\tau],\mathbb{R}^n)$ satisfying (3.1), for a.a. $t \in [0,\tau]$ (i.e., the one in the sense of Carathéodory).

Hence, if $x(\cdot,x_0)$ is a solution to (3.1) with $x(0,x_0) = x_0 \in \mathbb{R}^n$, then the translation operator $T_\tau : \mathbb{R}^n \multimap \mathbb{R}^n$ at time $\tau > 0$ along the trajectories of (3.1) is defined as follows:

$$T_\tau(x_0) := \{x(\tau,x_0) \mid x(\cdot,x_0) \text{ is a solution to (3.1) with } x(0,x_0) = x_0\}. \tag{3.2}$$

As already mentioned, T_τ defined by (3.2) can be proved (see [5] or [9]) to be a composition of an R_δ-mapping, namely

$$\varphi(x_0) : x_0 \multimap \{x(\cdot,x_0) \mid x(\cdot,x_0) \text{ is a solution to (3.1) with } x(0,x_0) = x_0\}, \tag{3.3}$$

and the continuous (single-valued) evaluation map $\psi(y) : y \to y(\tau)$, that is, $T_\tau = \psi \circ \varphi$.

Now, let $X \subset \mathbb{R}^n$ be a bounded subset satisfying conditions (i) and (ii) of part 2 and let $\mathcal{H} : X \to X$ be a homeomorphism. If $T_{\lambda\tau}$ is a self-map of X, that is, if $T_{\lambda\tau} : X \multimap X$, for each $\lambda \in [0,1]$, then we can still consider the composition

$$\begin{tikzcd} X \times [0,1] \arrow[r, "\varphi|"] & \mathrm{AC}([0,\lambda\tau], \operatorname{Im}\varphi(X), [0,1]) \arrow[d, "\psi|"] \\ \ddots & X \arrow[d, "\mathscr{H}"] \\ \arrow[r, "\mathscr{H}\circ T_{\lambda\tau}"] & X \end{tikzcd} \tag{3.4}$$

where $\varphi| := \varphi|_X$, $\psi| := \psi|_{\mathrm{AC}([0,\lambda\tau],\operatorname{Im}\varphi(X),[0,1])}$ denote the respective restrictions. Since X is, by hypothesis, bounded (i.e., $\mathscr{H} \circ T_{\lambda\tau} \in \mathbb{K}$), we can define the Nielsen number (see part 2 (a)) $N(\mathscr{H} \circ T_{\lambda\tau}) = N(\mathscr{H})$, where

$$\begin{aligned} N(\mathscr{H}) \le \#\{&x \in \mathrm{AC}([0,\tau],\mathbb{R}^n) \mid \\ &x \text{ is a solution to (3.1) with } \mathscr{H}(x(0,x_0)) = x(\tau,x_0) \in X, x(0,x_0) \in X\}. \end{aligned} \tag{3.5}$$

Two problems occur, namely,

(i) to guarantee that $T_{\lambda\tau}$ is a self-map of X, for each $\lambda \in [0,1]$,

(ii) to compute $N(\mathscr{H})$.

For the first requirement, we have at least two possibilities:

(i) $X := \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$,

(ii) the usage of Lyapunov (bounding) functions (cf. [9, Chapter III.8]).

If $X = \mathbb{T}^n$, then the requirement concerning a finitely generated abelian fundamental group $\pi_1(X)$ is satisfied and (cf. (3.5))

$$N(\mathscr{H} \circ T_\tau) = N(\mathscr{H}) = |\Lambda(\mathscr{H})|, \tag{3.6}$$

where Λ stands for the generalized Lefschetz number (see [11] and cf. [9]).

Hence, if

$$F(t,\dots,x_j,\dots) \equiv F(t,\dots,x_j+1,\dots), \tag{3.7}$$

for all $j = 1,\dots,n$, where $x = (x_1,\dots,x_n)$, then we can immediately give the following theorem.

Theorem 3.1. *System (3.1) admits, under (i)–(iv) and (3.7), at least* $|\Lambda(\mathscr{H})|$ *solutions* $x(\cdot,x_0)$ *such that* $\mathscr{H}(x(0,x_0)) = x(\tau,x_0) (\operatorname{mod} 1)$, *where* $\mathscr{H}$ *is a continuous self-map of* $\mathbb{T}^n$ *and* τ *is a positive number.*

Example 3.2. For $\mathscr{H} = -\mathrm{id}$, we obtain that $|\Lambda(-\mathrm{id})| = 2^n$, and so system (3.1) admits at least 2^n 2τ-periodic solutions $x(\cdot)$ on $\mathbb{T}^n$, that is, $x(t+2\tau) \equiv x(t) (\operatorname{mod} 1)$, provided still $F(t+\tau,-x) \equiv -F(t,x)$.

Lyapunov (bounding) functions can be employed for obtaining a positive flow-invariance of X under $T_{\lambda\tau}$ even in more general situations (cf., e.g., [9]).

It has also meaning to assume that T_τ has a compact attractor, that is, $T_\tau \in \mathbb{CA}$, which implies in $\mathbb{R}^n$ that $T_\tau \in \mathbb{CAC}$. Thus, a subinvariant subset $S \subset \mathbb{R}^n$ exists with respect to T_τ, namely $T_\tau(S) \subset S$, such that $\overline{T_\tau(S)}$ is compact. If, in particular, $S \in \mathrm{ANR}$, then the Nielsen number $N(T_\tau|_S)$ is well defined, but the same obstruction with its computation as above remains actual. Moreover, a number $\lambda \in [0,1]$ can exist such that $T_{\lambda\tau}|_S(x_0) \notin S$, for some $x_0 \in S$, by which the computation of $N(T_\tau|_S)$ need not be reduced to $N(\mathrm{id}|_S)$, and so forth.

As concerns the application of other Nielsen numbers, the situation is more complicated, especially with respect to their computation. In order to define relative Nielsen numbers, a closed connected subset $A \subset X$ should be positively flow-invariant under $T_{\lambda\tau}|_A$ (which can be guaranteed by means of bounding Lyapunov functions) and $\mathcal{H}(A) \subset A$. Then both the numbers $N(\mathcal{H} \circ T_\tau; A) + \#S(\mathcal{H} \circ T_\tau; A) = N(\mathcal{H}; A) + \#S(\mathcal{H}; A)$ and $NS(\mathcal{H} \circ T_\tau; A) = NS(\mathcal{H}; A)$ are well defined, provided the assumptions in the absolute case hold. For periodic coincidences, X was assumed to be still compact, for example, $X = \mathbb{T}^n$, but then the related Nielsen number $S_k(\widetilde{\mathcal{H} \circ T_\tau}) = S_k(\widetilde{\mathcal{H}})$ is again well defined. In particular, for $X = \mathbb{T}^n$, we obtain

$$S_k(\widetilde{\mathcal{H}}) \geq \left[\frac{1}{k}\sum_{m|k}\mu\left(\frac{k}{m}\right)|\Lambda(\mathcal{H}^m)|\right]^+, \tag{3.8}$$

provided $\Lambda(\mathcal{H}^k) \neq 0$, $k \in \mathbb{N}$, where $\Lambda(\mathcal{H}^m)$ denotes the Lefschetz number of $\mathcal{H}^m$, $[r]^+ = [r] + \mathrm{sgn}(r - [r])$ with $[r]$ being the integer part of r, and μ is the Möbius function, that is, for $d \in \mathbb{N}$,

$$\mu(d) = \begin{cases} 0 & \text{if } d = 1, \\ (-1)^l & \text{if } d \text{ is a product of } l \text{ distinct primes}, \\ 0 & \text{if } d \text{ is not square-free}. \end{cases} \tag{3.9}$$

In view of (3.8), we can get the following theorem.

Theorem 3.3. *System (3.1) admits, under (i)–(iv), (3.7), and $\Lambda(\mathcal{H}^k) \neq 0$, $k \in \mathbb{N}$, at least $[(1/k)\sum_{m|k}\mu(k/m)|\Lambda(\mathcal{H}^m)|]^+$ geometrically distinct k-tuples of solutions $x(\cdot, x_0)$ such that*

$$\mathcal{H} \circ x(\tau; \mathcal{H} \circ x(\tau; \ldots \mathcal{H} \circ x(\tau; x(0, x_0) \ldots))) = x(0, x_0) (\mathrm{mod}\, 1), \tag{3.10}$$

where $\mathcal{H}$ is a continuous self-map of $\mathbb{T}^n$ and τ is a positive number.

