ON THE UNIQUENESS OF THE FIXED POINT INDEX ON DIFFERENTIABLE MANIFOLDS

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It is well known that some of the properties enjoyed by the fixed point index can be chosen as axioms, the choice depending on the class of maps and spaces considered. In the context of finite-dimensional real differentiable manifolds, we will provide a simple proof that the fixed point index is uniquely determined by the properties of normalization, additivity, and homotopy invariance.

1. Introduction

The fixed point index enjoys a number of properties whose precise statement may vary in the literature. The prominent ones are those of normalization, additivity, homotopy invariance, commutativity, solution, excision, and multiplicativity (see, e.g., [4, 5, 6, 8, 9, 10]). It is well known that some of the above properties can be used as axioms for the fixed point index theory. For instance, in the manifold setting, it can be deduced from [3] that the first four, provided that the first three are stated as in Section 2, imply the uniqueness of the fixed point index. Actually the result of [3] is not merely confined to the context of (differentiable) manifold: it holds in the framework of metric ANRs. In this more general setting, other uniqueness results based on a stronger version of the normalization property are available for the class of compact maps (see, e.g., [6, Section 16, Theorem 5.1]).

Our goal here is to prove that in the framework of finite-dimensional manifolds the fixed point index is uniquely determined by three properties, namely, the Amann-Weiss-type properties of normalization, additivity, and homotopy invariance as enounced in Section 2. For this reason, these properties will be collectively referred to as the *fixed point index axioms (for manifolds)*.

The fact that in $\mathbb{R}^m$ any equation of the type $f(x) = x$ can be written as $f(x) - x = 0$ shows that in this context the theories of fixed point index and of topological degree are equivalent. Therefore, in this flat case, the uniqueness of the index could be deduced from the Amann-Weiss axioms of the topological degree given in [2]. Here we provide

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a simple proof of the uniqueness in $\mathbb{R}^m$ and we extend this result to the context of finite-dimensional manifolds.

Some technical lemmas are well known or belong to the folklore. Their proof is given for the sake of completeness.

2. Preliminaries

Given two sets X and Y, by a *local map* with *source* X and *target* Y we mean a triple $g = (X, Y, \Gamma)$, where Γ, the *graph* of g, is a subset of $X \times Y$ such that for any $x \in X$ there exists at most one $y \in Y$ with $(x, y) \in \Gamma$. The domain $\mathfrak{D}(g)$ of g is the set of all $x \in X$ for which there exists $y = g(x) \in Y$ such that $(x, y) \in \Gamma$; namely, $\mathfrak{D}(g) = \pi_1(\Gamma)$, where π_1 denotes the projection of $X \times Y$ onto the first factor. The *restriction* of a local map $g = (X, Y, \Gamma)$ to a subset C of X is the triple

$$g|_C = \big(C, Y, \Gamma \cap (C \times Y)\big). \tag{2.1}$$

Incidentally, we point out that sets and local maps (with the obvious composition) constitute a category.

Whenever it makes sense (e.g., when source and target spaces are manifolds), local maps are tacitly assumed to be continuous.

Throughout the paper M denotes a finite-dimensional, smooth, real, Hausdorff, second countable manifold. Given any $x \in M$, I_x denotes the identity on the tangent space T_xM of M at x.

By a *local map in M* we mean a local map having M both as source and target space. A local map in M is said to be smooth on a subset C of M if $C \subseteq \mathfrak{D}(f)$ and the restriction $f|_C$ admits a smooth extension to an open subset of M containing C.

Given an open subset U of M and a local map f in M, the pair (f, U) is said to be *admissible (in M)* if $U \subseteq \mathfrak{D}(f)$ and the set

$$\operatorname{Fix}(f, U) := \{x \in U : f(x) = x\} \tag{2.2}$$

of the fixed points of f in U is compact. In particular, (f, U) is admissible if the closure $\overline{U}$ of U is a compact subset of $\mathfrak{D}(f)$ and f is fixed-point-free on the boundary ∂U of U.

