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Research Article

On Exponential Dichotomy of Variational Difference Equations

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We give very general characterizations for uniform exponential dichotomy of variational difference equations. We propose a new method in the study of exponential dichotomy based on the convergence of some associated series of nonlinear trajectories. The obtained results are applied to difference equations and also to linear skew-product flows.

1. Introduction

The stability theory of evolution equations received a remarkable contribution when Zabczyk proved the following result (see [1, Theorem 5.1]).

Theorem 1.1. *Let X be a Banach space and let $T \in \mathcal{L}(X)$. If $N : [0,\infty) \to [0,\infty)$ is a continuous strictly increasing convex function with $N(0) = 0$ such that for every $x \in X$ there is $\alpha > 0$ such that*

$$\sum_{n=0}^{\infty} N(\alpha \|T^n x\|) < \infty, \tag{1.1}$$

then the spectral radius $r(T)$ is less than 1*.*

The original proof was based on the Banach-Steinhaus Theorem and on the construction of an auxiliary sequence space associated with the function N. As a consequence of this result it follows that a semigroup of linear operators $\{T(t)\}_{t\geq 0}$ on a Banach space X is exponentially stable (i.e., there are $K, \nu > 0$ such that $\|T(t)\| \leq Ke^{-\nu t}$, for all $t \geq 0$) if and only if there is a continuous strictly increasing convex function $N : [0,\infty) \to [0,\infty)$ with $N(0) = 0$

such that for every $x \in X$ there is $\alpha > 0$ such that

$$\sum_{n=0}^{\infty} N(\alpha \|T(n)x\|) < \infty. \tag{1.2}$$

Thus it was pointed out, for the first time in literature, that an asymptotic property, like stability, can be deduced from the convergence of a series of nonlinear trajectories.

A notable characterization for the stability of difference equations was obtained by Przyłuski and Rolewicz (see [2, Theorems 4.1 and 4.2]), where it is proved that a system of difference equations

$$x_{k+1} = A_k x_k, \quad k \geq k_0 \tag{1.3}$$

on a Banach space X is uniformly exponentially stable if and only if there is $p \in [1, \infty)$ such that for every $x \in X$

$$\sup_{k \geq k_0} \sum_{n=k}^{\infty} \left\| \left(\prod_{i=k}^{n-1} A_i \right) x \right\|^p < \infty. \tag{1.4}$$

This result can be considered the discrete-time counterpart of the stability theorem due to Datko (see [3, Theorem 1 and Remark 3]).

In recent years, a significant progress was made in the study of the qualitative properties of difference equations (see [4–26]). Stability of difference equations was studied from various perspectives, among which we mention the input-output techniques, the property that the trajectories lie in a certain space, the freezing method, the control type approaches (see [14–19]). The discrete input-output methods were extended to the case of exponential expansiveness (see [23]) as well as for uniform and exponential dichotomy (see [4–7, 9, 11, 12, 20–22, 25, 27]) and exponential trichotomy (see [8, 9, 24]).

Exponential dichotomy plays a very important role in the study of the asymptotic behavior of time-varying equations. Starting with classical works in this field (see [28–30]) many research studies have been done to define, characterize and extend diverse concepts of exponential dichotomy for various evolution equations (see [4–12, 20–22, 25, 27, 28, 31, 32]). In the last decades important results on exponential dichotomy in finite dimensional spaces were extended for the infinite dimensional case and valuable applications were provided (see [4–12, 27, 28, 31, 32]). The input-output methods or the so-called Perron type techniques were intensively used in the study of exponential dichotomy, the main idea being to associate with an evolution equation a linear control system and to characterize the existence of exponential dichotomy of the initial equation in terms of the solvability of the associated control system between various Banach sequence or function spaces (see [4–7, 9, 11, 12, 20–22, 25, 27, 28]). When one studies an asymptotic property via input-output methods, the solvability condition relative to the associated linear control system is expressed in terms of the upper boundedness of an associated linear operator or of a family of linear operators, and therefore the arguments involved are strongly connected with the qualitative theory of linear operators. In contrast with the theorems of Perron type, the Zabczyk and Przyluski-Rolewicz type conditions are more direct and the methods are not necessarily reduced only to the behavior of some associated linear operators. Recently, these techniques were applied to the study of exponential stability in [33, 34] and respectively, for exponential instability in [35].

Up to now, there is no Zabczyk-type result in literature concerning the general asymptotic property described by exponential dichotomy. This is motivated by the fact that due to the splitting on each fiber required by any dichotomy property (see e.g., [22, 27, 31]) the results concerning exponential stability and instability (see [33–35]) cannot be applied in order to deduce conditions for the existence of exponential dichotomy, because, generally, the dichotomy projections are not constant. In this framework, the natural question arises whether one can provide a characterization of Zabczyk type such that this is equivalent with the decomposition of the main space into a direct sum of two closed subspaces such that the behavior on these subspaces is modelled by exponential decay backward and forward in time.

The aim of this paper is to provide a complete resolution to this problem. We propose a new method in the study of exponential dichotomy of evolution equations and treat directly the case of variational difference equations. We do not only answer the above questions but also propose an approach at a greater level of generality than ever before. First, using constructive methods, we deduce necessary and sufficient conditions of nonlinear type for the existence of exponential dichotomy of variational difference equations, based on convergence conditions of some associated series. After that, the main results are applied to the study of the existence of exponential dichotomy for two main classes of systems: the difference equations and the linear skew-product flows. We show that the stability theorem of Zabczyk and also the Przyluski-Rolewicz theorem can be extended to the general case of exponential dichotomy and that the convergence of certain series of nonlinear trajectories may provide interesting information concerning the asymptotic behavior of the initial system.

2. Preliminary Results

Let X be a real or complex Banach space, let (Θ, d) be a metric space, and let $\mathcal{E} = X \times \Theta$. The norm on X and on $\mathcal{L}(X)$, the Banach algebra of all bounded linear operators on X, will be denoted by $\|\cdot\|$. Denote by I the identity operator on X and by $\mathcal{B}(\Theta, \mathcal{L}(X))$ the set of all mappings $A : \Theta \to \mathcal{L}(X)$ with $\sup_{\theta\in\Theta}\|A(\theta)\| < \infty$.

Throughout the paper $\mathbb{Z}$ denotes the set of real integers, $\mathbb{Z}_+$ is the set of all $m \in \mathbb{Z}$, $m \geq 0$, and $\mathbb{Z}_-$ is the set of all $m \in \mathbb{Z}$, $m \leq 0$.

