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Research Article

Global Asymptotic Stability for a Fourth-Order Rational Difference Equation

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Our aim is to investigate the global behavior of the following fourth-order rational difference equation: $x_{n+1} = (x_n x_{n-2} x_{n-3} + x_n + x_{n-2} + x_{n-3} + a)/(x_n x_{n-2} + x_n x_{n-3} + x_{n-2} x_{n-3} + 1 + a)$, $n = 0, 1, 2, \ldots$ where $a \in [0, \infty)$ and the initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$. To verify that the positive equilibrium point of the equation is globally asymptotically stable, we used the rule of the successive lengths of positive and negative semicycles of nontrivial solutions of the aforementioned equation.

1. Introduction

There has been a great interest in studying global behaviors of rational difference equations. One can easily see that it is hard to understand thoroughly the global behaviors of solutions of rational difference equations although they have simple forms. And there has not been any general method to identify the global behaviors of rational difference equations of order greater than one and so far [1–3].

Let us consider the following fourth-order difference equation:

$$x_{n+1} = \frac{x_n x_{n-2} x_{n-3} + x_n + x_{n-2} + x_{n-3} + a}{x_n x_{n-2} + x_n x_{n-3} + x_{n-2} x_{n-3} + 1 + a}, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where $a \in [0, \infty)$ and the initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$ in this paper. By determining the rule for the positive and negative semicycles, we assigned the global behavior of the positive equilibrium point. The unique positive equilibrium point $\overline{x}$ of (1.1) is obtained $\overline{x} = 1$ by solving

$$\overline{x} = \frac{\overline{x}^3 + 3\overline{x} + a}{3\overline{x}^2 + 1 + a}. \tag{1.2}$$

Definition 1.1. A solution $\{x_n\}_{n=-3}^{\infty}$ of (1.1) is said to be *eventually trivial* if x_n is eventually equal to $\overline{x} = 1$; otherwise, the solution is said to be *nontrivial* [4–6].

Definition 1.2. A solution $\{x_n\}_{n=-3}^{\infty}$ of (1.1) is said to be *eventually positive* if x_n is eventually greater than $\overline{x} = 1$.

A solution $\{x_n\}_{n=-3}^{\infty}$ of (1.1) is said to be *eventually negative* if x_n is eventually less than $\overline{x} = 1$ [1, 2, 4–6].

Definition 1.3. A *positive semicycle* of a solution $\{x_n\}_{n=-3}^{\infty}$ of (1.1) consists of a "string" of terms $\{x_l, x_{l+1}, \dots, x_m\}$, all greater than or equal to the equilibrium point $\overline{x}$, with $l \geq -3$ and $m < \infty$ such that

$$\begin{aligned} &\text{either } l = -3 \quad \text{or} \quad l > -3, \ x_{l-1} < \overline{x}, \\ &\text{either } m = \infty \quad \text{or} \quad m < \infty, \ x_{m+1} < \overline{x}. \end{aligned} \tag{1.3}$$

A *negative semicycle* of a solution $\{x_n\}_{n=-3}^{\infty}$ of (1.1) consists of a "string" of terms $\{x_l, x_{l+1}, \dots, x_m\}$, all greater than or equal to the equilibrium point $\overline{x}$, with $l \geq -3$ and $m < \infty$ such that

$$\begin{aligned} &\text{either } l = -3 \quad \text{or} \quad l > -3, \ x_{l-1} \geq \overline{x}, \\ &\text{either } m = \infty \quad \text{or} \quad m < \infty, \ x_{m+1} \geq \overline{x}. \end{aligned} \tag{1.4}$$

And also the *lengths of a semicycle* is $(m - l + 1)$, the number of the terms contained in it. And we denote that the lengths of a positive semicycle are by $(m - l + 1)^+$ and the lengths of a negative semicycle are by $(m - l + 1)^-$ [1, 2, 4–6].