Given an open subset U of M and a (continuous) local map H with source $M \times [0,1]$ and target M, we say that H is an *admissible homotopy in U* if $U \times [0,1] \subseteq \mathfrak{D}(H)$ and the set

$$\{(x, \lambda) \in U \times [0,1] : H(x, \lambda) = x\} \tag{2.3}$$

is compact. Thus, if $\overline{U}$ is compact and $\overline{U} \times [0,1] \subseteq \mathfrak{D}(H)$, a sufficient condition for H to be admissible in U is the following:

$$H(x, \lambda) \neq x, \quad \forall (x, \lambda) \in \partial U \times [0,1], \tag{2.4}$$

which, by abuse of terminology, will be referred to as "H is fixed-point-free on ∂U".

We will show that there exists at most one function that to any admissible pair (f,U) assigns an integer $\operatorname{ind}(f,U)$, called *fixed point index of f in U* or *index of the pair (f,U)*, that satisfies the following three axioms.

Normalization. Let $f: M \to M$ be constant. Then $\operatorname{ind}(f,M) = 1$.

Additivity. Given an admissible pair (f,U), if U_1 and U_2 are two disjoint open subsets of U such that $\operatorname{Fix}(f,U) \subseteq U_1 \cup U_2$, then

$$\operatorname{ind}(f,U) = \operatorname{ind}(f|_{U_1},U_1) + \operatorname{ind}(f|_{U_2},U_2). \tag{2.5}$$

Homotopy invariance. If H is an admissible homotopy in U, then

$$\operatorname{ind}(H(\cdot,0),U) = \operatorname{ind}(H(\cdot,1),U). \tag{2.6}$$

Remark 2.1. The pair $(f,\emptyset)$ is admissible. This includes the case when $\mathfrak{D}(f)$ is the empty set ($\mathfrak{D}(f) = \emptyset$ is coherent with the notion of local map). A simple application of the additivity property shows that $\operatorname{ind}(f|_\emptyset,\emptyset) = 0$ and $\operatorname{ind}(f,\emptyset) = 0$.

As a consequence of the additivity property and Remark 2.1, one easily gets the following (often neglected) property, which shows that the index of an admissible pair (f,U) does not depend on the behavior of f outside U.

Localization. If (f,U) is admissible, then $\operatorname{ind}(f,U) = \operatorname{ind}(f|_U,U)$.

Let (f,U) be admissible and let $U_1 \subseteq U$ be open and such that $\operatorname{Fix}(f,U) \subseteq U_1$. Then, by the additivity property, Remark 2.1, and localization, one gets

$$\operatorname{ind}(f,U) = \operatorname{ind}(f|_{U_1},U_1) + \operatorname{ind}(f|_\emptyset,\emptyset) = \operatorname{ind}(f,U_1). \tag{2.7}$$

Thus, we have the following important property of the fixed point index.

Excision. Given an admissible pair (f,U) and an open subset U_1 of U containing $\operatorname{Fix}(f,U)$, one has $\operatorname{ind}(f,U) = \operatorname{ind}(f,U_1)$.

From the excision, if $\operatorname{Fix}(f,U) = \emptyset$, taking $U_1 = \emptyset$, we get

$$\operatorname{ind}(f,U) = \operatorname{ind}(f,\emptyset) = 0, \tag{2.8}$$

and this implies the following property.

Solution. If $\operatorname{ind}(f,U) \neq 0$, then the fixed point equation $f(x) = x$ has a solution in U.

3. The fixed point index for linear maps

In this section, we will prove that, as a consequence of the properties of normalization, additivity and homotopy invariance, the index of an admissible pair $(A,\mathbb{R}^m)$, where A is a linear operator in $\mathbb{R}^m$, is either 1 or -1.