Definition 2.1. A mapping $\sigma : \Theta \times \mathbb{Z} \to \Theta$ is called a *discrete flow* on Θ if $\sigma(\theta, 0) = \theta$, for all $\theta \in \Theta$ and $\sigma(\theta, m+n) = \sigma(\sigma(\theta, m), n)$, for all $(\theta, m, n) \in \Theta \times \mathbb{Z}^2$.

Let $A \in \mathcal{B}(\Theta, \mathcal{L}(\mathrm{X}))$. We consider the linear system of variational difference equations:

$$x(\theta)(n+1) = A(\sigma(\theta, n))x(\theta)(n), \quad \forall(\theta, n) \in \Theta \times \mathbb{Z}_+. \tag{A}$$

The *discrete cocycle* associated with the system (A) is

$$\Phi : \Theta \times \mathbb{Z}_+ \longrightarrow \mathcal{L}(\mathrm{X}), \quad \Phi(\theta, n) = \begin{cases} A(\sigma(\theta, n-1)) \cdots A(\theta), & n \geq 1, \\ I, & n = 0. \end{cases} \tag{2.1}$$

Remark 2.2. One has $\Phi(\theta, m+n) = \Phi(\sigma(\theta, n), m)\Phi(\theta, n)$, for all $(\theta, m, n) \in \Theta \times \mathbb{Z}_+^2$.

Definition 2.3. The system (A) is said to be *uniformly exponentially dichotomic* if there are a family of projections $\{P(\theta)\}_{\theta\in\Theta} \subset \mathcal{L}(X)$ and two constants $K \geq 1$ and $\nu > 0$ such that

(i) $A(\theta)P(\theta) = P(\sigma(\theta,1))A(\theta)$, for all $\theta \in \Theta$;

(ii) $\|\Phi(\theta,n)x\| \leq Ke^{-\nu n}\|x\|$, for all $n \in \mathbb{Z}_+$, all $x \in \operatorname{Im}P(\theta)$ and all $\theta \in \Theta$;

(iii) $\|\Phi(\theta,n)y\| \geq (1/K)e^{\nu n}\|y\|$, for all $n \in \mathbb{Z}_+$, all $y \in \operatorname{Ker}P(\theta)$ and all $\theta \in \Theta$;

(iv) for every $\theta \in \Theta$, the restriction $A(\theta) : \operatorname{Ker}P(\theta) \to \operatorname{Ker}P(\sigma(\theta,1))$ is an isomorphism.

Remark 2.4. (i) If for every $\theta \in \Theta$, $P(\theta) = I$, then from Definition 2.3 we obtain the concept of uniform exponential stability.

(ii) If, in Definition 2.3, for every $\theta \in \Theta$, $P(\theta) = 0$, then we obtain the concept of uniform exponential expansiveness.

Remark 2.5. A remarkable progress in the study of the asymptotic behavior of variational equations modelled by cocycles over flows was done due to the work of Pliss and Sell (see [32]). The authors presented a complete study concerning significant robustness properties of linear skew-product semiflows on $W \times \mathcal{W}$, where W is a Banach space and $\mathcal{W}$ is a metric space. For important applications we also refer to the book of Sell and You (see [31]). In [32, Lemma 3.2] (see item (3)), the authors obtained the structure of the dichotomy projections of a linear skew-product semiflow on $W \times \mathcal{K}$, where $\mathcal{K}$ is an invariant set in $\mathcal{W}$ and also proved the invariance properties of the *stable set* and *unstable set* with respect to the linear skew-product flow (see item (4)). The structure of the dichotomy projections represents an important step in the study of an asymptotic property like dichotomy or trichotomy, because this anticipates the expectations concerning the decomposition of the central space at every point.

In what follows, as consequences of Definition 2.3, we will deduce the structure of the dichotomy projections of a system of variational difference equations, using certain invariant subspaces of the space X. Compared with [32] where is deduced a global decomposition of the central space $\mathcal{E}$, we will consider a decomposition of the space X via each fiber determined by the points of the space Θ. For the sake of clarity we will present short proofs of these auxiliary results.

For every $\theta \in \Theta$, we denote by $\mathcal{H}(\theta)$ the linear space of all sequences $\varphi : \mathbb{Z}_- \to X$ with the property that

$$\varphi(m) = A(\sigma(\theta,m-1))\varphi(m-1), \quad \forall m \in \mathbb{Z}_-. \tag{2.2}$$

For every $\theta \in \Theta$ we define *the stable subspace*

$$\mathcal{S}(\theta) = \left\{ x \in X : \sup_{n\in\mathbb{Z}_+}\|\Phi(\theta,n)x\| < \infty \right\} \tag{2.3}$$

and *the unstable subspace*

$$\mathcal{U}(\theta) = \left\{ x \in X : \exists \varphi \in \mathcal{H}(\theta) \text{ with } \varphi(0) = x \text{ and } \sup_{k\in\mathbb{Z}_-}\left\|\varphi(k)\right\| < \infty \right\}. \tag{2.4}$$

Remark 2.6. For every $\theta \in \Theta$, the mapping $n \mapsto \|\Phi(\theta,n)x\|$ is called *the trajectory* determined by $x \in X$ and the mapping $\varphi \in \mathscr{A}(\theta)$ with $\varphi(0) = x$ is called *a negative continuation* started at $x \in X$.

Lemma 2.7. *The following assertions hold:*

(i) $A(\theta)\mathcal{S}(\theta) \subset \mathcal{S}(\sigma(\theta,1))$*, for all* $\theta \in \Theta$*;*

(ii) $A(\theta)\mathcal{U}(\theta) = \mathcal{U}(\sigma(\theta,1))$*, for all* $\theta \in \Theta$*.*

Proof. The assertion (i) is immediate. To prove (ii), let $\theta \in \Theta$ and let $x \in \mathcal{U}(\theta)$. Then, there is $\varphi \in \mathscr{A}(\theta)$ with $\varphi(0) = x$ and $\sup_{k\in\mathbb{Z}_-}\|\varphi(k)\| < \infty$. Let

$$u : \mathbb{Z}_- \cup \{1\} \longrightarrow X, \quad u(k) = \begin{cases} A(\theta)x, & k = 1, \\ \varphi(k), & k \in \mathbb{Z}_-, \end{cases} \tag{2.5}$$

Then $u(j) = A(\sigma(\theta, j-1))u(j-1)$, for all $j \in \mathbb{Z}$, $j \le 1$. Taking $v : \mathbb{Z}_- \to X$, $v(k) = u(k+1)$ it follows that $v \in \mathscr{A}(\sigma(\theta,1))$ and $\sup_{k\in\mathbb{Z}_-}\|v(k)\| < \infty$. This shows that $v(0) = A(\theta)x \in \mathcal{U}(\sigma(\theta,1))$.