Example 3.4. For

$$\mathcal{H} = A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \Longrightarrow A^5 = \begin{pmatrix} 3 & 5 \\ 5 & 8 \end{pmatrix}, \tag{3.11}$$

we obtain that

$$\det(I-A)=\det\begin{pmatrix}1 & -1\\ -1 & 0\end{pmatrix}=-1,$$
$$\det(I-A^5)=\det\begin{pmatrix}-2 & -5\\ -5 & -7\end{pmatrix}=-11. \tag{3.12}$$

Since $\mu(1)=1$ and $\mu(5)=-1$, we arrive at

$$\frac{1}{5}\sum_{m|5}\mu\left(\frac{5}{m}\right)|\det(I-A^m)| = \frac{1}{5}(-1|-1|+1|-11|)=2, \tag{3.13}$$

and subsequently (3.1) with (3.7) admits at least two geometrically distinct 5-tuples of solutions x such that

$$A\circ x(\tau;A\circ x(\tau;A\circ x(\tau;A\circ x(\tau;A\circ x(\tau;x(0,x_0))))))=x(0,x_0)(\operatorname{mod}1). \tag{3.14}$$

For invariant and periodic sets, we must assume that the related Poincaré translation operators are continuous, namely we should consider (instead of (3.1)) Carathéodory systems of equations

$$x'=F(t,x),\quad x\in\mathbb{R}^n, \tag{3.15}$$

with uniquely solvable initial value problems (i.e., with F satisfying a uniqueness condition). Let us suppose that the related translation operator T_τ has a compact attractor, say $K\subset\mathbb{R}^n$, for which it is (in the single-valued case) sufficient to assume only that, for every $x_0\in\mathbb{R}^n$, we have

$$\overline{\{x_0,T_\tau(x_0),T_\tau^2(x_0),\dots\}}\cap K\neq\emptyset, \tag{3.16}$$

where the bar $\overline{\{\cdot\}}$ denotes the closure of the orbit $\{\cdot\}$ in $\mathbb{R}^n$. Then $T_\tau\in\mathbb{CAC}$, and subsequently a subset $K_0:=\bigcup_{k=0}^{n^*}T_\tau^k(K)\subset\mathbb{R}^n$ exists, for some $n^*\in\mathbb{N}$, such that $T_\tau(K_0)\subset K_0$ and $T_\tau|_{K_0}\in\mathbb{K}$. Since a continuous image of a locally connected set need not be locally connected, let K be such that K_0 is locally connected. Thus, $(\mathcal{K}(K_0),d_H)\in$ ANR, and so the Nielsen numbers $N(T_\tau^*|_{\mathcal{K}(K_0)})$ and $S_k(T_\tau^*|_{\mathcal{K}(K_0)})$ are well defined, satisfying

$$N(T_\tau^*|_{\mathcal{K}(K_0)})\le\#\{K_1\subset K_0\mid K_1\text{ is compact with }T_\tau(K_1)=K_1\}, \tag{3.17}$$

$$S_k(T_\tau^*|_{\mathcal{K}(K_0)})\le\#\{K_2\subset K_0\mid K_2\text{ is compact with }T_\tau^k(K_1)=K_2,\ T_\tau^j(K_2)\neq K_2,\text{ for }j<k\}. \tag{3.18}$$

However, the computation of these Nielsen numbers cannot be reduced in general to the one of $N(T_0^*|_{\mathcal{K}(K_0)})=N(\mathrm{id}|_{\mathcal{K}(K_0)})$ and $S_k(T_0^*|_{\mathcal{K}(K_0)})=N(\mathrm{id}|_{\mathcal{K}(K_0)})$, respectively. Moreover, it has good sense, as pointed out in part 2(f), only for disconnected K_0.

3.2. ad (b). Consider the upper-Carathéodory functional system

$$x'\in F(t,x_t),\quad x\in\mathbb{R}^n, \tag{3.19}$$

where $x_t(\cdot) = x(t+\cdot)$, for $t \in [0,\tau]$, denotes, as usual, a function from $[-\delta,0]$, $\delta \geq 0$, into $\mathbb{R}^n$, and $F : [0,\tau]\times\mathscr{C} \multimap \mathbb{R}^n$, where $\mathscr{C} := \mathrm{AC}([-\delta,0],\mathbb{R}^n)$, is an upper-Carathéodory map, that is,

(i) the set of values of $F(t,y)$ is nonempty, compact, and convex, for all $(t,y) \in [0,\tau]\times\mathscr{C}$,

(ii) $F(t,\cdot)$ is u.s.c., for a.a. $t \in [0,\tau]$,

(iii) $F(\cdot,y)$ is measurable for all $y \in \mathscr{C}$, that is, for any closed $U \subset \mathbb{R}^n$ and every $y \in \mathscr{C}$, the set $\{t \in [0,\tau] \mid F(\cdot,y)\cap U \neq \emptyset\}$ is measurable,

(iv) $|F(t,y)| \leq \alpha+\beta|y|$, for every $y \in \mathscr{C}$ and a.a. $t \in [0,\tau]$, where α and β are suitable nonnegative constants.

By a solution to (the initial problem of) (3.19), we mean again an absolutely continuous function $x \in \mathrm{AC}([-\delta,\tau],\mathbb{R}^n)$ (with $x(t) = x^*$, for $t \in [-\delta,0]$), satisfying (3.19), for a.a. $t \in [-\delta,\tau]$; such solutions exist on $[-\delta,\tau]$, for $\delta \geq 0$.

Hence, if $x(\cdot,x^*)$ is a solution of (3.8) with $x(t,x^*) = x^* \in E$, for $t \in [-\delta,0]$, where E consists of equicontinuous functions, then the translation operator $T_\tau : \mathrm{AC}([-\delta,0],\mathbb{R}^n) \multimap \mathrm{AC}([-\delta,0],\mathbb{R}^n)$ at the time $\tau > 0$ along the trajectories of (3.8) is defined as follows:

$$\begin{aligned} T_\tau(x^*) := \{&x(\tau+t,x^*), t \in [-\delta,0] \mid \\ &x(\cdot,x^*) \text{ is a solution to (3.19) with } x(t,x^*) = x^*, \text{ for } t \in [-\delta,0]\}. \end{aligned} \tag{3.20}$$

T_τ defined by (3.20) can be proved (see [5] or [9]) to be again a composition of an R_δ-mapping

$$\begin{aligned} \varphi(x^*) : x^* \multimap \{&x(\cdot,x^*) \mid \\ &x(\cdot,x^*) \text{ is a solution to (3.19) with } x(t,x^*) = x^*, \text{ for } t \in [-\delta,0]\} \end{aligned} \tag{3.21}$$

and the continuous (single-valued) evaluation mapping $\psi(y) : y \to y(\tau)$, that is, $T_\tau = \psi\circ\varphi$.

Now, let $X \subset \mathrm{AC}([-\delta,0],\mathbb{R}^n)$ be a bounded, closed subset consisting of equicontinuous functions satisfying conditions (i) and (ii) in part 2, and let $\mathscr{H} : X \to X$ be a homeomorphism. If $T_{\lambda\tau} : X \multimap X$, for each $\lambda \in [0,1]$, then we can still consider the composition

$$\begin{tikzcd} X\times[0,1] \arrow[r, "\varphi|"] \arrow[ddr, "\mathscr{H}\circ T_{\lambda\tau}"'] & \mathrm{AC}\big([-\delta,\lambda\tau],\mathrm{Im}\,\varphi(X),[0,1]\big) \arrow[d, "\psi|"] \\ & X \arrow[d, "\mathscr{H}"] \\ & X \end{tikzcd} \tag{3.22}$$

where $\varphi| := \varphi|_X$, $\psi| := \psi|_{\mathrm{AC}([-\delta,\lambda\tau],\mathrm{Im}\,\varphi(X),[0,1])}$ denote the respective restrictions. Since X is, by hypothesis, a bounded, closed subset consisting of equicontinuous functions, it is compact, and so is $\mathcal{H} \circ T_{\lambda\tau} : X \times [0,1] \multimap X$ (i.e., $\mathcal{H} \circ T_{\lambda\tau} \in \mathbb{K}$, for every $\lambda \in [0,1]$). Therefore, all the Nielsen numbers $N(\mathcal{H} \circ T_\tau) = N(\mathcal{H})$, $N(\mathcal{H} \circ T_\tau) + \#S(\mathcal{H} \circ T_\tau; A) = N(\mathcal{H}) + \#S(\mathcal{H}; A)$, $NS(\mathcal{H} \circ T_\tau; A) = NS(\mathcal{H}; A)$, $S_k(\widetilde{\mathcal{H} \circ T_\tau}) = S_k(\widetilde{\mathcal{H}})$, $N(T_\tau^* |_{\mathcal{K}(K_0)})$, and $S_k(T_\tau^* |_{\mathcal{K}(K_0)})$ are again well defined, satisfying the analogies of (3.5), (3.8), (3.17), and (3.18), respectively. On the other hand, there is one serious difference, namely since X is infinite-dimensional, we cannot take $X = \mathbb{T}^n$. Thus, there are no analogies of Theorems 3.1 and 3.3, whenever $\delta > 0$. For $\delta = 0$, the functional case reduces obviously to the ordinary one.