The Euclidean norm of a vector $v \in \mathbb{R}^m$ will be denoted by $|v|$. By $\mathrm{L}(\mathbb{R}^m)$ we will mean the normed space of linear endomorphisms of $\mathbb{R}^m$, and by $\mathrm{GL}(\mathbb{R}^m)$ we will distinguish the group of invertible ones. The identity on $\mathbb{R}^m$ is represented by the symbol I. An operator $A \in \mathrm{L}(\mathbb{R}^m)$ will be called *nondegenerate* if $I - A$ is invertible, and $\mathrm{N}(\mathbb{R}^m)$ will stand for the open subset of $\mathrm{L}(\mathbb{R}^m)$ of the nondegenerate operators. Observe that $A \in \mathrm{N}(\mathbb{R}^m)$ if and only if $\mathrm{Fix}(A,\mathbb{R}^m) = \{0\}$. Thus $(A,\mathbb{R}^m)$ is an admissible pair if and only if $A \in \mathrm{N}(\mathbb{R}^m)$.

It is well known (see, e.g., [1]) that the open subset $\mathrm{GL}(\mathbb{R}^m)$ of $\mathrm{L}(\mathbb{R}^m)$ has exactly two connected components:

$$\begin{aligned} \mathrm{GL}^+\left(\mathbb{R}^m\right) &= \left\{L \in \mathrm{GL}\left(\mathbb{R}^m\right) : \det(L) > 0\right\}, \\ \mathrm{GL}^-\left(\mathbb{R}^m\right) &= \left\{L \in \mathrm{GL}\left(\mathbb{R}^m\right) : \det(L) < 0\right\}. \end{aligned} \tag{3.1}$$

Therefore, $\mathrm{N}(\mathbb{R}^m)$ has two connected components, $\mathrm{N}^+(\mathbb{R}^m)$ and $\mathrm{N}^-(\mathbb{R}^m)$, consisting, respectively, of those $A \in \mathrm{GL}(\mathbb{R}^m)$ for which $\det(I-A) > 0$ and $\det(I-A) < 0$.

Since $\mathrm{N}^+(\mathbb{R}^m)$ and $\mathrm{N}^-(\mathbb{R}^m)$ are open in $\mathrm{L}(\mathbb{R}^m)$ and connected, they are actually path connected. Consequently, given $A \in \mathrm{N}(\mathbb{R}^m)$, the homotopy invariance implies that $\mathrm{ind}(A,\mathbb{R}^m)$ depends only on the component of $\mathrm{N}(\mathbb{R}^m)$ containing A. Therefore, given $A \in \mathrm{N}^+(\mathbb{R}^m)$, one has $\mathrm{ind}(A,\mathbb{R}^m) = \mathrm{ind}(\mathbf{0},\mathbb{R}^m)$, where $\mathbf{0}$ is the trivial operator. Thus, by normalization, we get

$$\mathrm{ind}\left(A,\mathbb{R}^m\right) = 1, \quad \forall A \in \mathrm{N}^+\left(\mathbb{R}^m\right). \tag{3.2}$$

We will prove that $\mathrm{ind}(A,\mathbb{R}^m) = -1$ for any $A \in \mathrm{N}^-(\mathbb{R}^m)$. As a distinguished representative in $\mathrm{N}^-(\mathbb{R}^m)$, we choose the linear operator $\hat{A}$ given by

$$\left(x_1,\ldots,x_{m-1},x_m\right) \longmapsto \left(0,\ldots,0,2x_m\right). \tag{3.3}$$

Lemma 3.1. *Let $\hat{A}$ be the above operator. Then* $\mathrm{ind}(\hat{A},\mathbb{R}^m) = -1$.