Conversely, let $y \in \mathcal{U}(\sigma(\theta,1))$. Then, there is $\psi \in \mathscr{A}(\sigma(\theta,1))$ with $\psi(0) = y$ and $\sup_{k\in\mathbb{Z}_-}\|\psi(k)\| < \infty$. Let $x = \psi(-1)$ and let $\delta : \mathbb{Z}_- \to X$, $\delta(k) = \psi(k-1)$. An easy computation shows that $\delta \in \mathscr{A}(\theta)$ and $\sup_{k\in\mathbb{Z}_-}\|\delta(k)\| < \infty$. This implies that $x = \delta(0) \in \mathcal{U}(\theta)$, so $y = A(\theta)x \in A(\theta)\mathcal{U}(\theta)$. □

Lemma 2.8. *If the system* (A) *is uniformly exponentially dichotomic with respect to the family of projections* $\{P(\theta)\}_{\theta\in\Theta}$*, then* $\sup_{\theta\in\Theta}\|P(\theta)\| < \infty$*.*

Proof. Let $K, \nu > 0$ be given by Definition 2.3 and let $M := \sup_{\theta\in\Theta}\|A(\theta)\|$. For every $(x,\theta) \in X \times \Theta$ and every $n \in \mathbb{N}$ we have that

$$\begin{aligned} \frac{1}{K}\, e^{\nu n}\|(I - P(\theta))x\| &\le \|\Phi(\theta,n)(I - P(\theta))x\| \\ &\le M^n\|x\| + Ke^{-\nu n}\|P(\theta)x\| \\ &\le (M^n + K)\|x\| + Ke^{-\nu n}\|(I - P(\theta))x\|, \end{aligned} \tag{2.6}$$

which implies that

$$\left(e^{2\nu n} - K^2\right)\frac{e^{-\nu n}}{K}\, \|(I - P(\theta))x\| \le (M^n + K)\|x\|, \quad \forall n \in \mathbb{N},\ \forall (x,\theta) \in X \times \Theta. \tag{2.7}$$

Let $h \in \mathbb{N}^*$ be such that $e^{2\nu h} - K^2 > 0$. Setting $\alpha := (e^{2\nu h} - K^2)e^{-\nu h}/K$ and $\delta := (M^h + K)$, from relation (2.7) it follows that $\|(I - P(\theta))x\| \le (\delta/\alpha)\|x\|$, for all $(x,\theta) \in X \times \Theta$. This implies that $\|I - P(\theta)\| \le \delta/\alpha$, for all $\theta \in \Theta$, so $\|P(\theta)\| \le 1 + (\delta/\alpha)$, for all $\theta \in \Theta$ and the proof is complete. □

Lemma 2.9. *If the system* (A) *is uniformly exponentially dichotomic with respect to the families of projections* $\{P(\theta)\}_{\theta\in\Theta}$, *then*

$$ImP(\theta) = \mathcal{S}(\theta), \quad \operatorname{Ker} P(\theta) = \mathcal{U}(\theta), \quad \forall\theta \in \Theta. \tag{2.8}$$

Proof. Let $K, \nu > 0$ be given by Definition 2.3. From Lemma 2.8 we have that $L := \sup_{\theta\in\Theta}\|P(\theta)\| < \infty$. Let $\theta \in \Theta$. Obviously, $\mathrm{Im}P(\theta) \subset \mathcal{S}(\theta)$. Conversely, if $x \in \mathcal{S}(\theta)$, then $\lambda_x := \sup_{\theta\in\mathbb{Z}_+}\|\Phi(\theta, n)x\| < \infty$ and we successively deduce that

$$\begin{aligned}
\|(I - P(\theta))x\| &\le Ke^{-\nu n}\|\Phi(\theta, n)(I - P(\theta))x\| \\
&\le Ke^{-\nu n}(\lambda_x + Ke^{-\nu n}\|P(\theta)x\|) \\
&\le Ke^{-\nu n}(\lambda_x + KL\|x\|), \quad \forall n \in \mathbb{Z}_+.
\end{aligned} \tag{2.9}$$

For $n \to \infty$ in (2.9) it follows that $(I - P(\theta))x = 0$, so $x = \mathrm{Im}P(\theta)x \in \mathrm{Im}P(\theta)$.

Let $y \in \operatorname{Ker} P(\theta)$. Since the system (A) is uniformly exponentially dichotomic, for every $k \in \mathbb{Z}_-$, the restriction $\Phi(\sigma(\theta, k), -k)_| : \operatorname{Ker} P(\sigma(\theta, k)) \to \operatorname{Ker} P(\theta)$ is an isomorphism. We define $\varphi : \mathbb{Z}_- \to X$, $\varphi(k) = \Phi(\sigma(\theta, k), -k)_|^{-1}y$ and using the inequality (iii) from Definition 2.3 it follows that $\|\varphi(k)\| \le Ke^{-\nu k}\|y\| \le K\|y\|$, for all $k \in \mathbb{Z}_-$. An easy computation shows that $\varphi \in \mathcal{A}(\theta)$. Then, using the above estimation we obtain that $y = \varphi(0) \in \mathcal{U}(\theta)$. Conversely, let $z \in \mathcal{U}(\theta)$. Then, there is $\psi \in \mathcal{A}(\theta)$ with $\psi(0) = z$ and $q_x := \sup_{k\in\mathbb{Z}_-}\|\psi(k)\| < \infty$. From

$$\|P(\theta)x\| = \|P(\theta)\psi(0)\| = \|\Phi(\sigma(\theta, j), -j)P(\sigma(\theta, j))\psi(j)\| \le KLq_x e^{\nu j}, \quad \forall j \in \mathbb{Z}_-, \tag{2.10}$$

we deduce that $P(\theta)x = 0$, so $x \in \operatorname{Ker} P(\theta)$. □

Remark 2.10. According to Lemma 2.9 we conclude that for variational difference equations, the family of dichotomy projections is uniquely determined.

3. Exponential Dichotomy of Variational Difference Equations

Let X be a real or complex Banach space, let (Θ, d) be a metric space and let σ be a discrete flow on Θ.