3.3. ad (c). Consider again system (3.19), where $F : [0,\tau] \times \mathscr{C} \multimap \mathbb{R}^n$ is the same as in part (b). For nonempty, compact, and convex set $K \subset \mathbb{R}^n$, the constraint, let us denote

$$\mathcal{K} := \{\xi \in \mathscr{C} \mid \xi(t) \in K, \text{ for } t \in [-\delta, 0]\} \tag{3.23}$$

and assume that the following Nagumo-type condition holds:

$$F(t,y) \cap T_K(y(0)) \neq \emptyset, \tag{3.24}$$

for all $(t,y) \in [0,\tau] \times \mathcal{K}$, where

$$T_K(y(0)) = \left\{ z \in \mathbb{R}^n \mid \liminf_{h \to 0^+} \frac{d(y(0) + hz, K)}{h} = 0 \right\} \tag{3.25}$$

is the tangent cone (in the sense of Bouligand).

Then, for every $x^* \in \mathcal{K}$, there exists at least one Carathéodory solution $x(\cdot, x^*)$ to (3.19) such that $x(t,x^*) = x^* \in E$, for $t \in [-\delta, 0]$, and $x(t,x^*) \in K$, for $t \in [0,\tau]$. Hence, we can define the associated translation operator $T_\tau : \mathcal{K} \multimap \mathcal{K}$ at the time $\tau > 0$ along the trajectories of (3.19) which makes the set $\mathcal{K}$ subinvariant, that is, $T_{\lambda\tau}(\mathcal{K}) \subset \mathcal{K}$, for every $\lambda \in [0,1]$. Moreover, $T_{\lambda\tau}$ can be shown to satisfy condition (iii); for more details, see [5] or [9].

Hence, although all the above Nielsen numbers can again be well defined, provided (ii), the convexity of K, and subsequent convexity of $\mathcal{K}$, makes most of the problems trivial, because $\mathcal{K} \in \mathrm{AR}$, that is, $N(T_\tau) = 1$, and so forth. Unfortunately, to avoid the convexity of K seems to be a difficult task. The only nontrivial situations seem to be those of the relative numbers $N(\mathcal{H} \circ T_\tau; A) + \#S(\mathcal{H} \circ T_\tau; A) = 1 + \#S(\mathcal{H}; A)$ and $NS(\mathcal{H} \circ T_\tau; A) = NS(\mathcal{H}; A)$.

3.4. ad (d). Consider the functional system

$$x' + Ax \in F(t, x_t), \quad x \in B, \tag{3.26}$$

where B is a separable Banach space, A is a closed, linear operator in B, generating an analytic semigroup, and $F : [0,\tau] \times \mathscr{C} \multimap B$ is an upper-Carathéodory map, where $\mathscr{C} = C([-\delta, 0], B)$. Under suitable restrictions in terms of the Hausdorff measure of noncompactness, imposed on A and F, one can show (see [5] or [9]) the existence of *mild solutions* $x \in C([-\delta, \tau], B)$, that is,

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-s)} f(s) ds, \tag{3.27}$$

for $t \in [0,\tau]$, with $x(t) = x^*$, for $t \in [-\delta,0]$, where f is a measurable selection of $F(s,x_s(t))$, $t \in [-\delta,0]$. Hence, we can define the associated translation operator $T_\tau : X \multimap X$, where $X \subset C([-\delta,0],B)$ is a subset satisfying conditions (i), (ii), and to show that $T_\tau \in \mathbb{C}$ is, under the mentioned restrictions, a γ-condensing composition (on equicontinuous sets) of an R_δ-mapping and the continuous evaluation mapping (see [5] or [9]). Nevertheless, in view of the difficulties discussed in part 2(e), it does not have much meaning to speak about applications of the Nielsen number to the above system in B before developing (even in the single-valued case) the appropriate Nielsen theory for condensing maps.

3.5. ad (e). Consider the random system

$$x'(\kappa,t) \in F(\kappa,t,x(\kappa,t)), \quad \kappa \in \Omega,\ x \in \mathbb{R}^n, \tag{3.28}$$

where Ω is a complete probabilistic space and

(i) the set of values of $F(\kappa,t,x)$ is nonempty, compact, and convex, for all $(\kappa,t,x) \in \Omega \times [0,\tau] \times \mathbb{R}^n$,

(ii) $F(\kappa,t,\cdot)$ is u.s.c., for a.a. $(\kappa,t) \in \Omega \times [0,\tau]$,

(iii) $F(\cdot,\cdot,x)$ is measurable, for all $x \in \mathbb{R}^n$, that is, for any closed $U \subset \mathbb{R}^n$ and every $x \in \mathbb{R}^n$, the set $\{(\kappa,t) \in \Omega \times [0,\tau] \mid F(\cdot,\cdot,x) \cap U \neq \emptyset\}$ is measurable,

(iv) $|F(\kappa,t,x)| \le \mu(\kappa,t)(1+|x|)$, for every $x \in \mathbb{R}^n$ and a.a. $(\kappa,t) \in \Omega \times [0,\tau]$, where $\mu : \Omega \times [0,\tau] \to [0,\infty)$ is a map such that $\mu(\cdot,t)$ is measurable and $\mu(\kappa,\cdot)$ is Lebesgue integrable on $[0,\tau]$.

The operator F satisfying (i)–(iv) is called the *random upper-Carathéodory operator*. Similarly, for metric spaces X and Y, we say that a multivalued mapping with nonempty, closed values $\varphi : \Omega \times X \multimap Y$ is a *random operator* if φ is product-measurable and $\varphi(\kappa,\cdot)$ is u.s.c., for every $\kappa \in \Omega$. By a *random homotopy* $\chi : \Omega \times X \times [0,1] \multimap Y$, we understand a product-measurable mapping with nonempty, closed values which is u.s.c. w.r.t. the last variable and which, for every $\lambda \in [0,1]$, satisfies that $\chi(\cdot,\cdot,\lambda)$ is a random operator. A measurable map (a random variable) $\hat{x} : \Omega \to X \cap Y$ is said to be a *random fixed-point* of a random operator $\varphi : \Omega \times X \multimap Y$ if $\hat{x}(\kappa) \in \varphi(\kappa,\hat{x}(\kappa))$, for a.a. $\kappa \in \Omega$.

By a *solution* to (3.28), we mean a function x such that $x(\cdot,t)$ is measurable, $x(\kappa,\cdot)$ is absolutely continuous, and $x(\kappa,t)$ satisfies (3.28), for a.a. $(\kappa,t) \in \Omega \times [0,\tau]$, where the derivative $x'(\kappa,t)$ is considered with respect to t. Under (i)–(iv), such solutions exist on $[0,\tau]$.

Besides (3.28), consider still the one-parameter family of deterministic systems

$$x' \in F_\kappa(t,x)[:= F(\kappa,t,x,)]. \tag{3.29}$$

This is because of the possibility to define the associated random translation operator T_τ in a deterministic way, just by means of the translation operator of (3.29).

Hence, defining

$$T_\tau(\kappa,x_0) := \{x(\tau,x_0) \mid x(\cdot,x_0) \text{ is a solution to (3.29) with } x(0,x_0) = x_0\}, \tag{3.30}$$

one can prove (see [5] or [9]) that T_τ is a random operator with compact values, composed of a random operator with R_δ-values and the continuous evaluation mapping. Thus, according to an important statement (see [9, Proposition 4.20, Chapter III.4]) allowing us to transform the investigation of random fixed points of T_τ to the one of $T_\tau(\kappa,\cdot)$, for every $\kappa \in \Omega$, it holds that T_τ possesses a random fixed point, whenever $T_\tau(\kappa,\cdot)$ has a fixed point, for every $\kappa \in \Omega$.

In view of the related deterministic obstructions in part (a), it will be useful to consider another composition with a homeomorphism $\mathcal{H} : X \multimap X$, namely $\mathcal{H} \circ T_{\lambda\tau}|_{\Omega\times X\times[0,1]} = \mathcal{H} \circ \psi| \circ \varphi|_{\Omega\times X\times[0,1]} : \Omega \times X \times [0,1] \multimap X$, that is,

$$
\begin{tikzcd}
\Omega\times X\times[0,1] \arrow[r, "\varphi|"] & \mathrm{AC}([0,\lambda\tau],\operatorname{Im}\varphi(\Omega,X),[0,1]) \arrow[d, "\psi|"] \\
\ddots & X \arrow[d, "\mathcal{H}"] \\
{} \arrow[r, "\mathcal{H}\circ T_{\lambda\tau}"] & X
\end{tikzcd}
\tag{3.31}
$$

where $X \subset \mathbb{R}^n$ is a bounded subset satisfying conditions (i) and (ii) of part 2,

$$\varphi(\kappa,x_0) : (\kappa,x_0) \multimap \{x(\cdot,x_0) \mid x(\cdot,x_0) \text{ is a solution to (3.29) with } x(0,x_0) = x_0\} \tag{3.32}$$

is an R_δ-mapping, for every $\kappa \in \Omega$, and $\varphi| := \varphi_{\Omega\times X}$, $\psi| := \psi|_{\mathrm{AC}([0,\lambda\tau],\operatorname{Im}\varphi(\Omega,X),[0,1])}$ denote the respective restrictions. Of course, $\hat{x} : \Omega \to X$ is a random fixed point of $\mathcal{H} \circ T_{\lambda\tau}$ if and only if the original system (3.28) has a *random solution* $x(\kappa,t)$ such that $\mathcal{H}(x(\kappa,0)) = x(\kappa,\tau) = \hat{x}(\kappa)$, for a.a. $\kappa \in \Omega$. Since $\mathcal{H} \circ T_{\lambda\tau} : \Omega \times X \times [0,1] \multimap X$ can be verified to be a compact random homotopy (see [9, Theorem 4.23, Chapter III.4]), we believe that one can define (via the mentioned transformation to the deterministic case) the *random Nielsen numbers* $N_\kappa(\mathcal{H} \circ T_\tau) = N_\kappa(\mathcal{H})$ and $S_k(\widetilde{\mathcal{H} \circ T_\tau})_\kappa = S_k(\widetilde{\mathcal{H}})_\kappa$, where the indices κ indicate the randomness, as the number of *essential random classes of coincidence points*, respectively of *essential random classes of irreducible cyclically different k-orbits of coincidences*. The random essentiality can be defined similarly as in [38, pages 156-157] by means of nontrivial related random coincidence indices. They should provide the lower bound of the numbers of random coincidence points and random irreducible cyclically different k-orbits of coincidences of $\mathcal{H} \circ T_\tau$, respectively.

