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Research Article

Strong Laws of Large Numbers for $\mathbb{B}$-Valued Random Fields

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We extend to random fields case, the results of Woyczynski, who proved Brunk's type strong law of large numbers (SLLNs) for $\mathbb{B}$-valued random vectors under geometric assumptions. Also, we give probabilistic requirements for above-mentioned SLLN, related to results obtained by Acosta as well as necessary and sufficient probabilistic conditions for the geometry of Banach space associated to the strong and weak law of large numbers for multidimensionally indexed random vectors.

1. Introduction

We will study the limiting behavior of multiple sums of random vectors indexed by lattice points, so called random fields. Such research has roots in statistical mechanics and arised in the context of ergodic theory. Almost 70 years ago, Wiener considered double sums over lattice points with applications to homogenous chaos. Many aspects of present investigations of models of critical phenomena in statistical physics, crystal physics or Euclidean quantum field theories involve multiple sums of random variables with multidimensional indices. Multiparameter processes arise in applied context such as brain data imaging, and so forth.

Let N^r, $r \geq 1$ be the positive integer r-dimensional lattice points with coordinate wise partial ordering, $\leq$. Points in N^r are denoted by $\mathbf{m}$, $\mathbf{n}$ or more explicitly $\mathbf{n} = (n_1, n_2, \ldots, n_r)$ and $\mathbf{1}$ stands for the r-tuple $(1, \ldots, 1)$. Also for $\mathbf{n} = (n_1, n_2, \ldots, n_r)$, we define $|\mathbf{n}| = \prod_{i=1}^{r} n_i$ and $(\mathbf{n}) = \{\mathbf{k} : \mathbf{k} \leq \mathbf{n}\}$. The notation $\mathbf{n} \to \infty$ means that $\max_{1\leq i\leq r} n_i \to \infty$ or equivalently $|\mathbf{n}| \to \infty$.

Let $(\Omega, \mathfrak{F}, P)$—be a probability space, $(\mathbb{B}, \|\cdot\|)$—a real separable Banach space, $\{X_{\mathbf{k}}, \mathbf{k} \in N^r\}$—a family of $\mathbb{B}$-valued random vectors and set

$$S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}. \tag{1.1}$$

If $E\|X\| < \infty$, then EX stands for the Bochner integral. Let $\{a_{\mathbf{k}}, \mathbf{k} \in N^r\}$ be a $\mathbb{B}$-valued net and $a \in \mathbb{B}$. We say that $a_{\mathbf{n}} \to a$ strongly as $\mathbf{n} \to \infty$ if for any $\varepsilon > 0$, there exists $\mathbf{N}_\varepsilon \in N^r$ such that $\mathbf{n} \not\leq \mathbf{N}_\varepsilon$ implies $\|a_{\mathbf{n}} - a\| < \varepsilon$ or shortly for any $\varepsilon > 0$, $\|a_{\mathbf{n}} - a\| \geq \varepsilon$ "occurs finitely often" (see Smythe [1], Fazekas [2]). Furthermore, let $\{\mathfrak{F}_{\mathbf{k}}, \mathbf{k} \in N^r\}$ be an increasing family of sub σ-algebras of $\mathfrak{F}$, that is,

$$\bigwedge_{\mathbf{k} \leq \mathbf{n}} \mathfrak{F}_{\mathbf{k}} \subset \mathfrak{F}_{\mathbf{n}} \subset \mathfrak{F}. \tag{1.2}$$

Now, we will introduce definition of martingale (submartingale) for real-valued random fields, Smythe [3] (for more information see Merzbach [4]). Through this paper $\{\mathfrak{F}_{\mathbf{k}}, \mathbf{k} \in N^r\}$ satisfies condition CI (conditional independence)

$$E\left(E(\cdot \mid \mathfrak{F}_{\mathbf{m}}) \mid \mathfrak{F}_{\mathbf{n}}\right) = E\left(\cdot \mid \mathfrak{F}_{\mathbf{m} \wedge \mathbf{n}}\right) \text{ a.s.,} \tag{1.3}$$

where $\mathbf{m} \wedge \mathbf{n}$ denotes the componentwise minimum of $\mathbf{m}$ and $\mathbf{n}$. A$\{\mathfrak{F}_{\mathbf{k}}, \mathbf{k} \in N^r\}$-adapted, integrable process $\{Z_{\mathbf{k}}, \mathbf{k} \in N^r\}$ is called martingale (submartingale) if

$$E\left(Z_{\mathbf{n}} \mid \mathfrak{F}_{\mathbf{m}}\right) = Z_{\mathbf{m} \wedge \mathbf{n}}\left(\geq Z_{\mathbf{m} \wedge \mathbf{n}}\right) \quad \text{a.s. } \forall m, \mathbf{n} \in N^r. \tag{1.4}$$

The main aim of this paper is to prove a couple Brunk type strong laws of large numbers for independent $\mathbb{B}$-valued random fields. To prove this we would like to apply, among others, maximal inequalities for real-valued submartingale fields. Main results concerning maximal inequalities for random variables indexed by multidimensional indices are due to Cairoli [5], Gabriel [6], Klesov [7], Smythe [3], Shorack and Smythe [8], as well as Wichura [9]. In [5], Cairoli gave counterexample that the well-known following Doob maximal inequality for submartingales

$$\lambda P\left(\max_{\mathbf{k} \leq \mathbf{n}} S_{\mathbf{k}} \geq \lambda\right) \leq ES_{\mathbf{n}}^{+} \tag{1.5}$$

cannot be proved for a discrete-time random fields, utilizing "one dimensional" idea, as well as Hajek-Renyi-Chow inequality [10] and in consequence Chow or Brunk type strong law of large numbers. This problem motivated us to make an effort to give some new results for strong law of large numbers for random fields.