Let $A \in \mathcal{B}(\Theta, \mathcal{L}(X))$. We consider the linear system of variational difference equations:

$$x(\theta)(n + 1) = A(\sigma(\theta, n))x(\theta)(n), \quad \forall(\theta, n) \in \Theta \times \mathbb{Z}_+. \tag{A}$$

In all what follows, we denote by $\mathcal{V}$ the set of all continuous nondecreasing functions $V : \mathbb{R}_+ \to \mathbb{R}_+$ with $V(0) = 0$ and $V(t) > 0$, for all $t > 0$,

Theorem 3.1. *If there are $V \in \mathcal{V}$ and $q > 0$ such that*

$$\sum_{n=0}^{\infty} V(\|\Phi(\theta,n)x\|) \leq qV(\|x\|), \quad \forall x \in \mathcal{S}(\theta),\ \forall \theta \in \Theta \tag{3.1}$$

then there are $K, \nu > 0$ such that

$$\|\Phi(\theta,n)x\| \leq Ke^{-\nu n}\|x\|, \quad \forall n \in \mathbb{N},\ \forall x \in \mathcal{S}(\theta),\ \forall \theta \in \Theta. \tag{3.2}$$

Proof. Let $M := \sup_{\theta\in\Theta}\|A(\theta)\|$ and let $h = [q] + 1$.

Let $\theta \in \Theta$ and let $x \in \mathcal{S}(\theta)$ with $\|x\| = 1$. For every $n \geq h$ we have that

$$\|\Phi(\theta,n)x\| \leq M^h \ \|\Phi(\theta,k)x\|, \quad \forall k \in \{n-h+1,\ldots,n\}. \tag{3.3}$$

Since V is nondecreasing, from relation (3.3) we deduce that

$$hV\left(\frac{\|\Phi(\theta,n)x\|}{M^h}\right) \leq \sum_{k=n-h+1}^{n} V(\|\Phi(\theta,k)x\|) \leq qV(1). \tag{3.4}$$

Since $h > q$ from relation (3.4), it follows that $\|\Phi(\theta,n)x\| \leq M^h$. In addition, we observe that $\|\Phi(\theta,n)x\| \leq M^h$, for all $n \in \{0,\ldots,h-1\}$. Then, setting $\lambda = M^h$ and taking into account that λ does not depend on x, θ, or n, we obtain that

$$\|\Phi(\theta,n)x\| \leq \lambda\|x\|, \quad \forall n \in \mathbb{N},\ \forall x \in \mathcal{S}(\theta),\ \forall \theta \in \Theta. \tag{3.5}$$

Let $p \in \mathbb{N}^*$ be such that

$$p > \frac{qV(1)}{V(1/\lambda e)}. \tag{3.6}$$

Let $\theta \in \Theta$ and let $x \in \mathcal{S}(\theta)$ with $\|x\| = 1$. Using relation (3.5) and Lemma 2.7(i) we deduce that

$$\left\|\Phi(\theta,p)x\right\| \leq \lambda \ \left\|\Phi(\theta,j)x\right\|, \quad \forall j \in \{1,\ldots,p\}, \tag{3.7}$$

which implies that

$$pV\left(\left\|\Phi(\theta,p)\frac{x}{\lambda}\right\|\right) \leq \sum_{j=1}^{p} V(\left\|\Phi(\theta,j)x\right\|) \leq qV(1). \tag{3.8}$$

From relations (3.6) and (3.8) it necessarily follows that

$$\left\|\Phi(\theta,p)x\right\| \leq \frac{1}{e}. \tag{3.9}$$

Taking into account that p does not depend on x or θ, it follows that

$$\left\|\Phi(\theta,p)x\right\| \le \frac{1}{e}\|x\|, \quad \forall x \in \mathcal{S}(\theta),\ \forall\theta \in \Theta. \tag{3.10}$$

Let $K = \lambda e$ and let $\nu = 1/p$. Let $(x,\theta) \in X \times \Theta$ and let $n \in \mathbb{N}$. Then, there are $k \in \mathbb{N}$ and $j \in \{0,\dots,p-1\}$ such that $n = kp + j$. Using relations (3.5) and (3.10) and Lemma 2.7(i) we successively deduce that

$$\|\Phi(\theta,n)x\| \le \lambda \left\|\Phi(\theta,kp)x\right\| \le \lambda e^{-k}\|x\| \le Ke^{-\nu n}\|x\| \tag{3.11}$$

and the proof is complete. □

Theorem 3.2. *If there are $V \in \mathcal{V}$ and $q > 0$ such that*

(i) $V(\|A(\theta)x\|) \ge (1/q)V(\|x\|)$, *for all $x \in \mathcal{U}(\theta)$ and all $\theta \in \Theta$;*

(ii) $\sum_{n=0}^{\infty} V(1/\|\Phi(\theta,n)x\|) \le q\, V(1/\|x\|)$, *for all $x \in \mathcal{U}(\theta) \setminus \{0\}$ and all $\theta \in \Theta$,*

then the following assertions hold:

(i) *for every $\theta \in \Theta$, the restriction $A(\theta)_| : \mathcal{U}(\theta) \to \mathcal{U}(\sigma(\theta,1))$ is an isomorphism;*

(ii) *there are $K, \nu > 0$ such that*

$$\|\Phi(\theta,n)x\| \ge \frac{1}{K}e^{\nu n}\|x\|, \quad \forall n \in \mathbb{N},\ \forall x \in \mathcal{U}(\theta),\ \forall\theta \in \Theta. \tag{3.12}$$

Proof. (i) Let $\theta \in \Theta$. From Lemma 2.7(ii) we have that the restriction $A(\theta)_| : \mathcal{U}(\theta) \to \mathcal{U}(\sigma(\theta,1))$ is correctly defined and it is surjective. To prove the injectivity, let $x \in \operatorname{Ker} A(\theta)$. Then, from our first hypothesis, we deduce that $V(\|x\|) = 0$. Since $V \in \mathcal{V}$, this implies that $x = 0$, and so $A(\theta)$ is also injective.

(ii) Let $M := \sup_{\theta\in\Theta}\|A(\theta)\|$ and let $h = [q] + 1$.