Proof. Consider the homotopy $H : \mathbb{R}^m \times [0,1] \to \mathbb{R}^m$ given by

$$\left(x_1,\ldots,x_m;\lambda\right) \longmapsto \left(0,\ldots,0,\left|x_m\right| + x_m + 2\lambda - 1\right). \tag{3.4}$$

Clearly, H is admissible and $\mathrm{Fix}\left(H(\cdot,1),\mathbb{R}^m\right) = \emptyset$. Thus, the solution and homotopy invariance properties imply

$$0 = \mathrm{ind}\left(H(\cdot,1),\mathbb{R}^m\right) = \mathrm{ind}\left(H(\cdot,0),\mathbb{R}^m\right). \tag{3.5}$$

Since

$$\mathrm{Fix}\left(H(\cdot,0),\mathbb{R}^m\right) = \left\{(0,\ldots,+1),(0,\ldots,-1)\right\}, \tag{3.6}$$

by additivity we get

$$0 = \mathrm{ind}\left(H(\cdot,0),\mathbb{R}^m\right) = \mathrm{ind}\left(H(\cdot,0),\mathbb{H}^m_+\right) + \mathrm{ind}\left(H(\cdot,0),\mathbb{H}^m_-\right), \tag{3.7}$$

where $\mathbb{H}^m_+$ and $\mathbb{H}^m_-$ denote the open half-spaces of $\mathbb{R}^m$ with positive and negative last coordinate. Since the restriction of $H(\cdot,0)$ to $\mathbb{H}^m_-$ is constantly equal to $(0,\ldots,0,-1)$, by normalization we get

$$\mathrm{ind}\left(H(\cdot,0),\mathbb{H}^m_-\right) = 1. \tag{3.8}$$

Hence, by (3.7),

$$\operatorname{ind}\left(H(\cdot,0),\mathbb{H}_+^m\right) = -1. \tag{3.9}$$

Notice that in $\mathbb{H}_+^m$ the map $H(\cdot,0)$ coincides with the affine operator

$$\Phi(x_1,\dots,x_{m-1};x_m) = (0,\dots,0,2x_m-1). \tag{3.10}$$

Thus, by localization and excision,

$$\operatorname{ind}\left(H(\cdot,0),\mathbb{H}_+^m\right) = \operatorname{ind}\left(\Phi,\mathbb{H}_+^m\right) = \operatorname{ind}\left(\Phi,\mathbb{R}^m\right). \tag{3.11}$$

Therefore, it is enough to show that $\operatorname{ind}(\hat{A},\mathbb{R}^m) = \operatorname{ind}(\Phi,\mathbb{R}^m)$, and this is true since the homotopy

$$(x_1,\dots,x_m,\lambda) \longmapsto (0,\dots,0,2x_m-\lambda) \tag{3.12}$$

is admissible. □

From the previous discussion and Lemma 3.1 one gets

$$\operatorname{ind}\left(A,\mathbb{R}^m\right) = -1, \quad \forall A \in \mathrm{N}^-\left(\mathbb{R}^m\right). \tag{3.13}$$

Formulas (3.2) and (3.13) can be summarized as follows.

Lemma 3.2. *If* $A \in \mathrm{N}(\mathbb{R}^m)$, *then* $\operatorname{ind}(A,\mathbb{R}^m) = \operatorname{sign}\det(I-A)$.

We conclude the section with a technical result regarding linearizable maps.

Lemma 3.3. *Let* $f: U \to \mathbb{R}^m$ *be a continuous map on an open subset of* $\mathbb{R}^m$. *Given* $p \in \operatorname{Fix}(f,U)$, *assume that* f *is differentiable at* p *with nondegenerate Fréchet derivative* $f'(p)$. *Then* p *is an isolated fixed point, and for any isolating neighborhood* $V \subseteq U$ *of* p,

$$\operatorname{ind}(f,V) = \operatorname{ind}\left(f'(p),\mathbb{R}^m\right). \tag{3.14}$$

Proof. By definition of differentiability we get

$$f(x) = p + f'(p)(x-p) + |x-p|\varepsilon(x-p), \quad x \in U, \tag{3.15}$$

where $\varepsilon: U-p \to \mathbb{R}^m$ is a continuous map with $\varepsilon(0)=0$. Thus

$$\begin{aligned} \left|x-f(x)\right| &\geq \left|\left(I-f'(p)\right)(x-p)\right| - |x-p|\left|\varepsilon(x-p)\right| \\ &\geq |x-p|\left(\inf_{|v|=1}\left|\left(I-f'(p)\right)v\right| - \left|\varepsilon(x-p)\right|\right). \end{aligned} \tag{3.16}$$

Since $f'(p)$ is nondegenerate, $\inf_{|v|=1}|(I-f'(p))v| > 0$, and this implies that p is an isolated fixed point of f.