To get the above-mentioned result we will exploit the idea of maximal inequality introduced by Christofides and Serfling [11].

Theorem 1.1. *Let $\{Y_{\mathbf{k}}, \mathfrak{F}_{\mathbf{k}}, \mathbf{k} \in N^r\}$ be a submartingale, $\{\mathfrak{F}_{\mathbf{k}}, \mathbf{k} \in N^r\}$ satisfies* (1.3)*, and let $\{C_{\mathbf{k}}, \mathbf{k} \in N^r\}$ be a nonincreasing array of nonnegative numbers. Then for $\lambda > 0$,*

$$\begin{aligned}\lambda P\left(\sup_{\mathbf{k}\leq\mathbf{n}} C_{\mathbf{k}} Y_{\mathbf{k}} \geq \lambda\right) &\leq \min_{1\leq s\leq r}\left\{\sum_{\mathbf{k}\leq\mathbf{n}}\left(C_{\mathbf{k}} - C_{\mathbf{k};s;k_s+1}\right) E Y_{\mathbf{k}}^{+} - \sum_{k_i, i\neq s} C_{\mathbf{k};s;n_s}\int_{\left[\bigcup_{k_s=1}^{n_s} B_{k_1,\ldots,k_r}^{(s)}\right]^c} Y_{\mathbf{k};s;n_s}^{+}\, dP\right\} \\ &\leq \min_{1\leq s\leq r}\left\{\sum_{\mathbf{k}\leq\mathbf{n}}\left(C_{\mathbf{k}} - C_{\mathbf{k};s;k_s+1}\right) E Y_{\mathbf{k}}^{+}\right\},\end{aligned} \tag{1.6}$$

where $C_{\mathbf{k};s;\alpha} = C_{k_1,\ldots,k_{s-1},\alpha,k_{s+1},\ldots,k_r}$ and $C_{\mathbf{k}} = 0$ if $k_i > n_i$ for some $i = 1, 2, \ldots, r$.

Proof. In the multidimensional martingale case, Theorem 1.1 was proved by Christofides and Serfling [11, Theorem 2.2] using properties of submartingale fields, thus assertion of Theorem 1.1 is true. □

The following remark concerns the technique of the proof of Theorem 1.1 in martingale case.

Remark 1.2. In the proof of Theorem 2.2 of [11], the authors construct the sets $B_{\mathbf{k}}^{(i)}$ (see the algorithm in [11]) and say "An explicit expression of the sets $B_{\mathbf{k}}^{(i)}$ in terms of the sets $A_{\mathbf{k}}$ is possible to derive, but such formula is notationally messy and complicated." It seems that in the proof, we can use the sets $\widetilde{B}_{\mathbf{k}}^{(i)}$ constructed as follows (in the case $r = 2$, for simplicity).

Let $\mathbf{n} = (n_1, n_2)$, set $Z_i(\omega) = \sup_{1\leq j\leq n_2} C_{ij} Y_{ij}(\omega)$,

$$\begin{aligned} I^{(1)}(\omega) &= \inf_{1\leq i\leq n_1}\{i : Z_i(\omega) \geq \lambda\} \quad (\text{or } n_1 + 1 \text{ if no such } i \text{ exists}), \\ J^{(1)}(\omega) &= \inf_{1\leq j\leq n_2}\{j : C_{Ij} Y_{Ij}(\omega) \geq \lambda\},\end{aligned} \tag{1.7}$$

and set

$$\widetilde{B}_{ij}^{(1)} = \left\{I^{(1)}(\omega) = i,\ J^{(1)}(\omega) = j\right\}. \tag{1.8}$$

We obtain the sets $\widetilde{B}_{\mathbf{k}}^{(2)}$ by changing the order of taking maximum. In this construction we used idea introduced by Zimmerman [12]. Similarly to the sets constructed by Christofides and Serfling, $\widetilde{B}_{\mathbf{k}}^{(1)}$, $\widetilde{B}_{\mathbf{k}}^{(2)}$ are disjoint, $\mathfrak{F}_{in_2}$ and $\mathfrak{F}_{n_1 j}$, respectively, measurable, and

$$\bigcup_{1\leq i\leq n_1, 1\leq j\leq n_2} \widetilde{B}_{ij}^{(1)} = \bigcup_{1\leq i\leq n_1, 1\leq j\leq n_2} \widetilde{B}_{ij}^{(2)} = \left[\sup_{1\leq i\leq n_1, 1\leq j\leq n_2} C_{ij} Y_{ij} \geq \lambda\right]. \tag{1.9}$$

Such construction gives a simple formula and is very intuitive.

2. The Main Results

We start from the following generalization of Theorem 1.1.