Definition 1.4. A solution $\{x_n\}_{n=-3}^{\infty}$ of (1.1) is called nonoscillatory about $\overline{x}$, or simply *nonoscillatory*, if there exists $N \geq -3$ such that either

$$x_n \geq \overline{x}, \quad \forall n \geq N \tag{1.5}$$

or

$$x_n < \overline{x}, \quad \forall n \geq N. \tag{1.6}$$

Otherwise, the solution $\{x_n\}_{n=-3}^{\infty}$ is called oscillatory about $\overline{x}$, or simply *oscillatory* [1, 2].

Lemma 1.5. *A positive solution $\{x_n\}_{n=-3}^{\infty}$ of (1.1) is eventually trivial if and only if*

$$(x_{-3} - 1)(x_{-2} - 1)(x_{-1} - 1)(x_0 - 1) = 0. \tag{1.7}$$

Proof. To prove the lemma, first we assume that $(x_{-3}-1)(x_{-2}-1)(x_{-1}-1)(x_0-1) \neq 0$ and then we must show that $x_n \neq 1$ for any $n \geq 1$.

Assume that for some $N \geq 1$, $x_N = 1$ and that $x_n \neq 1$ for $-3 \leq n \leq N-1$. So

$$x_N = \frac{x_{N-1}x_{N-3}x_{N-4} + x_{N-1} + x_{N-3} + x_{N-4} + a}{x_{N-1}x_{N-3} + x_{N-1}x_{N-4} + x_{N-3}x_{N-4} + 1 + a} = 1, \tag{1.8}$$

and we obtain $(x_{N-4}-1)(x_{N-3}-1)(x_{N-1}-1) = 0$; hence $x_{N-4} = 1$, $x_{N-3} = 1$, $x_{N-1} = 1$, when we solve the equation above. It is easy to see that $x_{N-4} = 1$, $x_{N-3} = 1$, or $x_{N-1} = 1$ contradicts with $x_n \neq 1$ for $-3 \leq n \leq N-1$.

If (1.7) holds, it is clear that the following conclusions hold:

(i) if $x_{-3} = 1$, $x_n = 1$ for $n \geq 1$,

(ii) if $x_{-2} = 1$, $x_n = 1$ for $n \geq 1$,

(iii) if $x_{-1} = 1$, $x_n = 1$ for $n \geq 2$,

(iv) if $x_0 = 1$, $x_n = 1$ for $n \geq 1$.

It is obvious that if the initial conditions do not satisfy (1.7), then the positive solution $\{x_n\}_{n=-3}^{\infty}$ of (1.1) is eventually nontrivial. □

Lemma 1.6. *If* $\{x_n\}_{n=-3}^{\infty}$ *is a nontrivial positive solution of* (1.1), *then the following conclusions are satisfied:*

(i) $(x_{n+1}-1)(x_n-1)(x_{n-2}-1)(x_{n-3}-1) > 0$,

(ii) $(x_{n+1}-x_n)(x_n-1) < 0$,

(iii) $(x_{n+1}-x_{n-2})(x_{n-2}-1) < 0$,

(iv) $(x_{n+1}-x_{n-3})(x_{n-3}-1) < 0$.

Proof. (i) The proof of the inequality (i) is obtained by subtracting 1 from (1.1)

$$\begin{aligned} x_{n+1} - 1 &= \frac{x_n x_{n-2} x_{n-3} + x_n + x_{n-2} + x_{n-3} + a}{x_n x_{n-2} + x_n x_{n-3} + x_{n-2} x_{n-3} + 1 + a} - 1 \\ &= \frac{(x_n - 1)(x_{n-2} - 1)(x_{n-3} - 1)}{x_n x_{n-2} + x_n x_{n-3} + x_{n-2} x_{n-3} + 1 + a}. \end{aligned} \tag{1.9}$$

The dominator of this fraction is positive so $(x_{n+1}-1)(x_n-1)(x_{n-2}-1)(x_{n-3}-1) > 0$ also.