Let $\theta \in \Theta$ and let $x \in \mathcal{U}(\theta)$ with $\|x\| = 1$. From (i) we have that $\Phi(\theta,n)x \neq 0$, for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. Then

$$\|\Phi(\theta,k)x\| \le M^h\|\Phi(\theta,n)x\|, \quad \forall k \in \{n,\dots,n+h-1\}. \tag{3.13}$$

We set $\lambda = M^h$, and since V is nondecreasing, from relation (3.13) we obtain that

$$hV\left(\frac{1}{\lambda\|\Phi(\theta,n)x\|}\right) \le \sum_{k=n}^{n+h-1} V\left(\frac{1}{\|\Phi(\theta,k)x\|}\right) \le qV(1). \tag{3.14}$$

This inequality implies that $\|\Phi(\theta,n)x\| \ge (1/\lambda)$. Since λ does not depend on x, θ, or n, we deduce that

$$\|\Phi(\theta,n)x\| \ge \frac{1}{\lambda}\|x\|, \quad \forall n \in \mathbb{N},\ \forall x \in \mathcal{U}(\theta),\ \forall\theta \in \Theta. \tag{3.15}$$

Let $p \in \mathbb{N}^*$ be such that

$$p > \frac{qV(1)}{V(1/\lambda e)}. \tag{3.16}$$

Let $\theta \in \Theta$ and let $x \in \mathcal{U}(\theta)$ with $\|x\| = 1$. Using Lemma 2.7 and relation (3.15) we have that

$$\|\Phi(\theta, p)x\| \geq \frac{1}{\lambda}\|\Phi(\theta, j)x\|, \quad \forall j \in \{1, \ldots, p\}. \tag{3.17}$$

Then, we obtain that

$$pV\left(\frac{1}{\lambda \, \|\Phi(\theta, p)x\|}\right) \leq \sum_{j=1}^{p} V\left(\frac{1}{\|\Phi(\theta, j)x\|}\right) \leq qV(1). \tag{3.18}$$

From relations (3.16) and (3.18) it follows that $\|\Phi(\theta, p)x\| \geq e$. Since p does not depend on x or θ we deduce that

$$\|\Phi(\theta, p)x\| \geq e\|x\|, \quad \forall x \in \mathcal{U}(\theta), \ \forall \theta \in \Theta. \tag{3.19}$$

Let $K = \lambda e$ and let $\nu = 1/p$. Let $(x, \theta) \in X \times \Theta$ and let $n \in \mathbb{N}$. Then, there are $k \in \mathbb{N}$ and $j \in \{0, \ldots, p-1\}$ such that $n = kp + j$. From relations (3.15) and (3.19) and using Lemma 2.7 we have that

$$\|\Phi(\theta, n)x\| \geq \frac{1}{\lambda} \ \|\Phi(\theta, kp)x\| \geq \frac{1}{\lambda}e^k\|x\| \geq \frac{1}{K}e^{\nu n}\|x\|. \tag{3.20}$$

□

The main result of this section is the following theorem.

Theorem 3.3. *The system (A) is uniformly exponentially dichotomic if and only if $\mathcal{S}(\theta) + \mathcal{U}(\theta) = X$, for all $\theta \in \Theta$, and there are $V \in \mathcal{V}$ and $q > 0$ such that the following assertions hold:*

(i) $\sum_{n=0}^{\infty} V(\|\Phi(\theta, n)x\|) \leq qV(\|x\|)$, *for all $x \in \mathcal{S}(\theta)$ and all $\theta \in \Theta$;*

(ii) $V(\|A(\theta)x\|) \geq (1/q)V(\|x\|)$, *for all $x \in \mathcal{U}(\theta)$ and all $\theta \in \Theta$;*

(iii) $\sum_{n=0}^{\infty} V(1/\|\Phi(\theta, n)x\|) \leq q \ V(1/\|x\|)$, *for all $x \in \mathcal{U}(\theta) \setminus \{0\}$ and all $\theta \in \Theta$.*

Proof. Necessity

Let $\{P(\theta)\}_{\theta \in \Theta}$ be the family of projections and let $K, \nu > 0$ be two constants given by Definition 2.3. From Lemma 2.9 we have that $\operatorname{Im} P(\theta) = \mathcal{S}(\theta)$ and $\operatorname{Ker} P(\theta) = \mathcal{U}(\theta)$, so $\mathcal{S}(\theta) + \mathcal{U}(\theta) = X$, for all $\theta \in \Theta$. Taking $V : \mathbb{R}_+ \to \mathbb{R}_+, V(t) = t$, it is easy to observe that the inequalities (i)–(iii) hold for $q = K/(1 - e^{-\nu})$.

Sufficiency

From Theorems 3.1 and 3.2 we have that there are $K, \nu > 0$ such that

$$\|\Phi(\theta, n)x\| \leq K e^{-\nu n}\|x\|, \quad \forall n \in \mathbb{N},\ \forall x \in \mathcal{S}(\theta),\ \forall \theta \in \Theta, \tag{3.21}$$

$$\|\Phi(\theta, n)x\| \geq \frac{1}{K}\ e^{\nu n}\|x\|, \quad \forall n \in \mathbb{N},\ \forall x \in \mathcal{U}(\theta),\ \forall \theta \in \Theta. \tag{3.22}$$

From relations (3.21) and (3.22) it follows that

$$\mathcal{S}(\theta) \cap \mathcal{U}(\theta) = \{0\}, \quad \forall \theta \in \Theta. \tag{3.23}$$

Step 1. We prove that $\mathcal{S}(\theta)$ is closed, for all $\theta \in \Theta$.

Let $\theta \in \Theta$ and let $(x_n) \subset \mathcal{S}(\theta)$ with $x_n \to x$ as $n \to \infty$. For every $h \in \mathbb{N}$ we have that

$$\sum_{j=0}^{h} V\left(\|\Phi(\theta, j)x_n\|\right) \leq qV(\|x_n\|), \quad \forall n \in \mathbb{N}. \tag{3.24}$$

For $n \to \infty$ in (3.24), since V is continuous, we deduce that

$$\sum_{j=0}^{h} V\left(\|\Phi(\theta, j)x\|\right) \leq qV(\|x\|), \quad \forall h \in \mathbb{N}, \tag{3.25}$$

which implies that

$$\sum_{j=0}^{\infty} V\left(\|\Phi(\theta, j)x\|\right) \leq qV(\|x\|). \tag{3.26}$$

From (3.26) we obtain that $V(\|\Phi(\theta, j)x\|) \to 0$, as $j \to \infty$. In particular, there is $n_0 \in \mathbb{N}$ such that $V(\|\Phi(\theta, j)x\|) < V(1)$, for all $j \geq n_0$. Since V is nondecreasing it follows that $\|\Phi(\theta, j)x\| \leq 1$, for all $j \geq n_0$. This implies that $\sup_{n\in\mathbb{N}}\|\Phi(\theta, n)x\| < \infty$, so $x \in \mathcal{S}(\theta)$.