If so, then we can randomize Theorems 3.1 and 3.3 as follows.

Conjecture 3.5. *System (3.28) admits, under (i)–(iv) and (3.7), at least* $|\Lambda(\mathcal{H})|$ *random solutions* $x(\kappa,t,x_0)$ *such that* $\mathcal{H}(x(\kappa,0,x_0)) = x(\kappa,\tau,x_0)(\operatorname{mod} 1)$, *for a.a.* $\kappa \in \Omega$, *and* $[(1/k)\sum_{m|k}\mu(k/m)|\Lambda(\mathcal{H}^m)|]^+$ *geometrically distinct k-tuples of random solutions* $x(\kappa,t,x_0)$ *such that*

$$\mathcal{H} \circ x(\kappa,\tau;\mathcal{H} \circ x(\kappa,\tau;\ldots\mathcal{H} \circ x(\kappa,\tau;x(0,x_0)\ldots))) = x(\kappa,0,x_0)(\operatorname{mod} 1), \tag{3.33}$$

for a.a. $\kappa \in \Omega$, *where* $\mathcal{H}$ *is a continuous self-map of* $\mathbb{T}^n$ *and* τ *is a positive number.*

Remark 3.6. Examples 3.2 and 3.4 can then be appropriately randomized as well.

3.6. ad (f). Let $M \in \mathbb{R}$ and let $\Gamma^M := \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \mid |x| \leq Mt\}$ be a closed, convex cone. We say that a multivalued mapping with nonempty, closed values $F : \mathbb{R} \times \mathbb{R}^n \multimap \mathbb{R}^n$ is Γ^M*-directionally u.s.c.* if, at each point $(t_0,x_0) \in \mathbb{R} \times \mathbb{R}^n$, and for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $(t,x) \in B((t_0,x_0),\delta)$ satisfying $|x - x_0| \leq M(t - t_0)$, the following holds: $F(t,x) \subset F(t_0,x_0) + \varepsilon B$.

Consider the Γ^M-directionally u.s.c. system (3.1). Since the solution set of (3.1) can be characterized by means of the Filippov-like regularization of (3.1), the related translation operator to (3.1) can be associated to the regularized system.

Thus, let $F : [0,\tau] \times \mathbb{R}^n \multimap \mathbb{R}^n$ in (3.1) be still convex-valued, locally bounded, and measurable. Then the mapping

$$\phi(t,x) = \bigcap_{\delta>0} \bigcap_{\substack{N\subset\mathbb{R}^{n+1}\\ \mu(N)=0}} \overline{\operatorname{conv}} F\big(B((t,x),\delta) \setminus N\big) \tag{3.34}$$

is called the *Filippov-like regularization* of the right-hand side of (3.1), where μ stands for the Lebesgue measure, and $\overline{\operatorname{conv}}$ for the closure of the convex hull of a set. The Filippov-like regularization can be proved to have the following properties (cf. [9]):

(i) $\phi(\cdot,\cdot)$ is u.s.c., for all $(t,x) \in [0,\tau] \times \mathbb{R}^n$,

(ii) $F(t,x) \subset \phi(t,x)$, for all $(t,x) \in [0,\tau] \times \mathbb{R}^n$,

(iii) ϕ is minimal in the following sense: if $\phi_0 : [0,\tau] \times \mathbb{R}^n \multimap \mathbb{R}^n$ satisfies (i) and (ii), then $\phi(t,x) \subset \phi_0(t,x)$, for all $(t,x) \in [0,\tau] \times \mathbb{R}^n$.

We have proved (see [9, Proposition 4.29, Chapter III.4]) that, under the above assumptions and $F([0,\tau] \times S) \subset B(0,L)$, where $0 < L < M$, every solution of the regularized inclusion

$$x' \in \phi(t,x), \quad x \in S, \tag{3.35}$$

is a solution to the original inclusion (3.1) with $F : [0,\tau] \times S \multimap \mathbb{R}^n$, where $S \subset \mathbb{R}^n$, and vice versa.

Hence, if $x(\cdot,x_0)$ is a solution to (3.1) with $x(0,x_0) = x_0 \in S$, then the translation operator $T_\tau : S \multimap S$ at time $\tau > 0$ along the trajectories of (3.1) can be defined as follows:

$$T_\tau(x_0) := \{x(\tau,x_0) \mid x(\cdot,x_0) \text{ is a solution to (3.1) with } x(0,x_0) = x_0 \in S\}, \tag{3.36}$$

and all that was presented in part (a) can be rewritten via (3.35) for (3.1). In particular, we can state the following corollary.

Corollary 3.7. *Let $F : [0,\tau] \times \mathbb{R}^n \multimap \mathbb{R}^n$ be Γ^M-directionally u.s.c., for some $M \in \mathbb{R}$, and let $F([0,\tau] \times \mathbb{R}^n) \subset B(0,L)$, where $0 < L < M$. Furthermore, let F be convex-valued and measurable. At last, let F satisfy (3.7). Then the conclusions of Theorems 3.1 and 3.3 hold for (3.1).*

Remark 3.8. Examples 3.2 and 3.4 can be appropriately rewritten under the assumptions of Corollary 3.7, too.

4. Hammerstein solution operator approach

For the sake of simplicity, we will concentrate on ordinary systems (3.1). Here, unlike in the foregoing section, fixed points of the associated (Hammerstein) operators will represent solutions to the given problems. Therefore, in view of Remark 2.1, these operators should be, in principle, "only" R_δ-maps.

The following multiplicity results for (3.1) will be considered:

(a) general multiplicity principle,
(b) multiplicity criterium for initial value problems,
(c) multiplicity criteria for boundary value problems.

4.1. ad (a). Consider the problem

$$\begin{aligned} x' \in F(t,x), \quad & \text{for a.a. } t \in I, \\ x \in S, & \end{aligned} \tag{4.1}$$

where $I \subset \mathbb{R}$ is a given interval, $F : I \times \mathbb{R} \multimap \mathbb{R}$ is an upper-Carathéodory map (cf. conditions (i)–(iv) in part 3(a)), and $S \subset AC_{\text{loc}}(I,\mathbb{R}^n)$, where $AC_{\text{loc}}(I,\mathbb{R}^n)$ denotes the class of locally absolutely continuous functions from I into $\mathbb{R}^n$.

Before applying the Nielsen theory presented in part 3(a) to (4.1), it will be convenient to introduce the following definition which follows R. F. Brown's modification of the Leray–Schauder boundary condition.

Definition 4.1. A mapping $T : Q \multimap U$, where U is an open subset of $C(I,\mathbb{R}^n)$ containing Q, is *retractible onto* Q, if there exists a (continuous) retraction $r : U \to Q$ such that $p \in U \setminus Q$ with $r(p) = q$ implies that $p \notin T(q)$.

The advantage of the above definition lies in the fact that, for a retractible mapping $T : Q \multimap U$ with a retraction r, the composition $r|_{T(Q)} \circ T : Q \to Q$ has a fixed point $\hat{q} \in Q$ if and only if $\hat{q}$ is a fixed point of T. Therefore, if $\hat{q} \in T(\hat{q})$ represents the solution to (4.1), where $T \in \mathbb{K}$ is an R_δ-map, so does $\hat{q} \in r|_{T(Q)} \circ T(\hat{q})$ and, in spite of Remark 2.1, $N(r|_{T(Q)} \circ T) \le \#\text{Fix}(T)$, whenever the Nielsen number $N(r|_{T(Q)} \circ T)$ is well defined (cf. conditions (i)–(iii) in part 3(a)).

The following statement is crucial (see [10] or [9]).

