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Oscillation of Certain Partial Difference Equations

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We first obtain sufficiency conditions for the oscillation of all solutions of linear partial difference equation

$$aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - dA_{m,n} + P_{m,n}A_{m-k,n-l} = 0.$$

Next, we establish a linearized oscillation result for the nonlinear partial difference equation

$$A_{m+1,n+1} + A_{m+1,n} + A_{m,n+1} - A_{m,n} + P_{m,n}f(A_{m-k,n-l}) = 0.$$

Keywords: Linear and nonlinear partial difference equations, Oscillation

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1 INTRODUCTION

Partial difference equations have been posed from various practical problems [3,8] and in the approximation of solutions of partial difference equations by finite difference methods [1,2,8]. Recently, the qualitative analysis of partial difference equations has received much attention, see [4–7,9,10,12].

In this paper, we first consider the linear partial difference equation

$$aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - dA_{m,n} + P_{m,n}A_{m-k,n-l} = 0, \qquad (1.1)$$

where $P_{m,n} > 0$ on N_0^2, k, $l \in N_0$, $N_i = \{i, i+1, \ldots\}$ and i is an integer. Equation (1.1) can be regarded

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as a discrete analogue of the delay partial difference equation

$$a\frac{\partial^2 u}{\partial y\partial x}+a_1\frac{\partial u}{\partial x}+a_2\frac{\partial u}{\partial y} + P(x,y)u(x-\sigma,y-\tau)=0. \qquad (1.2)$$

Next, we consider the nonlinear partial difference equation

$$A_{m+1,n+1}+A_{m+1,n}+A_{m,n+1} - A_{m,n}+P_{m,n}f(A_{m-k,n-l})=0, \qquad (1.3)$$

where $f\in C(\mathbb{R},\mathbb{R})$.

The general theory of partial functional differential equations can be seen from Wu [11].

A double sequence $\{A_{m,n}\}$ is said to be a *solution* of (1.1) if it satisfies (1.1) for $m\geq m_0$, $n\geq n_0$. A solution $\{A_{i,j}\}$ of (1.1) is said to be *eventually positive* if $A_{i,j}>0$ for all large i and j, and *eventually negative* if $A_{i,j}<0$ for all large i and j. It is said to be *oscillatory* if it is neither eventually positive nor eventually negative.

In Section 2, we shall obtain sufficiency conditions for all solutions of (1.1) to be oscillatory. In Section 3, we shall show a linearized oscillation theorem for (1.3), i.e., we shall show that under some assumptions, (1.3) has the same oscillatory character as an associated linear equation.

2 EQUATION (1.1)

We assume that a,b,c,d and $P_{m,n}$ are positive in Eq. (1.1). Define a set E by

$$E=\{\lambda>0\,|\,d-\lambda P_{m,n}>0,\ \text{eventually}\}. \qquad (2.1)$$

THEOREM 2.1 *Assume that*

(i) $\lim_{m,n\to\infty}\sup P_{m,n}>0$;
(ii) *for* $k\geq l\geq 1$, *there exist* $M,N\in N_1$ *such that*

$$\sup_{\lambda\in E,m\geq M,n\geq N}\lambda\prod_{i=1}^{l}(d-\lambda P_{m-i,n-i}) \times\prod_{j=1}^{k-l}(d-\lambda P_{m-l-j,n-l})<\left(a+\frac{2bc}{d}\right)^{l}b^{k-l} \qquad (2.2)$$

and for $l\geq k$

$$\sup_{\lambda\in E,m\geq M,n\geq N}\lambda\prod_{i=1}^{k}(d-\lambda P_{m-i,n-i}) \times\prod_{j=1}^{l-k}(d-\lambda P_{m-k,n-k-j})<\left(a+\frac{2bc}{d}\right)^{k}c^{l-k}. \qquad (2.3)$$

Then every solution of (1.1) *oscillates.*

Proof Suppose, to the contrary, we let $\{A_{m,n}\}$ be an eventually positive solution. We define a subset S of the positive numbers as follows:

$$S(A)=\{\lambda>0\,|\,aA_{m+1,n+1}+bA_{m+1,n} + cA_{m,n+1}(d-\lambda P_{m,n})A_{m,n}\leq 0,\ \text{eventually}\}.$$