Theorem 2.1. *Let* $\{Y_{\mathbf{k}}, \mathfrak{F}_{\mathbf{k}}, \mathbf{k} \in N^r\}$ *be a submartingale,* $\{\mathfrak{F}_{\mathbf{k}}, \mathbf{k} \in N^r\}$ *satisfies* (1.3), *and let* $\{C_{\mathbf{k}}, \mathbf{k} \in N^r\}$ *be a nonincreasing array of nonnegative numbers. Then for* $\lambda > 0$ *and* $\mathbf{m} \leq \mathbf{n}$,

$$\lambda P\left(\sup_{\mathbf{k}\in[(\mathbf{n})\setminus(\mathbf{m})]} C_{\mathbf{k}} Y_{\mathbf{k}} \geq \lambda\right) \leq \min_{1\leq s\leq r} \sum_{\mathbf{k}\in[(\mathbf{n})\setminus(\mathbf{m})]} \left(C_{\mathbf{k}} - C_{\mathbf{k};s;k_s+1}\right) E Y_{\mathbf{k}}^{+}, \tag{2.1}$$

where $C_{\mathbf{k};s;\alpha} = C_{k_1,\ldots,k_{s-1},\alpha,k_{s+1},\ldots,k_r}$, *and* $C_k = 0$ *if* $k_i > n_i$ *for some* $i = 1, 2, \ldots, r$.

Proof. Assume without loss of generality that the sum on right-hand side of (2.1) has minimum for $s_0 = 1$. Let us put $D = (\mathbf{n}) \setminus (\mathbf{m})$ and define disjoint partition of D as follow:

$$\begin{aligned} D_1 &= \left\{\mathbf{j} = \left(j_1, \ldots, j_r\right) : m_1 + 1 \leq j_1 \leq n_1,\ 1 \leq j_2 \leq m_2, \ldots, 1 \leq j_r \leq m_r\right\}, \\ D_i &= \left\{\mathbf{j} = \left(j_1, \ldots j_r\right) : 1 \leq j_1 \leq n_1,\ 1 \leq j_2 \leq n_2, \ldots, m_i + 1 \leq j_i \leq n_i, \right. \\ &\qquad \left. 1 \leq j_{i+1} \leq m_{i+1}, \ldots, 1 \leq j_r \leq m_r\right\}, \end{aligned} \tag{2.2}$$

for $i = 2, 3, \ldots, r$.

It is easy to see that $\bigcup_{i=1}^{r} D_i = D$ and $D_i \cap D_j = \varnothing$ for $i \neq j$. Now, let us observe that we can apply Theorem 1.1 to the "cubes" $\{\mathbf{k} \in \mathbf{N^r} : \mathbf{l} \leq \mathbf{k} \leq \mathbf{n}\}$, where $\mathbf{1} \leq \mathbf{l} < \mathbf{n}$. Thus we have

$$\begin{aligned} \lambda P\left(\sup_{\mathbf{k}\in[(\mathbf{n})\setminus(\mathbf{m})]} C_{\mathbf{k}} Y_{\mathbf{k}} \geq \lambda\right) &= \lambda P\left(\bigcup_{\mathbf{k}\in[(\mathbf{n})\setminus(\mathbf{m})]} \left[C_{\mathbf{k}} Y_{\mathbf{k}} \geq \lambda\right]\right) \\ &= \lambda P\left(\bigcup_{i=1}^{r} \bigcup_{\mathbf{k}\in D_i} \left[C_{\mathbf{k}} Y_{\mathbf{k}} \geq \lambda\right]\right) \\ &\leq \lambda \sum_{i=1}^{r} P\left(\bigcup_{\mathbf{k}\in D_i} \left[C_{\mathbf{k}} Y_{\mathbf{k}} \geq \lambda\right]\right) \\ &\leq \lambda \sum_{i=1}^{r} P\left(\sup_{\mathbf{k}\in D_i} C_{\mathbf{k}} Y_{\mathbf{k}} \geq \lambda\right) \\ &\leq \sum_{i=1}^{r} \min_{1\leq s\leq r} \sum_{\mathbf{k}\in D_i} \left\{\left(C_{\mathbf{k}} - C_{\mathbf{k};s;k_s+1}\right) E Y_{\mathbf{k}}^{+}\right\} \\ &\leq \min_{1\leq s\leq r} \sum_{i=1}^{r} \sum_{\mathbf{k}\in D_i} \left\{\left(C_{\mathbf{k}} - C_{\mathbf{k};s;k_s+1}\right) E Y_{\mathbf{k}}^{+}\right\} \\ &\leq \min_{1\leq s\leq r} \sum_{\mathbf{k}\in[(\mathbf{n})\setminus(\mathbf{m})]} \left\{\left(C_{\mathbf{k}} - C_{\mathbf{k};s;k_s+1}\right) E Y_{\mathbf{k}}^{+}\right\}. \end{aligned} \tag{2.3}$$

□

The next lemma is an equivalent version of the result obtained by Martikainen [13, page 435] of Kronecker lemma in multidimensional case. Let $\mathbf{l} = \left(l_1, \ldots, l_s\right)$, $\mathbf{m} = \left(m_1, \ldots, m_t\right)$ then $N^{s+t} \ni (\mathbf{l}, \mathbf{m}) := (l_1, \ldots, l_s, m_1, \ldots, m_t)$.