Proposition 4.2. *Let $G : I \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper-Carathéodory map and assume that*

(i) *there exists a closed, connected subset Q of $C(I,\mathbb{R}^n)$ with a finitely generated abelian fundamental group such that, for any $q \in Q$, the set $T(q)$ of all solutions of the linearized problem*

$$x' \in G(t,x,q(t)), \quad \text{for a.a. } t \in I,\ x \in S, \tag{4.2}$$

is R_δ,

(ii) *$T(Q)$ is bounded in $C(I,\mathbb{R}^n)$ and $\overline{T(Q)} \subset S$,*

(iii) *there exists a locally integrable function $\alpha : I \to \mathbb{R}$ such that $|G(t,x(t),q(t))| := \sup\{|y| \mid y \in G(t,x(t),q(t))\} \le \alpha(t)$, a.e. in I, for any pair $(q,x) \in \Gamma_T$, where Γ_T denotes the graph of T.*

Assume, furthermore, that

(iv) *the operator $T : Q \multimap U$, related to (4.2), is retractible onto Q with a retraction r in the sense of Definition 4.1.*

At last, let

$$G(t,c,c) \subset F(t,c), \tag{4.3}$$

for a.a. $t \in I$ and any $c \in \mathbb{R}^n$. Then the original problem (4.1) admits at least $N(r|_{T(Q)} \circ T)$ solutions belonging to Q, where N stands for the Nielsen number in part 2(a).

Remark 4.3. If Q is still compact and such that $\overline{T(Q)} \subset Q \cap S$ or if T is single-valued, then the fundamental group $\pi_1(Q)$ need not be abelian and finitely generated (see [2]).

In order to apply Proposition 4.2, the following main steps have to be taken:

(i) the R_δ-structure of the solution set to (4.2) must be verified,

(ii) the inclusion $\overline{T(Q)} \subset S$ or, most preferably, $\overline{T(Q)} \subset Q \cap S$ must be guaranteed, together with the retractibility of T,

(iii) $N(r|_{T(Q)} \circ T)$ must be computed.

4.2. ad (b). For initial value problems, condition (i) can be easily verified, provided G is still product-measurable (cf. [9]). In fact, since upper-Carathéodory inclusions with product-measurable right-hand sides G possess, for each $q \in Q \subset C(I,\mathbb{R}^n)$, an R_δ-set of solutions $x(\cdot,x_0)$ with $x(0,x_0) = x_0$, for every $x_0 \in \mathbb{R}^n$, such a requirement can be, in Proposition 4.2 with $S := \{x \in \mathrm{AC}_{\mathrm{loc}}(I,\mathbb{R}^n) \mid x(0,x_0) = x_0\}$, simply avoided. Moreover, if Q is still compact and such that $\overline{T(Q)} \subset Q \cap S$, then (see Remark 4.3) $\pi_1(Q)$ need not be abelian and finitely generated.

Thus, Proposition 4.2 simplifies, for initial value problems, as follows (cf. [2]).

Proposition 4.4. *Let $G : I \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper-Carathéodory product-measurable mapping, where $I = [0,\infty)$ or $I = [0,\tau]$, $\tau \in (0,\infty)$. Assume, furthermore, that there exists a (nonempty) compact, connected subset $Q \subset C(I,\mathbb{R}^n)$ which is a neighbourhood retract of $C(I,\mathbb{R}^n)$ such that $|G(t,x,q(t))| \le \mu(t)(|x|+1)$ holds, for every $(t,x,q) \in I \times \mathbb{R}^n \times Q$. Let the initial value problem*

$$x' \in G(t,x,q(t)), \quad \text{for a.a. } t \in I,\ x(0) = x_0, \tag{4.4}$$

have, for each $q \in Q$, a nonempty set of solutions $T(q)$ such that $\overline{T(Q)} \subset Q \cap S$, where $S := \{x \in \mathrm{AC}_{loc}(I,\mathbb{R}^n) \mid x(0) = x_0\}$. Then the original initial value problem

$$x' \in F(t,x), \quad \text{for a.a. } t \in I,\ x(0) = x_0, \tag{4.5}$$

admits at least $N(T)$ solutions, provided (4.3) holds a.e. on I, for any $c \in \mathbb{R}^n$.

Example 4.5. Consider the scalar ($n = 1$) initial value problem with $x_0 = 0$ and $I = [0,\tau]$, $\tau > 0$. Letting

$$\begin{aligned} Q := \{&q \in \mathrm{AC}([0,\tau],\mathbb{R}) \mid \\ &q(0) = 0 \text{ and } \delta_2 \le q'(t) \le \delta_1 \text{or } -\delta_1 \le q'(t) \le -\delta_2, \text{ for a.a. } t \in [0,\tau]\}, \end{aligned} \tag{4.6}$$

where $0 < \delta_2 < \delta_1$ are suitable constants, Q can be easily verified to be a disjoint (!) union of two convex, compact sets, and consequently Q is a compact ANR, that is, also a neighborhood retract of $C([0,\tau],\mathbb{R})$. Unfortunately, Q is disconnected, which excludes the direct application of Proposition 4.4.

Nevertheless, for example, the inclusion

$$x' \in \delta \operatorname{Sgn}(x), \qquad \text{for a.a. } t \in [0,\tau],\ \delta > 0, \tag{4.7}$$

where

$$\operatorname{Sgn}(x) = \begin{cases} -1, & \text{for } x \in (-\infty,0), \\ [-1,1], & \text{for } x = 0, \\ 1, & \text{for } x \in (0,\infty), \end{cases} \tag{4.8}$$

admits obviously two classical C^1-solutions $x_1(t) = \delta t$ with $x_1(0) = 0$, and $x_2(t) = -\delta t$ with $x_2(0) = 0$, satisfying the given inclusion everywhere.

The linearized inclusion

$$x' \in \delta \operatorname{Sgn}(q(t)), \qquad \text{for a.a. } t \in [0,\tau],\ \delta > 0, \tag{4.9}$$

possesses, for each $q \in Q$, either the solution $x_1(t) = \delta t$ with $x_1(0) = 0$ or $x_2(t) = -\delta t$ with $x_2(0) = 0$, depending on $\operatorname{sgn}(q(t))$, provided $\delta_2 \le \delta \le \delta_1$. Observe that there are no more solutions, for each $q \in Q$. Thus, we also have $\overline{T(Q)} \subset Q \cap S$ (i.e., condition (ii)), where $S := \{x \in AC_{\text{loc}}(I,\mathbb{R}^n) \mid x(0) = 0\}$.

The only handicap is related to the mentioned disconnectedness of Q. However, since $T : Q \to Q$, where

$$T(q) = \begin{cases} \delta t, & \text{for } q \ge 0, \\ -\delta t, & \text{for } q \le 0, \end{cases} \tag{4.10}$$

is obviously single-valued, the application of the multivalued Nielsen theory, as in part 2(a) (cf. [2]), in the proof of Proposition 4.4 can be replaced by the application of the single-valued one, where $Q \in$ ANR can already be disconnected (see, e.g., [55]). We can, therefore, conclude, on the basis of the appropriately modified Proposition 4.4, that the original inclusion (4.7) admits at least $N(T) = 2$ solutions $x(t)$ with $x(0) = 0$, as observed by the direct calculations. In fact, it must therefore have a nontrivial R_δ-set of infinitely many piece-wise linear solutions $x(t)$ with $x(0) = 0$. The computation of $N(T) = 2$ (i.e., condition (iii)) is trivial, because $Q = Q^+ \cup Q^-$, where

$$\begin{aligned} (\text{AR} \ni) Q^+ :=& \{q \in AC([0,\tau],\mathbb{R}) \mid \\ & \quad q(0) = 0, \delta_2 \le q'(t) \le \delta_1, \text{ for a.a. } t \in [0,\tau]\}, \\ (\text{AR} \ni) Q^- :=& \{q \in AC([0,\tau],\mathbb{R}) \mid \\ & \quad q(0) = 0, -\delta_1 \le q'(t) \le -\delta_2, \text{ for a.a. } t \in [0,\tau]\}, \end{aligned} \tag{4.11}$$

and so for the computation of the generalized Lefschetz numbers we have $\Lambda(T|_{Q^+}) = \Lambda(T|_{Q^-}) = 1$, where $T|_{Q^+} : Q^+ \to Q^+$ and $T|_{Q^-} : Q^- \to Q^-$.

The same is obviously true for the inclusion

$$x' \in [\delta + f(t,x)]\mathrm{Sgn}(x), \quad \text{for a.a. } t \in [0,\tau],\ \delta > 0, \tag{4.12}$$

where $f : [0,\tau] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory and locally Lipschitz function in x, for a.a. $t \in [0,\tau]$, such that $\delta_2 \le \delta + f(t,x) \le \delta_1$, for some $0 < \delta_2 < \delta_1$, because again $T : Q \to Q$.

Of course, we could arrive at the same conclusion even without an explicit usage of the Nielsen theory, just through double application (separately on Q^+ and Q^-) of the Lefschetz theory.

Remark 4.6. In view of Example 4.5, it is more realistic to suppose in Proposition 4.4 that (at least for $n = 1$) the solution operator T is single-valued and that Q can be disconnected and not necessarily compact. Naturally, the first requirement seems to be rather associated with differential equations than inclusions.