From (1.1), we have

$$aA_{m+1,n+1}+bA_{m+1,n}+cA_{m,n+1}<dA_{m,n}. \qquad (2.4)$$

If $k\geq l$, then

$$A_{m-k,n-l}>\left(\frac{a}{d}\right)^{l}A_{m-k+l,n}>\left(\frac{a}{d}\right)^{l}\left(\frac{b}{d}\right)^{k-l}A_{m,n}. \qquad (2.5)$$

If $l\geq k$, then

$$A_{m-k,n-l}>\left(\frac{a}{d}\right)^{k}A_{m,n-l+k}>\left(\frac{a}{d}\right)^{k}\left(\frac{c}{d}\right)^{l-k}A_{m,n}. \qquad (2.6)$$

Substituting (2.5) and (2.6) into (1.1) we obtain

$$aA_{m+1,n+1}+bA_{m+1,n}+cA_{m,n+1} - dA_{m,n}+\left(\frac{a}{d}\right)^{l}\left(\frac{b}{d}\right)^{k-l}P_{m,n}A_{m,n}<0 \qquad (2.7)$$

and

$$aA_{m+1,n+1}+bA_{m+1,n}+cA_{m,n+1} - dA_{m,n}+\left(\frac{a}{d}\right)^{k}\left(\frac{c}{d}\right)^{l-k}P_{m,n}A_{m,n}<0, \qquad (2.8)$$

respectively. Inequalities (2.7) and (2.8) show that $S(A)$ is nonempty. For $\lambda \in S$, we have eventually

$$d - \lambda P_{m,n} > 0,$$

which implies that $S \subset E$. Due to the condition (i), the set E is bounded, and hence $S(A)$ is bounded. Let $\mu \in S$. Then, from (2.4), we have

$$A_{m+1,n+1} \le \frac{d}{b} A_{m,n+1} \quad \text{and} \quad A_{m+1,n+1} \le \frac{d}{c} A_{m+1,n}.$$

Hence we obtain

$$\begin{aligned}\left(a + \frac{2bc}{d}\right) A_{m+1,n+1} &\le aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} \\ &\le (d - \mu P_{m,n}) A_{m,n}.\end{aligned}$$

If $k \ge l$, then

$$A_{m,n} \le \left(a + \frac{2bc}{d}\right)^{-l} \prod_{i=1}^{l} (d - \mu P_{m-i,n-i}) A_{m-l,n-l}$$

and

$$\begin{aligned}A_{m-l,n-l} &\le \frac{1}{b}(d - \mu P_{m-l-1,n-l}) A_{m-l-1,n-l} \\ &\le \cdots \le \left(\frac{1}{b}\right)^{k-l} \prod_{j=1}^{k-l} (d - \mu P_{m-l-j,n-l}) A_{m-k,n-l}.\end{aligned}$$

Hence

$$\begin{aligned}A_{m,n} \le &\left(a + \frac{2bc}{d}\right)^{-l} b^{l-k} \prod_{i=1}^{l} (d - \mu P_{m-i,n-i}) \\ &\times \prod_{j=1}^{k-l} (d - \mu P_{m-l-j,n-l}) A_{m-k,n-l}.\end{aligned} \tag{2.9}$$

Similarly, if $l \ge k$, then

$$\begin{aligned}A_{m,n} \le &\left(a + \frac{2bc}{d}\right)^{-k} c^{k-l} \prod_{i=1}^{k} (d - \mu P_{m-i,n-i}) \\ &\times \prod_{j=1}^{l-k} (d - \mu P_{m-k,n-k-j}) A_{m-k,n-l}.\end{aligned} \tag{2.10}$$

Substituting (2.9) and (2.10) into (1.3) we find respectively

$$\begin{aligned}&aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - dA_{m,n} \\ &\quad + P_{m,n}\left(a + \frac{2bc}{d}\right)^{k} c^{l-k} \left(\prod_{i=1}^{k} (d - \mu P_{m-i,n-i}) \right. \\ &\quad \left. \times \prod_{j=1}^{l-k} (d - \mu P_{m-k,n-k-j})\right)^{-1} A_{m,n} \le 0, \quad \text{for } l \ge k\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}&aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - dA_{m,n} \\ &\quad + P_{m,n}\left(a + \frac{2bc}{d}\right)^{l} b^{k-l} \left(\prod_{i=1}^{l} (d - \mu P_{m-i,n-i}) \right. \\ &\quad \left. \times \prod_{j=1}^{k-l} (d - \mu P_{m-l-j,n-l})\right)^{-1} A_{m,n} \le 0, \quad \text{for } k \ge l.\end{aligned} \tag{2.12}$$