Let $V \subseteq U$ be any neighborhood of p such that $\operatorname{Fix}(f,V) = \{p\}$, and consider the homotopy

$$H(x,\lambda) = p + f'(p)(x-p) + \lambda|x-p|\varepsilon(x-p). \tag{3.17}$$

The above argument shows that in some neighborhood $W \subseteq V$ of p one has

$$|x - H(x,\lambda)| > 0 \tag{3.18}$$

for any $x \in W \setminus \{p\}$ and $\lambda \in [0,1]$. Hence H is an admissible homotopy in W. By the homotopy and the excision properties, we get

$$\operatorname{ind}(f,W) = \operatorname{ind}\left(H(\cdot,0),W\right) = \operatorname{ind}\left(H(\cdot,0),\mathbb{R}^m\right). \tag{3.19}$$

Consequently, by excision,

$$\operatorname{ind}(f,V) = \operatorname{ind}(f,W) = \operatorname{ind}\left(H(\cdot,0),\mathbb{R}^m\right). \tag{3.20}$$

Since the affine map $H(x,0) = p + f'(p)(x-p)$ is admissibly homotopic in $\mathbb{R}^m$ to its linear part $x \mapsto f'(p)x$, the homotopy invariance property yields

$$\operatorname{ind}\left(H(\cdot,0),\mathbb{R}^m\right) = \operatorname{ind}\left(f'(p),\mathbb{R}^m\right). \tag{3.21}$$

The assertion follows from (3.20) and (3.21). □

4. The uniqueness result

Given a local map f in M and a relatively compact open subset U of M, the pair (f,U) will be called *nondegenerate* if f is smooth on $\overline{U}$, fixed-point-free on ∂U, and the Fréchet derivative of f at any fixed point in U is nondegenerate (as in the case of $\mathbb{R}^m$, an endomorphism of a vector space is nondegenerate if 1 is not an eigenvalue). Note that, in this case, $\operatorname{Fix}(f,U)$ is necessarily a discrete set, therefore finite, being closed in the compact set $\overline{U}$. In particular (f,U) is an admissible pair.

The following lemma shows that the computation of the fixed point index of any admissible pair can be reduced to that of a nondegenerate pair.

Lemma 4.1. *Let (f,U) be admissible and let V be a relatively compact open subset of M containing* $\operatorname{Fix}(f,U)$ *and such that $\overline{V} \subseteq U$. Then, there exists a local map g in M which is admissibly homotopic to f in V and such that (g,V) is a nondegenerate pair.*

Proof. Without loss of generality, we may assume that M is embedded in some $\mathbb{R}^k$. Thus, because of the ε-Neighborhood, Theorem (see, e.g., [7]) there exist an open neighborhood Ω of M in $\mathbb{R}^k$ and a smooth submersion $r : \Omega \to M$ such that $|x - r(x)| = \operatorname{dist}(x,M)$ for all x in Ω. In particular, M is a retract of Ω. Since $\overline{V}$ is compact, given $\delta > 0$, the Weierstrass approximation theorem implies the existence of a polynomial map $f^\delta : \mathbb{R}^k \to \mathbb{R}^k$ such that $|f(x) - f^\delta(x)| < \delta$ for all $x \in \overline{V}$. Again, by the compactness of $\overline{V}$, we may assume that δ is such that the homotopy

$$F^\delta(x,\lambda) := r\left((1-\lambda)f(x) + \lambda f^\delta(x)\right) \tag{4.1}$$

is well defined on $\overline{V} \times [0,1]$ and fixed-point-free on ∂V (where ∂V is the boundary of V relative to $M \subseteq \mathbb{R}^k$). Consequently, f is admissibly homotopic in V to the smooth map $h := F^\delta(\cdot,1)$.