4.3. ad (c). For boundary value problems, condition (i) is much more complicated to be verified (cf. [9, Chapter III.3]). Therefore, we restrict ourselves to semilinear Carathéodory inclusions with linear boundary conditions in the following form:

$$\begin{gathered} x' + A(t)x \in F(t,x), \quad \text{for a.a. } t \in I, \\ Lx = \Theta, \quad \Theta \in \mathbb{R}^n, \end{gathered} \tag{4.13}$$

where $A : [0,\tau] \to \mathbb{R}^n \times \mathbb{R}^n$ is a (single-valued) continuous $n \times n$ matrix and $F : [0,\tau] \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory (cf. conditions (i)–(iv) in part 3(a)) product-measurable mapping with nonempty, compact, and convex values.

In [10] (cf. [9]), we proved the following theorem by means of Proposition 4.2.

Theorem 4.7. *Let $A : [0,\tau] \to \mathbb{R}^n \times \mathbb{R}^n$ and $F : [0,\tau] \times \mathbb{R}^n \multimap \mathbb{R}^n$ be as above. Let, furthermore, $L : C([0,\tau],\mathbb{R}^n) \to \mathbb{R}^n$ be a linear operator such that the homogeneous problem*

$$x' + A(t)x = 0, \quad Lx = 0, \tag{4.14}$$

has only the trivial solution on $[0,\tau]$. Then the original problem (4.13) has at least $N(r|_{T(Q)} \circ T)$ solutions, where T denotes the solution operator to the linearized problem

$$\begin{gathered} x' + A(t)x \in F(t,q(t)), \quad \text{for a.a. } t \in I,\ q \in Q, \\ Lx = \Theta, \quad \Theta \in \mathbb{R}^n, \end{gathered} \tag{4.15}$$

provided there exists a closed, connected subset $Q \in C([0,\tau],\mathbb{R}^n)$ with a finitely generated abelian fundamental group such that

(i) *$T(Q)$ is bounded,*

(ii) *T is retractible onto Q with a retraction r, in the sense of Definition 4.1,*

(iii) *$\overline{T(Q)} \subset \{x \in \mathrm{AC}([0,\tau],\mathbb{R}^n) \mid Lx = \Theta\}$.*

Remark 4.8. In the single-valued case, we can assume the unique solvability of the linearized problem (4.15). Moreover, Q again need not have a finitely generated abelian fundamental group. In the multivalued case, the latter statement is true, provided Q is compact and $\overline{T(Q)} \subset Q$ (cf. Remark 4.3).

Remark 4.9. Although the solution operator T in the foregoing part (b) is rather Cauchy than Hammerstein, here T is indeed Hammerstein, which justifies the title of the whole section, because the focus is on boundary value problems.

Now, consider the planar ($n = 2$) inclusions

$$\begin{aligned} x' + ax &\in e(t,x,y)y^{(1/m)} + g(t,x,y), \\ y' + by &\in f(t,x,y)x^{(1/n)} + h(t,x,y), \end{aligned} \tag{4.16}$$

where a, b are positive constants and m, n are odd integers with $\min(m,n) \geq 3$. Let, furthermore, $e, f, g, h : \mathbb{R}^3 \multimap \mathbb{R}^2$ be product-measurable upper-Carathéodory maps with nonempty, compact, and convex values satisfying the inequalities

$$\begin{aligned} |e(t,x,y)| &\leq E_0, \qquad & |f(t,x,y)| &\leq F_0, \\ |g(t,x,y)| &\leq G, \qquad & |h(t,x,y)| &\leq H, \end{aligned} \tag{4.17}$$

for a.a. $t \in \mathbb{R}$ and all $(x,y) \in \mathbb{R}^2$, where E_0, F_0, G, and H are suitable constants. Let e_0, f_0, δ_1, and δ_2 be positive constants such that

$$0 < e_0 \leq e(t,x,y), \qquad \text{for a.a. } t \in \mathbb{R}, \text{ all } x \in \mathbb{R}, \ |y| \geq \delta_2, \tag{4.18}$$

jointly with

$$0 < f_0 \leq f(t,x,y), \qquad \text{for a.a. } t \in \mathbb{R}, \text{ all } y \in \mathbb{R}, \ |x| \geq \delta_1. \tag{4.19}$$

Applying Theorem 4.7, the following theorem was proved in [10] (cf. [9]).

Theorem 4.10. *If*

$$\begin{aligned} \left(\frac{1}{a}\right) \left| e_0 \delta_2^{1/m} - G \right| &\geq \delta_1 > \left(\frac{H}{f_0}\right)^n, \\ \left(\frac{1}{b}\right) \left| f_0 \delta_1^{1/n} - H \right| &\geq \delta_2 > \left(\frac{G}{e_0}\right)^m, \end{aligned} \tag{4.20}$$

then, under the above assumptions, system (4.16) admits at least two entirely bounded solutions. If the maps e, f, g, and h are still τ-periodic in t, then system (4.16) admits at least three τ-periodic solutions, provided sharp inequalities hold in (4.20).

Remark 4.11. In fact, there was a gap in our papers concerning conditions (4.18) and (4.19) (see [2, 10]). More precisely, we assumed (4.18) and (4.19) only on smaller domains than here, by which the Hammerstein solution operator T need not satisfy $T(Q) \subset Q$. On the other hand, if we assume (4.18) and (4.19) on the whole domains like here, then there evidently appear disjoint subinvariant subdomains with respect to T which are AR-spaces. Thus, the same result can also be obtained (similarly to Example 4.5) without the explicit usage of the Nielsen theory, for example, by means of the fixed-point index (cf. [29]).

The following example (which is due to our Ph.D. student Tomáš Fürst) brings a modification of Theorem 4.10 in the sense that the possible subinvariant subdomains (if any), mentioned in Remark 4.11, cannot be easily detected.

Example 4.12. Consider the planar system of integrodifferential equations

$$\begin{aligned} x_1'(t)+ax_1(t) &= \sqrt[3]{p_2(t)}-\sqrt[3]{p_1(t)}+h_1\big(t,x_1(t),x_2(t)\big),\\ x_2'(t)+ax_2(t) &= \sqrt[3]{p_1(t)}+\sqrt[3]{p_2(t)}+h_2\big(t,x_1(t),x_2(t)\big), \end{aligned} \tag{4.21}$$

for a.a. $t\in[0,\tau]$, $\tau>0$, where $a>0$ is a constant, $h_i:[0,\tau]\times\mathbb{R}^2\to\mathbb{R}^2$ are Carathéodory functions such that $h_i(t,x_1,x_2)\equiv h_i(t+\tau,x_1,x_2)$, $i=1,2$, and

$$p_i(t)=\frac{1}{\tau}\int_0^\tau x_i(s)ds-B\bigg(\frac{1}{\tau}\int_0^\tau x_i(s)ds-x_i(t)\bigg), \tag{4.22}$$

for $i=1,2$, with a sufficiently small constant $B\geq 0$ which will be specified below. Assume the existence of constants $D>0$ and $\delta>0$ such that $|h_i(t,x_1,x_2)|\leq(1/2)D\delta$, for a.a. $t\in[0,\tau]$ and all $(x_1,x_2)\in\mathbb{R}^2$, and

$$\frac{\sqrt[3]{32}\,(3\sqrt[3]{B}+D)}{a}<\sqrt[3]{\delta^2}<\frac{\sqrt{2}}{a(D+2)}. \tag{4.23}$$

Under the above assumptions, system (4.21) admits at least three τ-periodic solutions. The third solution can be proved, similarly to Theorem 4.10, by means of the fixed-point index. We note that the above conditions can be improved, but our goal was only to avoid the handicap mentioned in Remark 4.11. The last inequalities are satisfied, for example, for $a=0.5$, $B=0.0001$, $D=0.01$, $\delta=1$.

5. Concluding remarks

In this final part, we will briefly mention

(a) consequences and links (of the obtained results),
(b) some further possibilities,
(c) open problems.

5.1. ad (a). In Theorem 1.1, the existence of a subharmonic (i.e., $k\omega$-periodic with $k>1$) solution is assumed to the equation or, more generally, inclusion (see Remark 1.2) which must not be uniquely solvable for the initial value problem. The necessity of the uniqueness absence in Theorems 1.1 and 1.7 (see Remarks 1.2 and 1.8) can be expressed equivalently as a nontrivial R_δ-structure of the solutions of the initial value problems which was treated in Proposition 4.4 and (more appropriately) in Example 4.5. The existence of purely subharmonic (i.e., with $k > 1$) solution $x(t) \equiv x(t + k\omega)$ of Carathéodory differential equation de facto means the nontrivial R_δ-structure of solutions of the initial value problem. Generically, initial value problems of Carathéodory equations are uniquely solvable.