Hence

$$\begin{aligned}&aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} \\ &\quad - \left(d - P_{m,n}\left(a + \frac{2bc}{d}\right)^{k} c^{l-k} \right. \\ &\quad \times \sup_{m \ge M, n \ge N} \left[\left(\prod_{i=1}^{k} (d - \mu P_{m-i,n-i}) \right.\right. \\ &\quad \left.\left.\left. \times \prod_{j=1}^{l-k} (d - \mu P_{m-k,n-k-j})\right)^{-1}\right]\right) A_{m,n} \le 0, \quad l \ge k\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}&aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} \\ &\quad - \left(d - P_{m,n}\left(a + \frac{2bc}{d}\right)^{l} b^{k-l} \right. \\ &\quad \times \sup_{m \ge M, n \ge N} \left[\left(\prod_{i=1}^{l} (d - \mu P_{m-i,n-i}) \right.\right. \\ &\quad \left.\left.\left. \times \prod_{j=1}^{k-l} (d - \mu P_{m-l-j,n-l})\right)^{-1}\right]\right) A_{m,n} \le 0, \quad k \ge l.\end{aligned} \tag{2.14}$$

From (2.13) and (2.14) we obtain

$$\left(a+\frac{2bc}{d}\right)^k c^{l-k}\left(\sup_{m\geq M,n\geq N}\left[\left(\prod_{i=1}^{k}(d-\mu P_{m-i,n-i})\times\prod_{j=1}^{l-k}(d-\mu P_{m-k,n-k-j})\right)^{-1}\right]\right)\in S,\quad l>k \tag{2.15}$$

and

$$\left(a+\frac{2bc}{d}\right)^l b^{k-l}\left(\sup_{m\geq M,n\geq N}\left[\left(\prod_{i=1}^{l}(d-\mu P_{m-i,n-i})\times\prod_{j=1}^{k-l}(d-\mu P_{m-l-j,n-l})\right)^{-1}\right]\right)\in S,\quad k>l. \tag{2.16}$$

On the other hand, (2.2) implies that there exits $\beta\in(0,1)$ such that

$$\sup_{\lambda\in E,m\geq M,n\geq N}\lambda\prod_{i=1}^{l}(d-\lambda P_{m-i,n-i})\prod_{j=1}^{k-l}(d-\lambda P_{m-l-j,n-l})\leq\beta\left(a+\frac{2bc}{d}\right)^l b^{k-l},\quad k\geq l$$

and (2.3) implies that there exists $\beta\in(0,1)$ such that

$$\sup_{\lambda\in E,m\geq M,n\geq N}\lambda\prod_{i=1}^{k}(d-\lambda P_{m-i,n-i})\prod_{j=1}^{l-k}(d-\lambda P_{m-k,n-k-j})\leq\beta\left(a+\frac{2bc}{d}\right)^k c^{l-k},\quad l\geq k.$$

Hence, for $k\geq l$,

$$\sup_{m\geq M,n\geq N}\left(\prod_{i=1}^{l}(d-\mu P_{m-i,n-i})\prod_{j=1}^{k-l}(d-\mu P_{m-l-j,n-l})\right)\leq\frac{\beta}{\mu}\left(a+\frac{2bc}{d}\right)^l b^{k-l} \tag{2.17}$$

and for $l\geq k$

$$\sup_{m\geq M,n\geq N}\prod_{i=1}^{k}(d-\mu P_{m-i,n-i})\prod_{j=1}^{l-k}(d-\mu P_{m-k,n-k-j})\leq\frac{\beta}{\mu}\left(a+\frac{2bc}{d}\right)^k c^{l-k}. \tag{2.18}$$

From (2.15) and (2.18) for $l\geq k$, (2.16) and (2.17) for $k\geq l$, we have that $\mu/\beta\in S$. Repeating the above procedure, we conclude that $\mu(1/\beta)^r\in S$, $r=1,2,\ldots$, which contradicts the boundedness of S. The proof is complete.