It is enough to prove that h is admissibly homotopic in V to some local map g such that (g,V) is a nondegenerate pair. Observe first that an admissible pair (g,V), with g smooth on $\overline{V}$ and fixed-point-free on ∂V, is nondegenerate if and only if the graph map $x \mapsto (x,g(x))$ is transversal in V to the diagonal Δ of $M \times M$. We apply the transversality theorem (see, e.g., [7]) to the map

$$G(x,y) = \big(x, r\big(h(x)+y\big)\big), \tag{4.2}$$

defined on $\overline{V} \times B$, where B is an open ball about the origin so small that $h(x)+y \in \Omega$ for all $(x,y) \in \overline{V} \times B$ and the maps $x \mapsto r(h(x)+y)$ are all fixed-point-free on ∂V. This is possible since $\overline{V}$ is compact and $h(x) \neq x$ for all $x \in \partial V$.

Since r is a submersion, given any $(x,y) \in G^{-1}(\Delta)$, the derivative

$$G'(x,y) : T_xM \times \mathbb{R}^k \longrightarrow T_xM \times T_xM \tag{4.3}$$

is surjective, and this implies that G is transversal to Δ in $V \times B$. Consequently, the transversality theorem ensures the existence of a point $\bar{y} \in B$ such that the partial map

$$G(\cdot,\bar{y}) : x \longmapsto \big(x, r\big(h(x)+\bar{y}\big)\big) \tag{4.4}$$

is transversal to Δ in V. This, as pointed out before, means that any fixed point in V of the smooth map $g(x) := r(h(x)+\bar{y})$ is nondegenerate. The conclusion follows by observing that the assumption on B ensures that the homotopy $H : \overline{V} \times [0,1] \to M$ given by $H(x,\lambda) = r(h(x)+\lambda\bar{y})$ is fixed-point-free on ∂V, therefore admissible because of the compactness of $\overline{V}$. □

We will show that the properties of normalization, additivity, and homotopy invariance imply a formula for the computation of the fixed point index that is valid for any nondegenerate pair. Therefore, Lemma 4.1, the excision, and the homotopy invariance properties imply the existence of at most one real function on the set of admissible pairs that satisfies the fixed point index axioms. Moreover, since the function defined by this formula is integer valued, so is the fixed point index.

Theorem 4.2 (uniqueness of the fixed point index). *Let* ind *be a real function on the set of admissible pairs satisfying the properties of normalization, additivity, and homotopy invariance of the fixed point index. If (f,U) is a nondegenerate pair, then*

$$\operatorname{ind}(f,U) = \sum_{x \in \operatorname{Fix}(f,U)} \operatorname{sign}\big(\det\big(I_x - f'(x)\big)\big). \tag{4.5}$$

Consequently, there exists at most one function on the set of admissible pairs satisfying the fixed point index axioms, and this function is integer valued.

Proof. Consider first the case $M = \mathbb{R}^m$. Let (f,U) be a nondegenerate pair in $\mathbb{R}^m$ and, for any $x \in \operatorname{Fix}(f,U)$, let V_x be an isolating neighborhood of x. Since $\operatorname{Fix}(f,U)$ is finite, we may assume that the neighborhoods V_x's are pairwise disjoint. The additivity property, Lemmas 3.3 and 3.2 yield

$$\begin{aligned}\operatorname{ind}(f,U) &= \sum_{x\in \operatorname{Fix}(f,U)} \operatorname{ind}(f,V_x) = \sum_{x\in \operatorname{Fix}(f,U)} \operatorname{ind}(f'(x),\mathbb{R}^m)\\ &= \sum_{x\in \operatorname{Fix}(f,U)} \operatorname{sign}(\det(I-f'(x))).\end{aligned} \tag{4.6}$$

Now the uniqueness of the fixed point index on $\mathbb{R}^m$ follows immediately from Lemma 4.1, taking into account the properties of excision and homotopy invariance.