Although the generalization of inclusion (4.7), namely

$$x' \in [\delta + f(t,x)]\mathrm{Sgn}(x), \qquad \text{for a.a. } t \in [0,\tau],\ \delta > 0, \tag{5.1}$$

has under the above assumptions at least two (and subsequently a nontrivial R_δ-structure of) solutions $x(t)$ with $x(0) = 0$, it only gives a chance for the existence of a subharmonic solution. The sufficiency is far from being obvious. On the other hand, using the inverse limit method in [9, Chapter II.2], one can prove the R_δ-structure of solutions to the initial value problem on the whole half-line $[0,\infty)$, provided appropriate assumptions hold there. If the above inclusion appears in the triangular system considered in Theorem 1.7 (cf. Remark 1.8), then the same consequences are true.

The existence of three (harmonic) periodic solutions, required in Theorem 1.4, was proved in Theorem 4.10, but one should still verify the braid type of the conjugated trajectories in B_3/Z, to obtain infinitely many subharmonics. This is, in general, a difficult task. In our concrete situation, it means that the amplitude of the third harmonic solution, namely the one not located in the subinvariant subdomains mentioned in Remark 4.11, should be sufficiently large. Although it does not seem to be the case here, numerical simulations signalize, rather curiously, coexistence of infinitely many (sub)harmonics. This could mean that some further harmonics with not necessarily large amplitudes may exist.

5.2. ad (b). Our Nielsen theories concern the coincidence problems of the form $p(x) = q(x)$, where $p : \Gamma \Rightarrow X$ is a *Vietoris map*, that is, in particular that $p^{-1} : X \multimap \Gamma$ is "*onto*" (for more details, see [38]). Some other theories (see, e.g., [24, 25, 26, 27, 28, 32, 33, 34, 35, 36, 37]) deal with the problems of the form $Lx = Nx$, where $L : X \to Y$ is an isomorphism (cf. [24, 25, 26, 27, 28]) or, more generally, a Fredholm operator of index zero (cf. [32, 33, 36, 37]), or even more generally, a Fredholm operator of nonnegative index (cf. [34, 37]), between Banach spaces X and Y. We recall that $L : X \to Y$ is a *Fredholm operator*, if the image $\mathrm{Im}L$ of L is closed in Y and the kernel $\mathrm{Ker}L$ and cokernel $Y/\mathrm{Im}\,L$ of L are finite dimensional. Then the *index* of the Fredholm operator L is defined by $\mathrm{index}\,L := \dim\mathrm{Ker}\,L - \dim Y/\mathrm{Im}L$. Thus, the crucial assumptions here are that $\mathrm{Ker}\,L \neq \{0\}$ and index $L \geq 0$. Moreover, since $Lx = Nx$ is rewritten into a fixed-point problem by the classical Lyapunov–Schmidt reduction method, this construction is an extension of J. Mawhin's coincidence degree theory.

At first glance, it might seem that there is a lot in common for these two alternative approaches, but there is actually no real connection. Although Fredholm operators have acyclic (namely convex) fibres and are proper on bounded sets, they are not Vietoris maps, because they are not "onto." Of course, one may just restrict to the range of L, but this would dramatically decrease the number of admissible homotopies. In J. Mawhin's coincidence degree theory as well as M. Fečkan's papers, homotopies are not assumed to take their values in the range of L.

Nevertheless, it is definitely possible to combine both approaches to define a "generalized Nielsen number" for maps L, p, and q which would provide the lower bound of the number of coincidences of the maps L and $N := q \circ p^{-1}$, where L is a (not necessarily invertible) Fredholm operator of nonnegative index and p is a $\mathbb{Z}$-Vietoris map. Thus, one could consider unified coincidence problems of the form $Y \overset{L}{\leftarrow} X \overset{p}{\Leftarrow} \Gamma \overset{q}{\rightarrow} Y$, that is, $Lx \in Nx$, where $N := q \circ p^{-1}$, which reduce to $p(x) = q(x)$, for $L^{-1} = \mathrm{id}|_Y$, and to $Lx = Nx$, for $p^{-1} = \mathrm{id}|_X$ (which implies $N = q$). For the discussion concerning the related fixed-point index, see, for example, [19]. For some further possibilities, see [22, 23], where integral equations involving Urysohn-type operators are examined; [35], where small vector fields on manifolds are studied; and [56], where a relative Nielsen number on the closure of a complement is applied via Poincaré's translation operator on the lines of R. Srzednicki's geometric method.

In contrast to our papers [2, 3, 4, 10, 11, 12, 13] (cf. also [9]) and [56], all multiplicity results in [24, 25, 26, 27, 28, 32, 33, 34, 35, 36, 37] were obtained for differential equations involving necessarily small or tendentiously implemented parameters. For a survey of multiplicity criteria for concrete boundary value problems, see [27, 37].

In the fixed-point theory, there are besides the Nielsen theories only few more theorems for multiple fixed points (see, e.g., [1, 21, 30, 39, 41, 45, 46, 53, 54] and the references therein). Since many applications of the Nielsen theories to multiplicity results for differential equations and inclusions seem to be alternatively obtained by different techniques, it would be interesting, when comparing the results, to specify those available only by means of Nielsen numbers.

5.3. ad (c). We conclude by formulating some open problems.

For the Nielsen theories:

(i) to develop the *Nielsen theory for condensing maps* (single-valued or multivalued, with or without the explicit usage of the generalized Lefschetz number for condensing maps on noncontractible sets),

(ii) to develop the *Nielsen theory for periodic coincidences* available to *multivalued maps on noncompact domains*,

(iii) to develop the *relative Nielsen theories for periodic coincidences* available to multivalued maps (eventually, on noncompact domains),

(iv) to develop the *unified Nielsen theory for multivalued maps of the form* $Y \overset{L}{\leftarrow} X \overset{p}{\Leftarrow} \Gamma \overset{q}{\rightarrow} Y$, where L is a Fredholm operator of a nonnegative index and (p,q) is a pair satisfying condition (iii) in part 2(a),

(v) to verify the conjecture in [20] that *if φ satisfying condition (iii) in part 2(a) assumes only simply connected values, then $N(\varphi) \le \#Fix(\varphi)$*,

(vi) to show that the *random Nielsen numbers* can be defined via the transformation to the deterministic case as conjectured in part 3(e).

For applications of Nielsen theories to differential equations and inclusions:

(i) to obtain *nontrivial applications* of the Nielsen theory to initial or boundary value problems *with no subinvariant subdomain* with respect to the representing solution operators,

(ii) to prove (or, at least, to detect) the *existence of a subharmonic* (most preferably, 3ω-periodic) *solution to a scalar differential equation or inclusion*, as required in Theorem 1.1,

(iii) to apply the Nielsen theory for obtaining *at least three (harmonic) periodic solutions* of a planar Carathéodory system of equations, *whose trajectories are conjugated in B_3/Z to the admissible braid type*, as required in Theorem 1.4,

(iv) to verify Conjecture 3.5.

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References

[1] R. P. Agarwal and D. O'Regan, *A note on the existence of multiple fixed points for multivalued maps with applications*, J. Differential Equations **160** (2000), no. 2, 389–403.

[2] J. Andres, *Multiple bounded solutions of differential inclusions: the Nielsen theory approach*, J. Differential Equations **155** (1999), no. 2, 285–310.

[3] ———, *A nontrivial example of application of the Nielsen fixed-point theory to differential systems: problem of Jean Leray*, Proc. Amer. Math. Soc. **128** (2000), no. 10, 2921–2931.

[4] ———, *Nielsen number, Artin braids, Poincaré operators and multiple nonlinear oscillations*, Nonlinear Anal. **47** (2001), no. 2, 1017–1028, Proceedings of the Third World Congress of Nonlinear Analysts, Part 2 (Catania, 2000).

[5] ———, *Poincaré's translation multioperator revisited*, Lectures in Nonlinear Analysis, vol 3 (W. Kryszewski and A. Nowakowski, eds.), Juliusz Schauder Centre for Nonlinear Studies, Toruń, 2002, pp. 7–22.

[6] ———, *Period three implications for expansive maps in $\mathbb{R}^N$*, J. Differ. Equations Appl. **10** (2004), no. 1, 17–28.

[7] ———, *Applicable fixed point principles*, Handbook of Topological Fixed Point Theory (R. F. Brown, M. Furi, L. Górniewicz, and B. Jiang, eds.), Springer, Berlin, 2005, pp. 687–739.

[8] J. Andres, J. Fišer, and L. Jüttner, *On a multivalued version of the Sharkovskii theorem and its application to differential inclusions*, Set-Valued Anal. **10** (2002), no. 1, 1–14.

[9] J. Andres and L. Górniewicz, *Topological Fixed Point Principles for Boundary Value Problems*, Topological Fixed Point Theory and Its Applications, vol. 1, Kluwer Academic, Dordrecht, 2003.

[10] J. Andres, L. Górniewicz, and J. Jezierski, *Noncompact version of the multivalued Nielsen theory and its application to differential inclusions*, Differential Inclusions and Optimal Control (J. Andres, L. Górniewicz, and P. Nistri, eds.), Lecture Notes in Nonlinear Anal., vol. 2, Nicolaus Copernicus University, Toruń, 1998, pp. 33–50.