From Theorem 2.1, we can derive an explicit oscillation condition.

Corollary 2.1 *In addition to* (i) *of Theorem* 2.1, *assume that, for* $k\geq l$,

$$\lim_{m,n\to\infty}\inf P_{m,n}=P>d^{k+1}\left(\left(a+\frac{2bc}{d}\right)^l b^{k-l}\right)^{-1}\times\frac{k^k}{(1+k)^{1+k}} \tag{2.19}$$

and for $l\geq k$,

$$\lim_{m,n\to\infty}\inf P_{m,n}=P>d^{l+1}\left(\left(a+\frac{2bc}{d}\right)^k b^{l-k}\right)^{-1}\times\frac{l^l}{(1+l)^{1+l}}. \tag{2.20}$$

Then the conclusion of Theorem 2.1 *holds.*

Proof We see that

$$\max_{d/P>\lambda>0}\lambda(d-\lambda P)^k=\frac{d^{k+1}k^k}{P(1+k)^{1+k}}.$$

Hence (2.19) and (2.20) imply that (2.2) and (2.3) hold. By Theorem 2.1, every solution of (1.1) oscillates. The proof is complete.

Remark Obviously, Theorem 2.1 is true for $a=0$. Hence Theorem 2.1 includes Theorem 2.3 in

[9] as a special case. From (2.4) we have

$$\begin{aligned}A_{m,n} &< \frac{d}{b}A_{m-1,n} < \cdots < \left(\frac{d}{b}\right)^k A_{m-k,n}\\ &< \left(\frac{d}{b}\right)^k\left(\frac{d}{c}\right)A_{m-k,n-1}\\ &< \cdots < \left(\frac{d}{b}\right)^k\left(\frac{d}{c}\right)^l A_{m-k,n-l}.\end{aligned}$$

Let $\mu \in S$. Then

$$\begin{aligned}A_{m,n} &\le \frac{1}{b}(d-\mu P_{m-1,n})A_{m-1,n}\\ &\le \left(\frac{1}{b}\right)^k\prod_{i=1}^{k}(d-\mu P_{m-i,n})A_{m-k,n}\\ &\le \left(\frac{1}{b}\right)^k\left(\frac{1}{c}\right)\prod_{i=1}^{k}(d-\mu P_{m-i,n})\\ &\quad\times(d-\mu P_{m-k,n-1})A_{m-k,n-1}\\ &\le \left(\frac{1}{b}\right)^k\left(\frac{1}{c}\right)^l\prod_{i=1}^{k}(d-\mu P_{m-i,n})\\ &\quad\times\prod_{j=1}^{l}(d-\mu P_{m-k,n-j})A_{m-k,n-l}.\end{aligned}$$

Substituting the above inequality into (1.1) we obtain

$$\begin{aligned}&aA_{m+1,n+1}+bA_{m+1,n}+cA_{m,n+1}-dA_{m,n}\\ &\quad+P_{m,n}b^kc^l\left[\prod_{i=1}^{k}(d-\mu P_{m-i,n})\right.\\ &\quad\left.\times\prod_{j=1}^{l}(d-\mu P_{m-k,n-j})\right]^{-1}A_{m,n}\le 0.\end{aligned}$$

Hence

$$\begin{aligned}&aA_{m+1,n+1}+bA_{m+1,n}+cA_{m,n+1}\\ &\quad-\left(d-P_{m,n}b^kc^l\left[\sup_{m\ge M,n\ge N}\prod_{i=1}^{k}(d-\mu P_{m-i,n})\right.\right.\\ &\quad\left.\left.\times\prod_{j=1}^{l}(d-\mu P_{m-k,n-j})\right]^{-1}\right)A_{m,n}\le 0,\end{aligned}$$

which implies that

$$\begin{aligned}&b^kc^l\left[\sup_{m\ge M,n\ge N}\prod_{i=1}^{k}(d-\mu P_{m-i,n})\right.\\ &\quad\left.\times\prod_{j=1}^{l}(d-\mu P_{m-k,n-j})\right]^{-1}\in S.\end{aligned}$$

We are ready to state the following proposition.