(ii) If we subtract x_n from (1.1), we obtain

$$\begin{aligned} x_{n+1} - x_n &= \frac{x_n x_{n-2} x_{n-3} + x_n + x_{n-2} + x_{n-3} + a}{x_n x_{n-2} + x_n x_{n-3} + x_{n-2} x_{n-3} + 1 + a} - x_n \\ &= -\frac{(x_n - 1)[x_{n-2}(x_n + 1) + x_{n-3}(x_n + 1) + a]}{x_n x_{n-2} + x_n x_{n-3} + x_{n-2} x_{n-3} + 1 + a}. \end{aligned} \tag{1.10}$$

The expression $[x_{n-2}(x_n+1)+x_{n-3}(x_n+1)+a]/(x_nx_{n-2}+x_nx_{n-3}+x_{n-2}x_{n-3}+1+a)$ is positive, and so we get

$$(x_{n+1}-x_n)(x_n-1)<0. \tag{1.11}$$

The proofs for inequalities (iii) and (iv) are similar to the one for (ii). □

2. Main Results and Their Proofs

The trajectory of (1.1) and global asymptotic stability of the positive solution are considered in this part of the paper.

Theorem 2.1. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a strictly oscillatory solution of* (1.1). *Then the positive and negative semicycles of* (1.1) *are* $\ldots,3^+,3^-,3^+,3^-,3^+,3^-,3^+,3^-,\ldots$ *or* $\ldots,2^+,1^-,2^+,1^-,2^+,1^-,2^+,1^-,\ldots$ *or* $\ldots,2^-,1^+,2^-,1^+,2^-,1^+,2^-,1^+,\ldots$ *or* $\ldots,1^+,1^-,1^+,1^-,1^+,1^-,1^+,1^-,\ldots$.

Proof. Assume that $\{x_n\}_{n=-3}^{\infty}$ is a strictly oscillatory solution of (1.1), then the initial values must satisfy one of the following four cases:

(i) $x_{p-3}>1$, $x_{p-2}<1$, $x_{p-1}>1$, $x_p>1$,

(ii) $x_{p-3}>1$, $x_{p-2}<1$, $x_{p-1}>1$, $x_p<1$,

(iii) $x_{p-3}>1$, $x_{p-2}<1$, $x_{p-1}<1$, $x_p>1$,

(iv) $x_{p-3}>1$, $x_{p-2}<1$, $x_{p-1}<1$, $x_p<1$.

If (i) occurs, it follows from Lemma 1.6(i) that

$$\begin{gathered}
x_{p-3}>1,\quad x_{p-2}<1,\quad x_{p-1}>1,\quad x_p>1,\quad x_{p+1}<1,\quad x_{p+2}>1,\quad x_{p+3}>1,\\
x_{p+4}<1,\quad x_{p+5}>1,\quad x_{p+6}>1,\quad x_{p+7}<1,\quad x_{p+8}>1,\quad x_{p+9}>1,\\
x_{p+10}<1,\quad x_{p+11}>1,\quad x_{p+12}>1,\quad x_{p+13}<1,\quad x_{p+14}>1,\quad x_{p+15}>1,\\
x_{p+16}<1,\quad x_{p+17}>1,\quad x_{p+18}>1,\quad x_{p+19}<1,\quad x_{p+20}>1,\quad x_{p+21}>1,\ldots.
\end{gathered} \tag{2.1}$$

It means that the rule for the lengths of positive and negative semicycles of the solution of (1.1) occurs successively as $\ldots,2^+,1^-,2^+,1^-,2^+,1^-,2^+,1^-,\ldots$.