We now consider the general case and denote by m the dimension of M. Let W be an open subset of M which is diffeomorphic to the whole space $\mathbb{R}^m$ and let $\psi : W \to \mathbb{R}^m$ be any diffeomorphism onto $\mathbb{R}^m$. Denote by $\mathcal{U}$ the set of all pairs (f,U) which are admissible and such that $U \subseteq W$, $f(U) \subseteq W$. These pairs may be regarded as admissible in W, and the restriction of the index function to $\mathcal{U}$ still satisfies the fixed point index axioms. We claim that for any $(f,U)\in\mathcal{U}$ one necessarily has

$$\operatorname{ind}(f,U) = \mathrm{i}(\psi\circ f\circ\psi^{-1},\psi(U)), \tag{4.7}$$

where (for the moment) i denotes the (unique) fixed point index on $\mathbb{R}^m$. To show this, denote by $\mathcal{V}$ the set of pairs (g,V) which are admissible in $\mathbb{R}^m$ and consider the one-to-one correspondence $\omega : \mathcal{U} \to \mathcal{V}$ defined by

$$\omega(f,U) = (\psi\circ f\circ\psi^{-1},\psi(U)). \tag{4.8}$$

We need to prove that $\operatorname{ind} = \mathrm{i}\circ\omega$. Observe that

$$\omega^{-1}(g,V) = (\psi^{-1}\circ g\circ\psi,\psi^{-1}(V)), \tag{4.9}$$

and if two pairs $(f,U)\in\mathcal{U}$ and $(g,V)\in\mathcal{V}$ correspond under ω, then the sets $\operatorname{Fix}(f,U)$ and $\operatorname{Fix}(g,V)$ correspond under ψ. It is also evident that the function $\operatorname{ind}\circ\omega^{-1}$ satisfies the fixed point index axioms. Thus, i and $\operatorname{ind}\circ\omega^{-1}$ coincide on $\mathcal{V}$, and this implies $\operatorname{ind} = \mathrm{i}\circ\omega$, as claimed.

Let now (f,U) be a given nondegenerate pair in M. Let $\operatorname{Fix}(f,U) = \{x_1,\dots,x_n\}$ and let $W_1,\dots,W_n$ be n pairwise disjoint open subsets of U such that $x_j \in W_j$, for $j = 1,\dots,n$. Since any point of M has a fundamental system of neighborhoods which are diffeomorphic to the whole space $\mathbb{R}^m$, we may assume that each W_j is diffeomorphic to $\mathbb{R}^m$ under a diffeomorphism ψ_j. For any j, let U_j be an open subset of W_j such that $f(U_j)\subseteq W_j$. The additivity property yields

$$\operatorname{ind}(f,U) = \sum_{j=1}^{n}\operatorname{ind}(f,U_j), \tag{4.10}$$

and, by the above claim, we get

$$\sum_{j=1}^{n}\operatorname{ind}(f,U_j) = \sum_{j=1}^{n}\mathrm{i}(\psi_j\circ f\circ\psi_j^{-1},\psi_j(U_j)). \tag{4.11}$$

By the excision property, Lemma 3.2, and the chain rule for the derivative one has

$$\mathrm{i}\,(\psi_j \circ f \circ \psi_j^{-1}, \psi_j(U_j)) = \mathrm{i}\,(\psi_j \circ f \circ \psi_j^{-1}, \mathbb{R}^m) = \operatorname{sign}(\det(I_{x_j} - f'(x_j))), \tag{4.12}$$

for $j = 1, \dots, n$. Thus

$$\operatorname{ind}(f, U) = \sum_{j=1}^{n} \operatorname{sign}(\det(I_{x_j} - f'(x_j))). \tag{4.13}$$

As in the case when $M = \mathbb{R}^m$, the uniqueness of the fixed point index is now a consequence of Lemma 4.1. □

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