THEOREM 2.2 *In addition to* (i) *of Theorem* 2.1, *further assume that* (ii)′ *there exist* $M,N\in N_1$ *such that*

$$\begin{aligned}&\sup_{\lambda\in E,m\ge M,n\ge N}\lambda\prod_{i=1}^{k}(d-\lambda P_{m-i,n})\\ &\quad\times\prod_{j=1}^{l}(d-\lambda P_{m-k,n-j})<b^kc^l.\end{aligned}\tag{2.21}$$

Then every solution of (1.1) *oscillates.*

Since

$$\max_{d/q>\lambda>0}\lambda(d-\lambda P)^{k+l}=\frac{d^{k+l+1}(k+l)^{k+l}}{p(1+k+l)^{1+k+l}}$$

and (2.21), we have the following result.

COROLLARY 2.2 *In addition to* (i) *of Theorem* 2.1, *assume that*

$$\liminf_{m,n\to\infty}P_{m,n}=P>\frac{d^{1+k+l}(k+l)^{k+l}}{b^kc^l(1+k+l)^{1+k+l}}.\tag{2.22}$$

Then every solution of (1.1) *is oscillatory.*

Example 2.1 Consider the partial difference equation

$$\begin{aligned}&A_{m+1,n+1}+eA_{m+1,n}+A_{m,n+1}\\ &\quad-A_{m,n}+\frac{1+e}{e^4}A_{m-2,n-2}=0.\end{aligned}\tag{2.23}$$

It is easy to see that (2.23) satisfies the conditions of Corollary 2.1, so every solution of this equation is oscillatory. In fact, $A_{m,n}=(-e)^{m+n}$ is such a solution.

3 EQUATION (1.3)

We consider (1.3) together with the associate linear equation

$$A_{m+1,n+1} + A_{m+1,n} + A_{m,n+1} - A_{m,n} + pA_{m-k,n-l} = 0, \tag{3.1}$$

where $p > 0$, k and l are positive integers.

For (3.1), the following result is known [12].

LEMMA 3.1 *Every solution of* (3.1) *oscillates if and only if its characteristic equation*

$$\lambda\mu + \lambda + \mu - 1 + p\lambda^{-k}\mu^{-l} = 0 \tag{3.2}$$

has no positive roots.

LEMMA 3.2 *Assume that every solution* (3.1) *oscillates. Then there exists* $\varepsilon_0 \in (0, p)$ *such that for each* $\varepsilon \in [0, \varepsilon_0]$ *every solution of the equation*

$$A_{m+1,n+1} + A_{m+1,n} + A_{m,n+1} - A_{m,n} + (p-\varepsilon)A_{m-k,n-l} = 0 \tag{3.3}$$

also oscillates.

Proof By Lemma 3.1, it is sufficient to prove that

$$\lambda\mu + \lambda + \mu - 1 + (p-\varepsilon)\lambda^{-k}\mu^{-l} = 0 \tag{3.4}$$

has no positive roots. Obviously, (3.4) has no positive roots in the region $\lambda\mu + \lambda + \mu - 1 \geq 0$.

Let

$$F(\lambda, \mu) = \lambda\mu + \lambda + \mu - 1 + p\lambda^{-k}\mu^{-l}.$$

Since (3.2) has no positive roots, so $F(\lambda, \mu) > 0$ for $(\lambda, \mu) \in (0, \infty) \times (0, \infty)$. Thus

$$\min_{\lambda>0,\mu>0} F(\lambda, \mu) = m > 0.$$

Let

$$G(s, t) = st + s + t - 1 + \tfrac{1}{2}ps^{-k}t^{-l},$$

$\varepsilon_0 \in (0, p/2)$ and $\varepsilon_0\alpha^{-(k+l)} \leq m/2$, where

$$0 < \alpha < \min\left\{\left(\frac{p}{2}\right)^{1/(k+l)}, \left(\frac{p}{2}\right)^{1/k}, \left(\frac{p}{2}\right)^{1/l}\right\}.$$