Step 2. We show that $\mathcal{U}(\theta)$ is closed, for all $\theta \in \Theta$.

Let $\theta \in \Theta$ and let $(y_n) \subset \mathcal{U}(\theta)$ with $y_n \to y$ as $n \to \infty$. Then, for every $n \in \mathbb{N}$ there is $\varphi_n \in \mathcal{H}(\theta)$ with $\varphi_n(0) = y_n$ and $\sup_{k\in\mathbb{Z}_-}\|\varphi_n(k)\| < \infty$.

It is easy to see that $\varphi_n(j) \in \mathcal{U}(\sigma(\theta, j))$, for every $j \in \mathbb{Z}_-$ and every $n \in \mathbb{N}$. Then, from relation (3.22) we have that

$$\begin{aligned}\|y_n - y_m\| &= \left\|\Phi(\sigma(\theta, j), -j)(\varphi_n(j) - \varphi_m(j))\right\| \\ &\geq \frac{1}{K}\ e^{-\nu j}\left\|\varphi_n(j) - \varphi_m(j)\right\|, \quad \forall j \in \mathbb{Z}_-,\ \forall m, n \in \mathbb{N}.\end{aligned} \tag{3.27}$$

From relation (3.27) we deduce that for every $j \in \mathbb{Z}_-$, the sequence $(\varphi_n(j))$ is convergent. Taking

$$\varphi : \mathbb{Z}_- \longrightarrow X, \qquad \varphi(j) = \lim_{n\to\infty} \varphi_n(j) \tag{3.28}$$

from $\varphi_n \in \mathcal{A}(\theta)$, for all $n \in \mathbb{N}$, it follows that $\varphi \in \mathcal{A}(\theta)$. Moreover

$$\varphi(0) = \lim_{n\to\infty} \varphi_n(0) = \lim_{n\to\infty} y_n = y. \tag{3.29}$$

Let $n \in \mathbb{N}$. For $m \to \infty$ in (3.27) we have that

$$\|\varphi_n(j) - \varphi(j)\| \leq K e^{\nu j} \|y_n - y\| \leq K \|y_n - y\|, \quad \forall j \in \mathbb{Z}_- \tag{3.30}$$

which implies that

$$\|\varphi(j)\| \leq \sup_{j\in\mathbb{Z}_-} \|\varphi_n(j)\| + K\|y_n - y\|, \quad \forall j \in \mathbb{Z}_-, \tag{3.31}$$

Since $\varphi \in \mathcal{A}(\theta)$, from relation (3.31), we obtain that $y = \varphi(0) \in \mathcal{U}(\theta)$, and so $\mathcal{U}(\theta)$ is closed.

According to our hypothesis, relation (3.23), and *Steps 1* and 2 it follows that

$$\mathcal{S}(\theta) \oplus \mathcal{U}(\theta) = X, \quad \forall \theta \in \Theta. \tag{3.32}$$

Then, for every $\theta \in \Theta$ there is a projection $P(\theta)$ with $\operatorname{Im} P(\theta) = \mathcal{S}(\theta)$ and $\operatorname{Ker} P(\theta) = \mathcal{U}(\theta)$. From Lemma 2.7 we have that $A(\theta)P(\theta) = P(\sigma(\theta,1))A(\theta)$, for all $\theta \in \Theta$. Finally, from relations (3.21) and (3.22) and Theorem 3.2(i) we conclude that the system (A) is uniformly exponentially dichotomic. □

Remark 3.4. The proof of the sufficiency part of Theorem 3.3 shows that conditions (i)–(iii) imply for each $\theta \in \Theta$ the closure of the subspaces $\mathcal{S}(\theta)$ and $\mathcal{U}(\theta)$ as well as the fact that at every point $\theta \in \Theta$, their intersection contains only the zero vector, which are specific properties of the dichotomic behavior. These facts together with hypothesis $\mathcal{S}(\theta) + \mathcal{U}(\theta) = X$, for all $\theta \in \Theta$ led to the existence of the family of projections.

Corollary 3.5. *The system (A) is uniformly exponentially dichotomic if and only if $\mathcal{S}(\theta) + \mathcal{U}(\theta) = X$, for all $\theta \in \Theta$, and there are $p \in [1,\infty)$ and $L > 0$ such that*

(i) $\sum_{n=0}^{\infty} \|\Phi(\theta,n)x\|^p \leq L\|x\|^p$, *for all $x \in \mathcal{S}(\theta)$ and all $\theta \in \Theta$;*

(ii) $\|A(\theta)x\|^p \geq (1/L)\|x\|^p$, *for all $x \in \mathcal{U}(\theta)$ and all $\theta \in \Theta$;*

(iii) $\sum_{n=0}^{\infty}(1/\|\Phi(\theta,n)x\|^p) \leq L/\|x\|^p$, *for all $x \in \mathcal{U}(\theta) \setminus \{0\}$ and all $\theta \in \Theta$.*

Proof. This immediately follows from Theorem 3.3 for $V(t) = t^p$. □

Remark 3.6. Generally, the necessity part of Theorem 3.3 does not hold for any function $V \in \mathcal{V}$. More exactly, if (A) is uniformly exponentially dichotomic and $V \in \mathcal{V}$ is an arbitrary function it does not follow that all the conditions (i)–(iii) are fulfilled. This fact is illustrated by the following example.

Example 3.7. Let Y be a Banach space and let $X = Y \times Y$, which is a Banach space with respect to the norm $\|(x_1, x_2)\| = \|x_1\|_Y + \|x_2\|_Y$. We consider the operator

$$A : X \longrightarrow X, \quad A(x_1, x_2) = \left(\frac{1}{e}\, x_1, e x_2\right). \tag{3.33}$$

Let (Θ, d) be a metric space and let σ be a discrete flow on Θ. For every $\theta \in \Theta$, let $A(\theta) = A$. The discrete cocycle associated with the system (A) is

$$\Phi(\theta, n)(x_1, x_2) = \left(e^{-n} x_1, e^{n} x_2\right), \quad \forall (\theta, n) \in \Theta \times \mathbb{N}, \ \forall x = (x_1, x_2) \in X. \tag{3.34}$$