If (ii) happens, the positive and negative semicycles are

$$\begin{gathered}
x_{p-3}>1,\quad x_{p-2}<1,\quad x_{p-1}>1,\quad x_p<1,\quad x_{p+1}>1,\quad x_{p+2}<1,\quad x_{p+3}>1,\quad x_{p+4}<1,\\
x_{p+5}>1,\quad x_{p+6}<1,\quad x_{p+7}>1,\quad x_{p+8}<1,\quad x_{p+9}>1,\quad x_{p+10}<1,\quad x_{p+11}>1,\\
x_{p+12}<1,\quad x_{p+13}>1,\quad x_{p+14}<1,\quad x_{p+15}>1,\quad x_{p+16}<1,\quad x_{p+17}>1,\\
x_{p+18}<1,\quad x_{p+19}>1,\quad x_{p+20}<1,\quad x_{p+21}>1,\quad x_{p+22}<1,\quad x_{p+23}>1,\ldots.
\end{gathered} \tag{2.2}$$

The regulation for the lengths of positive and negative semicycles which occur successively is $\ldots,1^+,1^-,1^+,1^-,1^+,1^-,1^+,1^-,\ldots$.

The other cases can be shown similarly. □

Theorem 2.2. *The positive equilibrium point of* (1.1) *is globally asymptotically stable.*

Proof. Let us show that the positive equilibrium point $\overline{x} = 1$ of (1.1) is locally asymptotically stable and also globally attractive, then it is globally asymptotically stable. The linearized equation of (1.1) about the positive equilibrium $\overline{x} = 1$ is

$$y_{n+1} = q_0 y_n + q_1 y_{n-1} + q_2 y_{n-2} + q_3 y_{n-3}, \quad (2.3)$$

where $q_i = (\partial F/\partial u_i)(\overline{x}, \overline{x}, \overline{x}, \overline{x})$ and $F(u_0, u_1, u_2, u_3) = (u_0 u_2 u_3 + u_0 + u_2 + u_3 + a)/(u_0 u_2 + u_0 u_3 + u_2 u_3 + 1 + a)$.

And we obtain

$$y_{n+1} = q_0 y_n + q_1 y_{n-1} + q_2 y_{n-2} + q_3 y_{n-3} = 0 \cdot y_n + 0 \cdot y_{n-1} + 0 \cdot y_{n-2} + 0 \cdot y_{n-3} = 0, \quad (2.4)$$

thereby $\overline{x} = 1$ is locally asymptotically stable.

Now we must show that $\lim_{n\to\infty} x_n = \overline{x} = 1$. The proof is as follows.

(1) If initial values of (1.1) satisfy (1.7), then $x_n = 1$ according to Lemma 1.5, so $\lim_{n\to\infty} x_n = \overline{x} = 1$.

(2) If the initial values of (1.1) do not satisfy (1.7), then for any solution of (1.1), $x_n \neq 1$ for $n \geq -3$.

  (i) If the solution is nonoscillatory about the positive equilibrium point of (1.1), then $\{x_n\}$ is monotonic and bounded because of Lemma 1.6. So, the limit

$$\lim_{n\to\infty} x_n = L \quad (2.5)$$

  exists and is finite. If we take limits on both sides of (1.1), then we obtain $L = (L^3 + 3L + a)/(3L^2 + 1 + a)$ and thereby $L = 1$. So $\lim_{n\to\infty} x_n = L = 1$.

  (ii) If the solution is strictly oscillatory, then trajectory structure of nontrivial solutions of (1.1) is $\ldots, 3^+, 3^-, 3^+, 3^-, 3^+, 3^-, 3^+, 3^-, \ldots$ or $\ldots, 2^+, 1^-, 2^+, 1^-, 2^+, 1^-, 2^+, 1^-, \ldots$ or $\ldots, 2^-, 1^+, 2^-, 1^+, 2^-, 1^+, 2^-, 1^+, \ldots$ or $\ldots, 1^+, 1^-, 1^+, 1^-, 1^+, 1^-, 1^+, 1^-, \ldots$.