For $0 < \lambda \leq \alpha$, $0 < \mu \leq \alpha$, we have

$$\begin{aligned}&\lambda\mu + \lambda + \mu - 1 + (p-\varepsilon)\lambda^{-k}\mu^{-l}\\ &> \lambda\mu + \lambda + \mu - 1 + \frac{p}{2}\lambda^{-k}\mu^{-l}\\ &> -1 + \frac{p}{2}\alpha^{-(k+l)} > 0.\end{aligned}$$

For $\lambda > \alpha$, $\mu > \alpha$, we have

$$\begin{aligned}&\lambda\mu + \lambda + \mu - 1 + (p-\varepsilon)\lambda^{-k}\mu^{-l}\\ &\geq F(\lambda, \mu) - \varepsilon_0\lambda^{-k}\mu^{-l}\\ &\geq m - \varepsilon_0\alpha^{-(k+l)} \geq \frac{m}{2} > 0.\end{aligned}$$

For $0 < \lambda \geq \alpha$, $\mu > \alpha$, $\lambda\mu + \lambda + \mu < 1$, we have

$$\begin{aligned}&\lambda\mu + \lambda + \mu - 1(p-\varepsilon)\lambda^{-k}\mu^{-l}\\ &> \lambda\mu + \lambda + \mu - 1 + \frac{p}{2}\lambda^{-k}\mu^{-l}\\ &> 1 - \frac{p}{2}\alpha^{-k} > 0.\end{aligned}$$

Similarly, for $\lambda > \alpha$, $0 < \mu \leq \alpha$, $\lambda\mu + \lambda + \mu < 1$, we have

$$\lambda\mu + \lambda + \mu - 1 + (p-\varepsilon)\lambda^{-k}\mu^{-l} > 0.$$

We have proved that (3.4) has no positive roots.

We are ready to state the following result.

THEOREM 3.1 *Assume that*

(i) $\lim_{m,n\to\infty} \inf P_{m,n} = P > 0$,

(ii) $f \in C(\mathbb{R}, \mathbb{R})$, $xf(x) > 0$ *as* $x \neq 0$, $\lim_{x\to 0} f(x)/x = 1$.

Then every solution of (3.1) *oscillates implies that every solution of* (1.3) *oscillates.*

Proof If not, let $A_{m,n} > 0$, $m \geq m_0$, $n \geq n_0$ be a solution (1.3). Then $A_{m,n}$ is decreasing in m and n, and hence $\lim_{m,n\to\infty} A_{m,n} = 0$, $\lim_{m\to\infty} A_{m,n} = 0$ and $\lim_{n\to\infty} A_{m,n} = 0$. Let

$$\bar{P}_{m,n} = P_{m,n}\frac{f(A_{m-k,n-l})}{A_{m-k,n-l}}.$$

Then $\lim_{m,n\to\infty}\inf \bar{P}_{m,n} = P$. For each $\varepsilon \in (0, \epsilon_0]$, there exist M and N such that $\bar{P}_{m,n} > p - \varepsilon$, for $m \geq M$, $n \geq N$. Therefore

$$A_{m+1,n+1} + A_{m+1,n} + A_{m,n+1} - A_{m,n} + (p-\varepsilon)A_{m-k,n-l} \leq 0, \quad m \geq M, \ n \geq N.$$

Summing it in n from n ($\geq N$) to ∞, we have

$$\sum_{i=n}^{\infty} A_{m+1,i+1} + \sum_{i=n}^{\infty} A_{m+1,i} - A_{m,n} + (p-\varepsilon)\sum_{i=n}^{\infty} A_{m-k,i-l} \leq 0.$$

We rewrite the last inequality in the form

$$\sum_{i=n}^{\infty} A_{m+1,i+1} + \sum_{i=n+1}^{\infty} A_{m+1,i} + A_{m+1,n} - A_{m,n} + (p-\varepsilon)\sum_{i=n}^{\infty} A_{m-k,i-l} \leq 0.$$

Summing it in m from m ($\geq M$) to ∞, we obtain

$$\sum_{j,i=m,n}^{\infty} A_{j+1,i+1} + \sum_{j=m}^{\infty}\sum_{i=n+1}^{\infty} A_{j+1,i} - A_{m,n} + (p-\varepsilon)\sum_{j,i=m,n}^{\infty} A_{j-k,i-l} \leq 0.$$