Lemma 2.2. *Let $s, t \geq 1$ be natural numbers, with $s + t = r$ and $\{a_{\mathbf{l}}, \mathbf{l} \in N^s\}, \{b_{\mathbf{m}}, \mathbf{m} \in N^t\}$ families of increasing, positive numbers such that $a_{\mathbf{l}} \to \infty$, $b_{\mathbf{m}} \to \infty$ strongly as* $\mathbf{l}$ *and* $\mathbf{m} \to \infty$. *Furthermore $\{x_{(\mathbf{l},\mathbf{m})}, (\mathbf{l},\mathbf{m}) \in N^r\}$ be an array of positive numbers such that*

$$\sum_{(\mathbf{l},\mathbf{m})\in N^r} \frac{x_{(\mathbf{l},\mathbf{m})}}{a_{\mathbf{l}} b_{\mathbf{m}}} < \infty. \tag{2.4}$$

Then

$$\frac{1}{a_{\mathbf{N}_1}} \sum_{(\mathbf{l},\mathbf{m})\leq(\mathbf{N}_1,\mathbf{N}_2)} \frac{x_{(\mathbf{l},\mathbf{m})}}{b_{\mathbf{m}}} \longrightarrow 0 \quad \textit{strongly as } N^s \ni \mathbf{N}_1 \longrightarrow \infty \tag{2.5}$$

for every $\mathbf{N}_2 \in N^t$.

Proof. By applying the Martikainen lemma to

$$y_{(\mathbf{l},\mathbf{N}_2)} = \sum_{\mathbf{m}\leq\mathbf{N}_2} \frac{x_{(\mathbf{l},\mathbf{m})}}{b_{\mathbf{m}}} \quad \text{where } \mathbf{N}_2 \in N^t. \tag{2.6}$$

□

Lemma 2.3. *Let $(\mathbb{B}, \|\cdot\|)$ be a real separable Banach space and $1 \leq p \leq 2$ and $q \geq 1$, then the following properties are equivalent.*

(i) *$\mathbb{B}$ is R-type p.*

(ii) *There exists positive constant C such that for every $\mathbf{n} \in N^r$ and for any family $\{X_{\mathbf{k}}, \mathbf{k} \in N^r\}$ of independent random vectors in $\mathbb{B}$ with $EX_{\mathbf{k}} = 0$,*

$$E\left\|\sum_{\mathbf{k}\leq\mathbf{n}} X_{\mathbf{k}}\right\|^q \leq CE\left(\sum_{\mathbf{k}\leq\mathbf{n}} \|X_{\mathbf{n}}\|^p\right)^{q/p}. \tag{2.7}$$

Proof. For $r = 1$ (Woyczyński [14]) and since $\{X_{\mathbf{k}}, \mathbf{k} \in N^r\}$ are independent, the lemma is also valid for $r > 1$. □

Theorem 2.4. *Let $1 \leq p \leq 2$, $q > 1$ and $\{X_{\mathbf{k}}, \mathbf{k} \in N^r\}$ be field of independent, zero-mean, $\mathbb{B}$-valued random vectors such that*

$$\min_{1\leq s\leq r} \sum_{\mathbf{k}\in N^r} \frac{E\|X_{\mathbf{k}}\|^{pq}}{k_s |\mathbf{k}|^{pq-q}} < \infty. \tag{2.8}$$

If $\mathbb{B}$ is R-type p, then

$$\frac{S_{\mathbf{n}}}{|\mathbf{n}|} \longrightarrow 0 \quad \textit{strongly a.s. as } \mathbf{n} \longrightarrow \infty. \tag{2.9}$$

Proof. Let $\mathfrak{F}_{\mathbf{n}} = \sigma(X_{\mathbf{k}}, \mathbf{k} \le \mathbf{n})$, since $X_{\mathbf{k}}$ are independent, $\{\mathfrak{F}_{\mathbf{k}}, \mathbf{k} \in N^r\}$ satisfies (1.3) and $\{\|S_{\mathbf{k}}\|^{pq}, \mathfrak{F}_{\mathbf{k}}, \mathbf{k} \in N^r\}$ is real, nonnegative submartingale. By the definition of strong convergence of elements of $\mathbb{B}$ and "event occurs finitely (infinitely) often," it is enough to prove

$$\lim_{N\to\infty} P\left(\sup_{\mathbf{k}\nleq\mathbf{N}} \frac{\|S_{\mathbf{k}}\|}{|\mathbf{k}|} \ge \lambda\right) = 0 \quad \text{where } N^r \ni \mathbf{N} = (N,\dots,N) \tag{2.10}$$

for any $\lambda > 0$. Let us observe, that by Theorem 2.1, Lemma 2.3 and Hölder's inequality for some constants C, we get