It is easy to see that the system (A) is uniformly exponentially dichotomic and

$$\mathcal{S}(\theta) = \{(x_1, 0) : x_1 \in Y\}, \quad \forall \theta \in \Theta, \tag{3.35}$$

and, respectively,

$$\mathcal{U}(\theta) = \{(0, x_2) : x_2 \in Y\}, \quad \forall \theta \in \Theta. \tag{3.36}$$

We consider the function

$$V : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \quad V(t) = \begin{cases} 0, & t = 0, \\ -\dfrac{1}{\ln t}, & t \in \left(0, \dfrac{1}{e}\right], \\ et, & t \geq \dfrac{1}{e}, \end{cases} \tag{3.37}$$

and it is obvious that $V \in \mathcal{V}$. Let $\theta \in \Theta$ and let $x \in \mathcal{S}(\theta) \setminus \{0\}$. Then, there is $x_1 \in Y \setminus \{0\}$ such that $x = (x_1, 0)$. From relation (3.34) we deduce that

$$\|\Phi(\theta, n) x\| = e^{-n} \|x_1\|_Y, \quad \forall n \in \mathbb{N}. \tag{3.38}$$

Let $n_0 \in \mathbb{N}^*$ be such that $e^{-n}\|x_1\| < e^{-1}$, for all $n \geq n_0$. Then, we have that

$$\sum_{n=n_0}^{\infty} V(\|\Phi(\theta, n) x\|) = \sum_{n=n_0}^{\infty} \frac{1}{n - \ln \|x_1\|_Y} = \infty. \tag{3.39}$$

It follows that

$$\sum_{n=0}^{\infty} V(\|\Phi(\theta, n) x\|) = \infty, \quad \forall x \in \mathcal{S}(\theta) \setminus \{0\}, \ \forall \theta \in \Theta, \tag{3.40}$$

which shows that relation (i) does not hold, even if (A) is uniformly exponentially dichotomic.

4. Consequences for Two Classes of Evolution Equations

In what follows, by applying the main results from the previous section we will deduce characterizations of Zabczyk type and, respectively, of Przyluski-Rolewicz type for the property of uniform exponential dichotomy for two classes of evolution equations: the difference equations and the linear skew-product flows.

4.1. The Case of Difference Equations

First, we will apply the main results in order to deduce characterizations of uniform exponential dichotomy of difference equations in terms of the convergence of some associated series of nonlinear trajectories.

Let X be a real or complex Banach space and let I denote the identity operator on X. Let $\mathcal{B}(\mathbb{Z}, \mathcal{L}(X)) = \{A : \mathbb{Z} \to \mathcal{L}(X) \mid \sup_{n\in\mathbb{Z}} \|A(n)\| < \infty\}$.

Let $A \in \mathcal{B}(\mathbb{Z}, \mathcal{L}(X))$. We consider the linear system of difference equations:

$$x(n+1) = A(n)x(n), \quad n \in \mathbb{Z}. \tag{A}$$

Let $\Delta = \{(m,n) \in \mathbb{Z}\times\mathbb{Z} : m \geq n\}$ and let $\Phi = \{\Phi(m,n)\}_{(m,n)\in\Delta}$ be the discrete evolution operator associated with (A), that is,

$$\Phi(m,n) := \begin{cases} A(m-1)\cdots A(n), & m > n, \\ I, & m = n. \end{cases} \tag{4.1}$$

Remark 4.1. One has $\Phi(m,n)\Phi(n,k) = \Phi(m,k)$, for all $(m,n), (n,k) \in \Delta$.

Definition 4.2. The system (A) is said to be *uniformly exponentially dichotomic* if there are a family of projections $\{P(n)\}_{n\in\mathbb{Z}}$ and two constants $K, \nu > 0$ such that the following properties are satisfied:

(i) $A(n)P(n) = P(n+1)A(n)$, for all $n \in \mathbb{Z}$;

(ii) $\|\Phi(m,n)x\| \leq Ke^{-\nu(m-n)}\|x\|$, for all $x \in \operatorname{Im} P(n)$ and all $(m,n) \in \Delta$;

(iii) $\|\Phi(m,n)y\| \geq (1/K)e^{\nu(m-n)}\|y\|$, for all $y \in \operatorname{Ker} P(n)$ and all $(m,n) \in \Delta$;

(iv) for every $n \in \mathbb{Z}$, the restriction $A(n)_| : \operatorname{Ker} P(n) \to \operatorname{Ker} P(n+1)$ is an isomorphism.

For every $k \in \mathbb{Z}$ we denote by $\mathcal{H}(k)$ the linear space of all sequences $\varphi : \mathbb{Z}_- \to X$ with the property

$$\varphi(m) = A(k+m-1)\varphi(m-1), \quad \forall m \in \mathbb{Z}_-. \tag{4.2}$$

For every $k \in \mathbb{Z}$, we consider *the stable subspace*

$$X_s(k) = \left\{x \in X : \sup_{n\geq k}\|\Phi(n,k)x\| < \infty\right\} \tag{4.3}$$

and *the unstable subspace*

$$X_u(k) = \left\{ x \in X : \exists \varphi \in \mathcal{A}(k) \text{ with } \varphi(0) = x \text{ and } \sup_{j \in \mathbb{Z}_-} \|\varphi(j)\| < \infty \right\}. \tag{4.4}$$

We denote by $\mathcal{V}$ the set of all continuous nondecreasing functions $V : \mathbb{R}_+ \to \mathbb{R}_+$ with $V(0) = 0$ and $V(t) > 0$, for all $t > 0$.

As an application of the results from the previous section, we obtain the folowing.

Corollary 4.3. *The system (A) is uniformly exponentially dichotomic if and only if $X_s(k) + X_u(k) = X$, for all $k \in \mathbb{Z}$ and there are $V \in \mathcal{V}$ and $q > 0$ such that the following assertions hold:*

(i) $\sum_{n=k}^{\infty} V(\|\Phi(n,k)x\|) \leq qV(\|x\|)$, *for all $x \in X_s(k)$ and all $k \in \mathbb{Z}$;*

(ii) $V(\|A(k)x\|) \geq (1/q)V(\|x\|)$, *for all $x \in X_u(k)$ and all $k \in \mathbb{Z}$;*

(iii) $\sum_{n=k}^{\infty} V(1/\|\Phi(n,k)x\|) \leq qV(1/\|x\|)$, *for all $x \in X_u(k) \setminus \{0\}$ and all $k \in \mathbb{Z}$.*

Proof. Let $\Theta = \mathbb{Z}$, let d be the Euclidean metric on $\mathbb{Z}$ and let $\sigma : \Theta \times \mathbb{Z} \to \Theta$, $\sigma(\theta, m) = \theta + m$. We have that σ is a discrete flow on Θ. We observe that

$$A(\sigma(k, n-1)) \cdots A(k) = \Phi(n+k, k), \quad \forall k \in \Theta,\ \forall n \in \mathbb{Z}_+. \tag{4.5}$$

Since $A \in \mathcal{B}(\mathbb{Z}, \mathcal{L}(X))$, by applying Theorem 3.3 we obtain the conclusion. □

Remark 4.4. Theorem 4.12 is a Zabzcyk type theorem for the case of exponential dichotomy.