Hence

$$A_{m,n} \geq \sum_{j,i=m,n}^{\infty} A_{j+1,i+1} + \sum_{j=m}^{\infty}\sum_{i=n+1}^{\infty} A_{j+1,i} + (p-\varepsilon)\sum_{j,i=m,n}^{\infty} A_{j-k,i-l}. \tag{3.5}$$

Define a set of real double sequences

$$X = \{\{B_{m,n}\} \mid 0 \leq B_{m,n} \leq 1, \ m \geq M-k, \ n \geq N-l\}$$

and an operator T on X by

$$(TB)_{m,n} = \begin{cases} \dfrac{1}{A_{m,n}}\left[2\displaystyle\sum_{j,i=m+1,n+1}^{\infty} A_{j,i}B_{j,i} + (p-\varepsilon)\sum_{j,i=m,n}^{\infty} A_{j-k,i-l}B_{j-k,i-l}\right], & m \geq M, \ n \geq N, \\ 1, & \text{otherwise}. \end{cases} \tag{3.6}$$

In view of (3.5), it is easy to see that $TX \subset X$. Define a set of sequences $\{B^{(r)}_{m,n}\}$, $r = 0, 1, 2, \ldots$ as follows:

$$B^{(0)}_{m,n} \equiv 1, \qquad B^{(r)}_{m,n} = (TB)^{(r-1)}_{m,n}, \quad r = 1, 2, \ldots$$

In view of (3.5), we have

$$B^{(0)}_{m,n} \geq B^{(1)}_{m,n} \geq \cdots \geq B^{(r)}_{m,n} \geq \cdots, \quad \text{for } m \geq M-k, \ n \geq N-l.$$

Hence $\lim_{r\to\infty} B^{(r)}_{m,n} = B_{m,n}$ exists, for $m \geq M-k$, $n \geq N-l$. From (3.6),

$$B_{m,n} = \begin{cases} \dfrac{1}{A_{m,n}}\left[2\displaystyle\sum_{j,i=m+1,n+1}^{\infty} A_{j,i}B_{j,i} + (p-\varepsilon)\sum_{j,i=m,n}^{\infty} A_{j-k,i-l}B_{j-k,i-l}\right], & m \geq M, \ n \geq N, \\ 1, & \text{otherwise}. \end{cases}$$

Clearly, $B_{m,n} > 0$ for $m \geq M-k$, $n \geq N-l$. Let $x_{m,n} = A_{m,n}B_{m,n}$. Then $x_{m,n} > 0$ for $m \geq M-k$, $n \geq N-l$ and

$$x_{m,n} = 2\sum_{j=m+1}^{\infty}\sum_{i=n+1}^{\infty} x_{j,i} + (p-\varepsilon)\sum_{j=m}^{\infty}\sum_{i=n}^{\infty} x_{j-k,i-l}.$$

Hence

$$x_{m+1,n} - x_{m,n} = -2\sum_{i=n+1}^{\infty} x_{m+1,i} - (p-\varepsilon)\sum_{i=n}^{\infty} x_{m-k,i-l}$$

or

$$x_{m,n} = \sum_{i=n+1}^{\infty} x_{m+1,i} + \sum_{i=n}^{\infty} x_{m+1,i} + (p-\varepsilon)\sum_{i=n}^{\infty} x_{m-k,i-l}.$$

Then we have

$$x_{m,n+1} - x_{m,n} = -x_{m+1,n+1} - x_{m+1,n} - (p-\varepsilon)x_{m-k,n-l} = 0,$$

which implies that (3.3) has a positive solution $\{x_{m,n}\}$. By Lemma 3.2, (3.1) has a positive solution, which is a contradiction. The proof is complete.

THEOREM 3.2 *Assume that*

(i) $0 \le P_{m,n} \le P$,
(ii) *there exists* $h > 0$ *such that* $f(x)$ *is nondecresing and* $0 \le f(x)/x \le 1$, *for* $0 < |x| < h$.