$$\begin{aligned}
\lambda^{pq} P\left(\sup_{\mathbf{k}\nleq\mathbf{N}} \frac{\|S_{\mathbf{k}}\|}{|\mathbf{k}|} \ge \lambda\right) &\le \min_{1\le s\le r} \sum_{\mathbf{k}\nleq\mathbf{N}} \left(\frac{1}{|\mathbf{k}|^{pq}} - \frac{1}{(|\mathbf{k}|^{pq}; s; k_s+1)}\right) E\|S_{\mathbf{k}}\|^{pq} \\
&\le C \min_{1\le s\le r} \sum_{\mathbf{k}\nleq\mathbf{N}} \left(\frac{1}{|\mathbf{k}|^{pq}} - \frac{1}{(|\mathbf{k}|^{pq}; s; k_s+1)}\right) E\left(\sum_{\mathbf{j}\le\mathbf{k}} \|X_{\mathbf{j}}\|^{p}\right)^{q} \\
&\le C \min_{1\le s\le r} \sum_{\mathbf{k}\nleq\mathbf{N}} \left(\frac{1}{|\mathbf{k}|^{pq}} - \frac{1}{(\mathbf{k}^{pq}; s; k_s+1)}\right) |\mathbf{k}|^{q-1} \sum_{\mathbf{j}\le\mathbf{k}} E\|X_{\mathbf{j}}\|^{pq} \\
&\le C \min_{1\le s\le r} \sum_{\mathbf{k}\nleq\mathbf{N}} \frac{1}{k_s |\mathbf{k}|^{pq-q+1}} \sum_{\mathbf{j}\le\mathbf{k}} E\|X_{\mathbf{j}}\|^{pq},
\end{aligned} \tag{2.11}$$

where $(|\mathbf{k}|^{\beta}; s; \alpha) := k_1^{\beta} \cdot, \dots, \cdot k_{s-1}^{\beta} \alpha^{\beta} k_{s+1}^{\beta} \cdot, \dots, \cdot k_r^{\beta}$.

Now, it is enough to prove that appropriate multiple series is finite. Changing the order of summation and comparing to integrals, for some constant $C > 0$ and for every s, $1 \le s \le r$, we have

$$\begin{aligned}
\sum_{\mathbf{k}\le\mathbf{N}} \frac{1}{k_s |\mathbf{k}|^{pq-q+1}} \sum_{\mathbf{j}\le\mathbf{k}} E\|X_{\mathbf{j}}\|^{pq} &= \sum_{\mathbf{k}\le\mathbf{N}} E\|X_{\mathbf{k}}\|^{pq} \sum_{\mathbf{k}\le\mathbf{j}\le\mathbf{N}} \frac{1}{j_s |\mathbf{j}|^{pq-q+1}} \\
&\le C \sum_{\mathbf{k}\le\mathbf{N}} E\|X_{\mathbf{k}}\|^{pq} \left(\frac{1}{N^{pq-q+1}} - \frac{1}{k_s^{pq-q+1}}\right) \prod_{i=1, i\ne s}^{r} \left(\frac{1}{N^{pq-q}} - \frac{1}{k_i^{pq-q}}\right).
\end{aligned} \tag{2.12}$$

The above expression contains the following types of sums:

$$\sum_{\mathbf{k}\le\mathbf{N}} \frac{E\|X_{\mathbf{k}}\|^{pq}}{N^{pq-q+1} N^{l(pq-q)} (k_{i_1} \cdot \dots \cdot k_{i_m})^{pq-q}}, \tag{2.13}$$

$$\sum_{\mathbf{k}\le\mathbf{N}} \frac{E\|X_{\mathbf{k}}\|^{pq}}{k_s^{pq-q+1} N^{l(pq-q)} (k_{i_1} \cdot \dots \cdot k_{i_m})^{pq-q}}, \tag{2.14}$$

where $l, m = 0, 1, \dots, r-1$, $l + m = r - 1$, and $\{i_1, \cdot \dots \cdot, i_m\}$ is any subset of $\{1, 2, \dots, r\} \setminus \{s\}$.

Now, by Lemma 2.2, (2.13) tends to 0 as $N \to \infty$ as well as (2.14) for $l \neq 0$. Hence, we have

$$\sum_{\mathbf{k}\in N^r} \frac{1}{k_s|\mathbf{k}|^{pq-q+1}} \sum_{\mathbf{j}\leq\mathbf{k}} E\left\|X_{\mathbf{j}}\right\|^{pq} \leq C \sum_{\mathbf{k}\in N^r} \frac{E\left\|X_{\mathbf{k}}\right\|^{pq}}{k_s|\mathbf{k}|^{pq-q}}. \tag{2.15}$$

We complete the proof by taking the minimum over $s \in \{1,\ldots,r\}$ of both sides of (2.15), combined with (2.8) and (2.11). □

For $r = 1$, we let obtain the following result.

Corollary 2.5. *Let $1 \leq p \leq 2$, $q > 1$ and let $\{X_k,\ k \in N\}$ be sequence of independent, zero-mean, $\mathbb{B}$-valued random vectors such that*

$$\sum_{n=1}^{\infty} \frac{E\left\|X_n\right\|^{pq}}{n^{pq-q+1}} < \infty. \tag{2.16}$$

If $\mathbb{B}$ is R-type p, then

$$\frac{S_n}{n} \longrightarrow 0 \quad \text{a.s. as } n \longrightarrow \infty. \tag{2.17}$$

Corollary 2.5 for $q \geq 1$ is due to Woyczyński [14], which generalized results of Hoffman-Jørgensen, Pisier and Woyczyński [15] ($1 \leq p \leq 2$, $q = 1$), and results due to Brunk [16], Prohorov [17] ($\mathbb{B} = \mathbf{R}$, $p = 2$, $q \geq 1$).