[11] ———, *A generalized Nielsen number and multiplicity results for differential inclusions*, Topology Appl. **100** (2000), no. 2-3, 193–209.

[12] ———, *Relative versions of the multivalued Lefschetz and Nielsen theorems and their application to admissible semi-flows*, Topol. Methods Nonlinear Anal. **16** (2000), no. 1, 73–92.

[13] ______, *Periodic points of multivalued mappings with applications to differential inclusions on tori*, Topology Appl. **127** (2003), no. 3, 447–472.
[14] J. Andres and L. Jüttner, *Period three plays a negative role in a multivalued version of Sharkovskii's theorem*, Nonlinear Anal. **51** (2002), no. 6, 1101–1104.
[15] J. Andres, L. Jüttner, and K. Pastor, *On a multivalued version of the Sharkovskii theorem and its application to differential inclusions. II*, Set-Valued Anal. **13** (2005), no. 1, 47–68.
[16] J. Andres and K. Pastor, *On a multivalued version of the Sharkovskii theorem and its application to differential inclusions. III*, Topol. Methods Nonlinear Anal. **22** (2003), no. 2, 369–386.
[17] ______, *A version of Sharkovskii's theorem for differential equations*, Proc. Amer. Math. Soc. **133** (2005), no. 2, 449–453.
[18] ______, *A multivalued version of Sharkovskii's theorem holds with at most two exceptions*, in preparation.
[19] J. Andres and M. Väth, *Coincidence index for noncompact mappings in nonconvex sets*, Nonlinear Funct. Anal. Appl. **7** (2002), no. 4, 619–658.
[20] ______, *Two topological definitions of a Nielsen number for coincidences of noncompact maps*, Fixed Point Theory Appl. **2004** (2004), no. 1, 49–69.
[21] R. I. Avery, *A generalization of the Leggett-Williams fixed point theorem*, Math. Sci. Res. Hot-Line **3** (1999), no. 7, 9–14.
[22] A. Yu. Borisovich, Z. Kucharski, and W. Marzantowicz, *Nielsen numbers and lower estimates for the number of solutions to a certain system of nonlinear integral equations*, Applied Aspects of Global Analysis, Novoe Global. Anal., vol. 14, Voronezh University Press, Voronezh, 1994, pp. 3–10, 99 (Russian).
[23] ______, *A multiplicity result for a system of real integral equations by use of the Nielsen number*, Nielsen Theory and Reidemeister Torsion (Warsaw, 1996) (J. Jezierski, ed.), Banach Center Publ., vol. 49, Polish Academy of Sciences, Warsaw, 1999, pp. 9–18.
[24] R. F. Brown, *Multiple solutions to parametrized nonlinear differential systems from Nielsen fixed point theory*, Nonlinear Analysis, World Scientific, Singapore, 1987, pp. 89–98.
[25] ______, *Nielsen fixed point theory and parametrized differential equations*, Fixed Point Theory and Its Applications (Berkeley, Calif, 1986) (R. F. Brown, ed.), Contemp. Math., vol. 72, American Mathematical Society, Rhode Island, 1988, pp. 33–46.
[26] ______, *Topological identification of multiple solutions to parametrized nonlinear equations*, Pacific J. Math. **131** (1988), no. 1, 51–69.
[27] ______, *More about Nielsen theories and their applications*, Handbook of Topological Fixed Point Theory (R. F. Brown, M. Furi, L. Górniewicz, and B. Jiang, eds.), Springer, Berlin, in press.
[28] R. F. Brown and P. Zezza, *Multiple local solutions to nonlinear control processes*, J. Optim. Theory Appl. **67** (1990), no. 3, 463–485.
[29] M. J. Capiński and K. Wójcik, *Isolating segments for Carathéodory systems and existence of periodic solutions*, Proc. Amer. Math. Soc. **131** (2003), no. 8, 2443–2451.
[30] K.-C. Chang, *Infinite-Dimensional Morse Theory and Multiple Solution Problems*, Progress in Nonlinear Differential Equations and their Applications, vol. 6, Birkhäuser Boston, Massachusetts, 1993.
[31] D. W. Curtis, *Hyperspaces of noncompact metric spaces*, Compositio Math. **40** (1980), no. 2, 139–152.
[32] M. Fečkan, *Multiple perturbed solutions near nondegenerate manifolds of solutions*, Comment. Math. Univ. Carolin. **34** (1993), no. 4, 635–643.
[33] ______, *Nielsen fixed point theory and nonlinear equations*, J. Differential Equations **106** (1993), no. 2, 312–331.
[34] ______, *Differential equations with nonlinear boundary conditions*, Proc. Amer. Math. Soc. **121** (1994), no. 1, 103–111.

[35] ______, *Multiple periodic solutions of small vector fields on differentiable manifolds*, J. Differential Equations **113** (1994), no. 1, 189–200.
[36] ______, *Parametrized singular boundary value problems*, J. Math. Anal. Appl. **188** (1994), no. 2, 417–425.
[37] ______, *Multiple solutions of nonlinear equations via Nielsen fixed-point theory: a survey*, Nonlinear Analysis in Geometry and Topology (T. M. Rassias, ed.), Hadronic Press, Florida, 2000, pp. 77–97.
[38] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Mathematics and Its Applications, vol. 495, Kluwer Academic, Dordrecht, 1999.
[39] C. Golé, *Symplectic Twist Maps. Global Variational Techniques*, Advanced Series in Nonlinear Dynamics, vol. 18, World Scientific, New Jersey, 2001.
[40] P. R. Heath, *A survey of Nielsen periodic point theory (fixed n)*, Nielsen Theory and Reidemeister Torsion (Warsaw, 1996) (J. Jezierski, ed.), Banach Center Publ., vol. 49, Polish Academy of Sciences, Warsaw, 1999, pp. 159–188.
[41] H. Hofer and E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser Verlag, Basel, 1994.
[42] B. J. Jiang, *Lectures on Nielsen Fixed Point Theory*, Contemporary Mathematics, vol. 14, American Mathematical Society, Rhode Island, 1983.
[43] ______, *Nielsen theory for periodic orbits and applications to dynamical systems*, Nielsen Theory and Dynamical Systems (South Hadley, Mass, 1992) (C. K. McCord, ed.), Contemp. Math., vol. 152, American Mathematical Society, Rhode Island, 1993, pp. 183–202.
[44] B. Jiang, *Applications of Nielsen theory to dynamics*, Nielsen Theory and Reidemeister Torsion (Warsaw, 1996) (J. Jezierski, ed.), Banach Center Publ., vol. 49, Polish Academy of Sciences, Warsaw, 1999, pp. 203–221.
[45] P. Le Calvez, *Dynamical Properties of Diffeomorphisms of the Annulus and of the Torus*, SMF/AMS Texts and Monographs, vol. 4, American Mathematical Society, Rhode Island, 2000, translated from the 1991 French original by Philippe Mazaud.
[46] R. W. Leggett and L. R. Williams, *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana Univ. Math. J. **28** (1979), no. 4, 673–688.
[47] T. Matsuoka, *The number and linking of periodic solutions of periodic systems*, Invent. Math. **70** (1983), no. 3, 319–340.
[48] ______, *Waveform in the dynamical study of ordinary differential equations*, Japan J. Appl. Math. **1** (1984), no. 2, 417–434.
[49] ______, *The number and linking of periodic solutions of nondissipative systems*, J. Differential Equations **76** (1988), no. 1, 190–201.
[50] ______, *Braid invariants and instability of periodic solutions of time-periodic 2-dimensional ODE's*, Topol. Methods Nonlinear Anal. **14** (1999), no. 2, 261–274.
[51] ______, *Periodic points and braid theory*, Handbook of Topological Fixed Point Theory (R. F. Brown, M. Furi, L. Górniewicz, and B. Jiang, eds.), Springer, Berlin, in press.
[52] C. K. McCord, ed., *Nielsen Theory and Dynamical Systems*, Contemporary Mathematics, vol. 152, American Mathematical Society, Rhode Island, 1993, Proceedings of the AMS-IMS-SIAM Summer Research Conference held at Mount Holyoke College, Massachusetts, 1992.
[53] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, 2nd ed., Oxford Mathematical Monographs, Clarendon Press, Oxford University Press, New York, 1998.
[54] R. McGehee and K. R. Meyer, eds., *Twist Mappings and Their Applications*, The IMA Volumes in Mathematics and Its Applications, vol. 44, Springer, New York, 1992.
[55] U. K. Scholz, *The Nielsen fixed point theory for noncompact spaces*, Rocky Mountain J. Math. **4** (1974), 81–87, collection of articles on fixed point theory.

[56] K. Wójcik, *Periodic segments and Nielsen numbers*, Conley Index Theory (Warsaw, 1997) (K. Mischaikow, M. Mrozek, and P. Zgliczyński, eds.), Banach Center Publ., vol. 47, Polish Academy of Sciences, Warsaw, 1999, pp. 247–252.

[57] X. Zhao, *Periodic points for relative maps*, Chinese J. Contemp. Math. **24** (2003), no. 2, 169–174.

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