If (3.1) *has a positive solution, then* (1.3) *also has a positive solution.*

Proof If (3.1) has a positive solution, by Lemma 3.1,

$$\lambda\mu + \lambda + \mu - 1 + p\lambda^{-k}\mu^{-l} = 0$$

has a positive root (λ, μ) with $0 < \lambda < 1$, $0 < \mu < 1$ and that $\{\lambda^m \mu^n\}$ is a positive solution of (3.1). Choose $a > 0$ such that

$$A_{m,n} = a\lambda^m \mu^n < h, \quad \text{for } m \ge -k, n - l.$$

$\{A_{m,n}\}$ is a positive solution of (3.1) and satisfies $f(A_{m,n}) \le A_{m,n}$ by condition (ii). Similar to Theorem 3.1, summing (3.1) we get

$$A_{m,n} = 2\sum_{j=m+1}^{\infty}\sum_{i=n+1}^{\infty} A_{j,i} + p\sum_{j=m}^{\infty}\sum_{i=n}^{\infty} A_{j-k,i-l}, \quad m \ge 0, \ n \ge 0.$$

Hence

$$A_{m,n} \ge 2\sum_{j=m+1}^{\infty}\sum_{i=n+1}^{\infty} A_{j,i} + \sum_{j=m}^{\infty}\sum_{i=n}^{\infty} P_{j,i} f(A_{j-k,i-l}). \qquad (3.7)$$

Define

$$X = \{\{B_{m,n}\} \mid 0 \le B_{m,n} \le 1, m \ge -k, n \ge -l\}$$

and an operator T on X by

$$(TB_{m,n}) = \begin{cases} \dfrac{1}{A_{m,n}}\left[2\displaystyle\sum_{j=m+1}^{\infty}\sum_{i=n+1}^{\infty} A_{j,i}B_{j,i} + \sum_{j=m}^{\infty}\sum_{i=n}^{\infty} P_{j,i} f(A_{j-k,i-l}B_{j-k,i-l})\right], & m \ge 0, \ n \ge 0, \\ 1, & \text{otherwise.} \end{cases}$$

In view of (3.7), $TX \subset X$. Similar to Theorem 3.1, we can prove that there exists $\{B_{m,n}\} \in X$ such that $(TB)_{m,n} = B_{m,n}$ for $m \ge 0, n \ge 0$. Let $x_{m,n} = A_{m,n}B_{m,n} > 0, m \ge 0, n \ge 0$. Then

$$x_{m,n} = 2\sum_{j=m+1}^{\infty}\sum_{i=n+1}^{\infty} x_{j,i} + \sum_{j=m}^{\infty}\sum_{i=n}^{\infty} P_{j,i} f(x_{j-k,i-l}). \qquad (3.8)$$

Similar to the proof of Theorem 3.1, (3.8) implies that $\{x_{m,n}\}$ is a positive solution of (1.3). The proof is complete.

From Theorems 3.1 and 3.2 we obtain

COROLLARY 3.1 *Assume that* $P_{m,n} \equiv P > 0$, (ii) *of Theorem* 3.1 *and* (ii) *of Theorem* 3.2 *hold. Then every solution of* (1.3) *oscillates if and only if every solution of* (3.1) *oscillates.*

Consider (1.3) together with the equation

$$A_{m+1,n+1} + A_{m+1,n} + A_{m,n+1} - A_{m,n} + q_{m,n} f(A_{m-k,n-l}) = 0, \qquad (3.9)$$

where $f \in C(\mathbb{R}, \mathbb{R})$, $xf(x) > 0$ as $x \ne 0$.

We have the comparison theorem as follows:

THEOREM 3.3 *Assume that* $p_{m,n} \ge q_{m,n} > 0$ *for all large* m, n *and that every solution of* (3.9) *oscillates. Then every solution of* (1.3) *oscillates.*

Proof Suppose to the contrary, $\{A_{m,n}\}$ is a positive solution, of (1.3). Summing (1.3), we obtain

$$A_{m,n} = 2\sum_{j=m+1}^{\infty}\sum_{i=n+1}^{\infty} A_{j,i} + \sum_{j=m}^{\infty}\sum_{i=n}^{\infty} p_{j,i} f(A_{j-k,i-l})$$
$$\geq 2\sum_{j=m+1}^{\infty}\sum_{i=n+1}^{\infty} A_{j,i} + \sum_{j=m}^{\infty}\sum_{i=n}^{\infty} q_{j,i} f(A_{j-k,i-l}),$$

from which and using the method in the proof of Theorem 3.2, we obtain a positive solution of (3.9). This contradiction proves the theorem.

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