Example 2.6. Let $r = 2$ and let $\{X_{\mathbf{k}},\ \mathbf{k} \in N^2\}$ be a field of random vectors fulfilling the assumptions of Theorem 2.4. For $\theta \in R_+$, we define 2-dimensional sector N_θ^2 as follow:

$$N_\theta^2 = \left\{(k_1,k_2) \in N^2 : 1 \leq k_2 \leq \theta k_1\right\}. \tag{2.18}$$

Assume that $\{E\|X_{\mathbf{n}}\|^{pq},\ \mathbf{n} \in N^2\}$ are uniformly bounded by constant M and $pq - q > 1/2$. Hence by comparison to integrals, we have

$$\sum_{(k_1,k_2)\in N_\theta^2} \frac{E\left\|X_{(k_1,k_2)}\right\|^{pq}}{k_s(k_1k_2)^{pq-q}} \leq M \sum_{(k_1,k_2)\in N_\theta^2} \frac{1}{k_s(k_1k_2)^{pq-q}} < \infty, \tag{2.19}$$

for $s = 1,2$.

Thus, the condition (2.8) of Theorem 2.4 is met and we have

$$\frac{\sum_{N_\theta^2 \ni \mathbf{k}\leq\mathbf{n}} X_{\mathbf{k}}}{|\mathbf{n}|} \longrightarrow 0 \quad \text{strongly a.s. as } \mathbf{n} \longrightarrow \infty. \tag{2.20}$$

In Theorem 2.7, we will give necessary and sufficient probabilistic condition for the geometry of Banach space associated with the above-mentioned strong law. In Theorem 2.12

we will replace geometric condition of Theorem 2.4 mentioned by probabilistic one to obtain SLLN (2.9).

Theorem 2.7. *Let $1 \leq p \leq 2$, $q \geq 1$. The following conditions are equivalent:*

(i) *$\mathbb{B}$ is R-type p.*

(ii) *For every $\lambda > 0$ there exists C_λ such that for any independent, $\mathbb{B}$-valued, zero-mean random vectors $\{X_{\mathbf{k}}, \mathbf{k} \in N^r\}$,*

$$\sum_{\mathbf{k}\in N^r} |\mathbf{k}|^{-1} P\left(\frac{\|S_{\mathbf{k}}\|}{|\mathbf{k}|} \geq \lambda\right) \leq C_\lambda \sum_{\mathbf{k}\in N^r} \frac{E\|X_{\mathbf{k}}\|^{pq}}{|\mathbf{k}|^{pq-q+1}}. \tag{2.21}$$

For $r = 1$, the theorem is due to Woyczyński [14].

Proof. (i)⇒(ii) Using Chebyshev inequality ,Lemma 2.3 and Hölder's inequality ,we have

$$\begin{aligned}
\sum_{\mathbf{k}\in N^r} |\mathbf{k}|^{-1} P\left(\frac{\|S_{\mathbf{k}}\|}{|\mathbf{k}|} \geq \lambda\right) &\leq \lambda^{-pq} \sum_{\mathbf{k}\in N^r} \frac{E\|S_{\mathbf{k}}\|^{pq}}{|\mathbf{k}|^{pq+1}} \\
&\leq C\lambda^{-pq} \sum_{\mathbf{k}\in N^r} |\mathbf{k}|^{-pq+q-2} \sum_{\mathbf{i}\leq\mathbf{k}} E\|X_{\mathbf{i}}\|^{pq} \\
&\leq C\lambda^{-pq} \sum_{\mathbf{k}\in N^r} E\|X_{\mathbf{k}}\|^{pq} \sum_{\mathbf{i}\geq\mathbf{k}} |\mathbf{i}|^{-pq+q-2} \\
&\leq C\lambda^{-pq} \sum_{\mathbf{k}\in N^r} \frac{E\|X_{\mathbf{k}}\|^{pq}}{|\mathbf{k}|^{pq-q+1}}.
\end{aligned} \tag{2.22}$$

(ii)⇒(i) Let $\mathbf{n} = (n_1, \ldots, n_r)$ and let $\{Y_{n_1}, n_1 \geq 1\}$ be an arbitrary sequence of independent random vectors in $\mathbb{B}$, with $EY_{n_1} = 0$ and $T_m = \sum_{n_1=1}^{m} Y_{n_1}$. Set

$$X_{\mathbf{n}} = \begin{cases} Y_{n_1} & \text{for } (n_1, n_2, \ldots, n_r) = (n_1, 1, \ldots, 1), \\ 0 & \text{for } (n_1, n_2, \ldots, n_r) \neq (n_1, 1, \ldots, 1). \end{cases} \tag{2.23}$$

Then

$$\begin{aligned}
\sum_{n_1=1}^{\infty} n_1^{-1} P\left(\frac{\|T_{n_1}\|}{n_1} \geq \lambda\right) &= \sum_{\mathbf{n}\in N^r} |\mathbf{n}|^{-1} P\left(\frac{\|S_{\mathbf{n}}\|}{|\mathbf{n}|} \geq \lambda\right) \\
&\leq C_\lambda \sum_{\mathbf{n}\in N^r} \frac{E\|X_{\mathbf{n}}\|^{pq}}{|\mathbf{n}|^{pq-q+1}} \\
&= C_\lambda \sum_{n_1=1}^{\infty} \frac{E\|Y_{n_1}\|^{pq}}{n_1^{pq-q+1}}.
\end{aligned} \tag{2.24}$$

Thus, (i) follows directly from Theorem 3.1 of Woyczyński [14]. □

Combining Theorem 2.7 with the result of Rosalsky and Van Thanh [18, Theorem 3.1], we get the following corollary.