First, we investigate the case where the rule of the trajectory structure is $\ldots, 3^+, 3^-, 3^+, 3^-, 3^+, 3^-, 3^+, 3^-, \ldots$ in a period. We denote positive semicycles by $\{x_p, x_{p+1}, x_{p+2}\}^+$ and negative semicycles by $\{x_{p+3}, x_{p+4}, x_{p+5}\}^-$. The rule for the positive and negative semicycles can be "periodically" expressed as follows:

$$\{x_{p+6n}, x_{p+6n+1}, x_{p+6n+2}\}^+, \quad \{x_{p+6n+3}, x_{p+6n+4}, x_{p+6n+5}\}^-, \quad n = 0, 1, \ldots. \quad (2.6)$$

By using Lemma 1.6 we obtain

(i) $x_{p+6n+2} < x_{p+6n+1} < x_{p+6n}$; $x_{p+6n+3} < x_{p+6n+4} < x_{p+6n+5}$,

(ii) $x_{p+6n+6} < x_{p+6n+2}$; $x_{p+6n+9} > x_{p+6n+5}$.

This relations give rise to

$$x_{p+6n+6} < x_{p+6n+2} < x_{p+6n+1} < x_{p+6n}, \qquad x_{p+6n+3} < x_{p+6n+4} < x_{p+6n+5} < x_{p+6n+9}. \tag{2.7}$$

It is easy to see that $\{x_{p+6n}\}_{n=0}^{\infty}$ is decreasing with its lower bound 1 because of the inequality $x_{p+6n+6} < x_{p+6n+2} < x_{p+6n+1} < x_{p+6n}$. Hence the limit exists and is finite. The limit is then

$$\lim_{n\to\infty} x_{p+6n} = \lim_{n\to\infty} x_{p+6n+1} = \lim_{n\to\infty} x_{p+6n+2} = L. \tag{2.8}$$

Similarly $\{x_{p+6n+3}\}_{n=0}^{\infty}$ is increasing and its upper bound 1 because of the inequality $x_{p+6n+3} < x_{p+6n+4} < x_{p+6n+5} < x_{p+6n+9}$. So, the limit $\lim_{n\to\infty} x_{p+6n+3} = M$ exists and is finite. And we derive $\lim_{n\to\infty} x_{p+6n+4} = \lim_{n\to\infty} x_{p+6n+5} = M$.

Now, we must show that $L = M = 1$. To do this, let us take

$$x_{p+6n+6} = \frac{x_{p+6n+5}x_{p+6n+3}x_{p+6n+2} + x_{p+6n+5} + x_{p+6n+3} + x_{p+6n+2} + a}{x_{p+6n+5}x_{p+6n+3} + x_{p+6n+5}x_{p+6n+2} + x_{p+6n+3}x_{p+6n+2} + 1 + a}, \tag{2.9}$$

and taking the limit on both sides of the above equality, we obtain

$$L = \frac{M \cdot M \cdot L + M + M + L + a}{M \cdot M + M \cdot L + M \cdot L + 1 + a}. \tag{2.10}$$

By solving this equation we have

$$(L-1)[2M(L+1) + a] = 0, \tag{2.11}$$

and then $L = 1$. Similarly to obtain M by using $1 \le x_{p+6n+2} < x_{p+6n+1}$ we are taking

$$x_{p+6n+5} = \frac{x_{p+6n+4}x_{p+6n+2}x_{p+6n+1} + x_{p+6n+4} + x_{p+6n+2} + x_{p+6n+1} + a}{x_{p+6n+4}x_{p+6n+2} + x_{p+6n+4}x_{p+6n+1} + x_{p+6n+2}x_{p+6n+1} + 1 + a}. \tag{2.12}$$

And taking the limit on both sides of the above equality we get

$$M = \frac{M \cdot L \cdot L + M + L + L + a}{M \cdot L + M \cdot L + L \cdot L + 1 + a}. \tag{2.13}$$

By solving this equation we have $(M-1)[2L(M+1) + a] = 0$ and $M = 1$. Hence, we derive $L = M = 1$, so

$$\lim_{n\to\infty} x_n = 1. \tag{2.14}$$

It can be shown that $\lim_{n\to\infty} x_n = 1$ for the other rules of the trajectory structures with the same manner. Therefore, the positive equilibrium point is globally asymptotically stable. □

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