Corollary 2.8. *Let $1 \le p \le 2$, $q \ge 1$, and let $\mathbb{B}$ be a separable Banach space. If $\{X_{\mathbf{k}}, \mathbf{k} \in N^r\}$ is family of independent, $\mathbb{B}$-valued, zero-mean random vectors, then the following conditions are equivalent.*

(i) *for every $q \ge 1$ and $\lambda > 0$, there exists C_λ such that for any vectors $\{X_{\mathbf{k}}, \mathbf{k} \in N^r\}$,*

$$\sum_{\mathbf{k}\in N^r} |\mathbf{k}|^{-1} P\left(\frac{\|S_{\mathbf{k}}\|}{|\mathbf{k}|} \ge \lambda\right) \le C_\lambda \sum_{\mathbf{k}\in N^r} \frac{E\|X_{\mathbf{k}}\|^{pq}}{|\mathbf{k}|^{pq-q+1}}. \tag{2.25}$$

(ii) *For every random vectors $\{X_{\mathbf{k}}, \mathbf{k} \in N^r\}$, the condition*

$$\sum_{\mathbf{k}\in N^r} \frac{E\|X_{\mathbf{k}}\|^{p}}{|\mathbf{k}|^{p}} < \infty \tag{2.26}$$

implies that the SLLN holds.

Before we state the next theorem, we need more notations and present some useful lemmas. Let $N_0 = (0, 1, 2, \ldots)$ and $2^{\mathbf{n}} = (2^{n_1}, \ldots, 2^{n_r})$, where 2^{-1} is defined as 0 and

$$\text{for } \mathbf{k} < \mathbf{n} \text{ denote } S_{\mathbf{k}}^{\mathbf{n}} = \sum_{\mathbf{k}<\mathbf{j}\le\mathbf{n}} X_{\mathbf{j}}. \tag{2.27}$$

Lemma 2.9 (Fazekas [2, Lemma 2.5]). *Let $\{X_{\mathbf{k}}, \mathbf{k} \in N^r\}$ be independent symmetric $\mathbb{B}$-valued random vectors. Assume that for all $\lambda > 0$,*

$$\begin{gathered} \sum_{\mathbf{k}\in N_0^r} P\left(\frac{\left|\|S_{2^{\mathbf{k}-1}}^{2^{\mathbf{k}}}\| - E\|S_{2^{\mathbf{k}-1}}^{2^{\mathbf{k}}}\|\right|}{|2^{\mathbf{k}-1}|} > \lambda\right) < \infty, \\ E\left(\frac{S_{\mathbf{n}}}{|\mathbf{n}|}\right) \longrightarrow 0 \quad \textit{strongly as } \mathbf{n} \longrightarrow \infty, \end{gathered} \tag{2.28}$$

then SLLN (2.9) *holds.*

Lemma 2.10 (Fazekas [2, Lemma 2.3] with $a_{\mathrm{k}} = |\mathrm{k}|$). *Let $\{X_{\mathbf{k}}, \mathbf{k} \in N^r\}$ be a field of independent, symmetric, $\mathbb{B}$-valued random vectors. If*

$$\begin{gathered} \|X_{\mathbf{k}}\| \le |\mathbf{k}| \quad \textit{a.s. for every } \mathbf{k} \in N^r, \\ \frac{S_{\mathbf{n}}}{|\mathbf{n}|} \longrightarrow 0 \quad \textit{strongly in probability}, \end{gathered} \tag{2.29}$$

then $E\|S_{\mathbf{n}}\|/|\mathbf{n}| \to 0$ strongly as $\mathbf{n} \to \infty$.

Lemma 2.11. *For $q \geq 1$, there exists a positive constant C_q such that for any separable Banach space $\mathbb{B}$ and any finite set $\{X_{\mathbf{k}}, \mathbf{k} \leq \mathbf{n}\}$ of independent $\mathbb{B}$-valued random vectors with $X_{\mathbf{k}} \in L^q$ for all $\mathbf{k} \leq \mathbf{n}$, the following holds.*

For $1 \leq q \leq 2$,

$$E\big|\|S_{\mathbf{n}}\| - E\|S_{\mathbf{n}}\|\big|^q \leq C_q \sum_{\mathbf{k}\leq\mathbf{n}} E\|X_{\mathbf{k}}\|^q, \tag{2.30}$$

if $q = 2$, then it is possible to take $C_2 = 4$.

For $q > 2$,

$$E\big|\|S_{\mathbf{n}}\| - E\|S_{\mathbf{n}}\|\big|^q \leq C_q \left[\left(\sum_{\mathbf{k}\leq\mathbf{n}} E\|X_{\mathbf{k}}\|^2\right)^{q/2} + \sum_{\mathbf{k}\leq\mathbf{n}} E\|X_{\mathbf{k}}\|^q\right]. \tag{2.31}$$

Proof. For $r = 1$, the result is due to de Acosta [19, Theorem 2.1]. Since $\{X_{\mathbf{k}}, \mathbf{k} \in N^r\}$ are independent the theorem is true in the multidimensional case. □

Theorem 2.12. *Let $\{X_{\mathbf{k}}, \mathbf{k} \in N^r\}$ be a family of independent $\mathbb{B}$-valued zero-mean, random vectors and assume $\|S_{\mathbf{n}}\|/|\mathbf{n}| \to 0$ strongly in probability. Then*

(i) *if $1 \leq q \leq 2$, then $\sum_{\mathbf{n}\in N^r} (E\|X_{\mathbf{n}}\|^q/|\mathbf{n}|^q) < \infty$ implies SLLN* (2.9),

(ii) *if $2 \leq q$, then $\sum_{\mathbf{n}\in N^r} (E\|X_{\mathbf{n}}\|^q/|\mathbf{n}|^{q/2+1}) < \infty$ implies SLLN* (2.9).

Theorem 2.12 is multiple sum analogue of a strong law of large numbers, Theorem 3.2 of de Acosta [19].

Proof. (i) Let us assume that $\{X_{\mathbf{k}}, \mathbf{k} \in N^r\}$ are symmetric (desymmetryzation is standard) and put

$$Y_{\mathbf{k}} = X_{\mathbf{k}} I\big(\|X_{\mathbf{k}}\| \leq |\mathbf{k}|\big), \qquad T_{\mathbf{n}} = \sum_{\mathbf{n}\in N^r} Y_{\mathbf{k}}. \tag{2.32}$$

By assumption, it follows that $\sum_{\mathbf{n}\in N^r} P(\|X_{\mathbf{k}}\| \geq |\mathbf{k}|) < \infty$ and by the Borell-Cantelli lemma, it is enough to prove $T_{\mathbf{n}}/|\mathbf{n}| \to 0$ *strongly a.s.* It follows from assumption that

$$\frac{T_{\mathbf{n}}}{|\mathbf{n}|} \longrightarrow 0 \quad \text{strongly in probability}, \tag{2.33}$$

thus by Lemma 2.10

$$E\frac{\|T_{\mathbf{n}}\|}{|\mathbf{n}|} \longrightarrow 0 \quad \text{strongly as } \mathbf{n} \longrightarrow \infty, \tag{2.34}$$

and on the virtue of Lemma 2.9 and the Borell-Cantelli lemma, the proof will be completed if we show that for any $\lambda > 0$,

$$\sum_{\mathbf{n}\in N^r} P\left(\frac{|V_{\mathbf{k}}|}{|2^{\mathbf{k}-1}|} > \lambda\right) < \infty, \quad \text{where } V_{\mathbf{k}} = \left\|T^{2^{\mathbf{k}}}_{2^{\mathbf{k}-1}}\right\| - E\left\|T^{2^{\mathbf{k}}}_{2^{\mathbf{k}-1}}\right\|. \tag{2.35}$$

Now, for any $\lambda > 0$ by Chebyshev inequality and Lemma 2.11, we have

$$\begin{aligned}\sum_{\mathbf{k}\in N^r} P\left(\frac{|V_{\mathbf{k}}|}{|2^{\mathbf{k}-1}|} > \lambda\right) &\le \sum_{\mathbf{k}\in N^r} \frac{E|V_{\mathbf{k}}|^q}{\lambda^q|2^{\mathbf{k}-1}|^q} \\ &\le \frac{C_q 2^{rq}}{\lambda^q} \sum_{\mathbf{k}\in N^r} \sum_{2^{\mathbf{k}-1}\le \mathbf{j}<2^{\mathbf{k}}} \frac{E\|Y_{\mathbf{j}}\|^q}{|\mathbf{j}|^q} \\ &\le \frac{C_q 2^{rq}}{\lambda^q} \sum_{\mathbf{k}\in N^r} \frac{E\|X_{\mathbf{k}}\|^q}{|\mathbf{k}|^q} < \infty.\end{aligned} \tag{2.36}$$

(ii) The same arguments and Hölder's inequality

$$\begin{aligned}\sum_{\mathbf{k}\in N^r} P\left(\frac{|V_{\mathbf{k}}|}{|2^{\mathbf{k}-1}|} > \lambda\right) &\le \frac{C_q}{\lambda_q} \sum_{\mathbf{k}\in N^r} \frac{1}{|2^{\mathbf{k}-1}|^q}\left[\left(\sum_{2^{\mathbf{k}-1}\le \mathbf{j}<2^{\mathbf{k}}} E\|Y_{\mathbf{j}}\|^2\right)^{q/2} + \sum_{2^{\mathbf{k}-1}\le \mathbf{j}<2^{\mathbf{k}}} E\|Y_{\mathbf{j}}\|^q\right] \\ &\le \frac{C_q}{\lambda_q} \sum_{\mathbf{k}\in N^r} \frac{1}{|2^{\mathbf{k}-1}|^q}\left[|2^{\mathbf{k}-1}|^{q/2-1} \sum_{2^{\mathbf{k}-1}\le \mathbf{j}<2^{\mathbf{k}}} E\|Y_{\mathbf{j}}\|^q + \sum_{2^{\mathbf{k}-1}\le \mathbf{j}<2^{\mathbf{k}}} E\|Y_{\mathbf{j}}\|^q\right] \\ &\le \frac{2C_q}{\lambda_q} 2^{r(q/2+1)} \sum_{\mathbf{k}\in N^r} \frac{E\|X_{\mathbf{k}}\|^q}{|\mathbf{k}|^{q/2+1}} < \infty.\end{aligned} \tag{2.37}$$

□

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