Corollary 4.5. *The system (A) is uniformly exponentially dichotomic if and only if $X_s(k) + X_u(k) = X$, for all $k \in \mathbb{Z}$, and there are $p \in [1, \infty)$ and $L > 0$ such that the following assertions hold:*

(i) $\sum_{n=k}^{\infty} \|\Phi(n,k)x\|^p \leq L\|x\|^p$, *for all $x \in X_s(k)$ and all $k \in \mathbb{Z}$;*

(ii) $\|A(k)x\|^p \geq (1/L)\|x\|^p$, *for all $x \in X_u(k)$ and all $k \in \mathbb{Z}$;*

(iii) $\sum_{n=k}^{\infty} 1/\|\Phi(n,k)x\|^p \leq L(1/\|x\|^p)$, *for all $x \in X_u(k) \setminus \{0\}$ and all $k \in \mathbb{Z}$.*

Proof. This follows from Corollary 4.3 for $V(t) = t^p$. □

Remark 4.6. Corollary 4.5 shows that the characterizations of Przyluski-Rolewicz type for uniform exponential stability can be extended to the case of uniform exponential dichotomy.

4.2. The Case of Linear Skew-Product Flows

In what follows, by applying the results from the third section we obtain necessary and sufficient conditions of Zabczyk type for the study of the existence of exponential dichotomy of linear skew-product flows.

Let X be a real or complex Banach space and let I be the identity operator on X. Let (Θ, d) be a metric space and let $\mathcal{E} = X \times \Theta$.

Definition 4.7. A continuous mapping $\sigma : \Theta \times \mathbb{R} \to \Theta$ is called a *flow* on Θ if $\sigma(\theta, 0) = \theta$ and $\sigma(\theta, s+t) = \sigma(\sigma(\theta, s), t)$, for all $(\theta, s, t) \in \Theta \times \mathbb{R}^2$.

Definition 4.8. A pair $\pi = (\Phi, \sigma)$ is called a *linear skew-product flow* on $\mathcal{E}$ if σ is a flow on Θ and $\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{L}(X)$ satisfies the following conditions:

(i) $\Phi(\theta, 0) = I$, for all $\theta \in \Theta$;

(ii) $\Phi(\theta, s+t) = \Phi(\sigma(\theta, s), t)\Phi(\theta, s)$, for all $(\theta, t, s) \in \Theta \times \mathbb{R}_+^2$ (*the cocycle identity*);

(iii) there are $M \geq 1$ and $\omega > 0$ such that $\|\Phi(\theta, t)\| \leq Me^{\omega t}$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$.

For examples of skew-product flows we refer, for example, to [27, 31–35].

Remark 4.9. If $\pi = (\Phi, \sigma)$ is a linear skew-product flow on $\mathcal{E}$, then one associates with π the linear system of variational difference equations:

$$x(\theta)(n+1) = \Phi(\sigma(\theta, n), 1)x(\theta)(n), \quad \forall(\theta, n) \in \Theta \times \mathbb{Z}_+. \qquad (A_\pi)$$

The discrete cocycle associated with the system (A_π) is $\Phi_d(\theta, n) = \Phi(\theta, n)$, for all $(\theta, n) \in \Theta \times \mathbb{Z}_+$.

Definition 4.10. A linear skew-product flow $\pi = (\Phi, \sigma)$ is said to be *uniformly exponentially dichotomic* if there exist a family $\{P(\theta)\}_{\theta \in \Theta} \subset \mathcal{L}(X)$ and two constants $K \geq 1$, $\nu > 0$ such that the following properties hold:

(i) $\Phi(\theta, t)P(\theta) = P(\sigma(\theta, t))\Phi(\theta, t)$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$;

(ii) $\|\Phi(\theta, t)x\| \leq Ke^{-\nu t}\|x\|$, for all $x \in \operatorname{Im} P(\theta)$, all $\theta \in \Theta$ and all $t \geq 0$;

(iii) $\|\Phi(\theta, t)y\| \geq (1/K)\ e^{\nu t}\|y\|$, for all $y \in \operatorname{Ker} P(\theta)$, all $\theta \in \Theta$ and all $t \geq 0$;

(iv) for every $(\theta, t) \in \Theta \times \mathbb{R}_+$ the operator $\Phi(\theta, t) : \operatorname{Ker} P(\theta) \to \operatorname{Ker} P(\sigma(\theta, t))$ is an isomorphism.

Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E}$. For every $\theta \in \Theta$ we denote by $\mathcal{Q}(\theta)$ the linear space of all functions $f : \mathbb{R}_- \to X$ with

$$f(t) = \Phi(\sigma(\theta, s), t-s)f(s), \quad \forall s \leq t \leq 0. \tag{4.6}$$

For every $\theta \in \Theta$ we consider the *stable subspace*

$$\mathcal{S}(\theta) = \left\{x \in X : \sup_{t \geq 0}\|\Phi(\theta, t)x\| < \infty\right\} \tag{4.7}$$

and the *unstable subspace*

$$\mathcal{U}(\theta) = \left\{x \in X : \exists\varphi \in \mathcal{Q}(\theta) \text{ with } \varphi(0) = x \text{ and } \sup_{s \leq 0}\left\|\varphi(s)\right\| < \infty\right\}. \tag{4.8}$$

Proposition 4.11. *Let $\theta \in \Theta$. If $\mathcal{S}_d(\theta)$ and $\mathcal{U}_d(\theta)$ are the stable and the unstable subspaces corresponding to the system (A_π), then*

$$\mathcal{S}(\theta) = \mathcal{S}_d(\theta), \qquad \mathcal{U}(\theta) = \mathcal{U}_d(\theta). \tag{4.9}$$

Proof. The inclusions $\mathcal{S}(\theta) \subseteq \mathcal{S}_d(\theta)$ and $\mathcal{U}(\theta) \subseteq \mathcal{U}_d(\theta)$ are obvious and the inclusion $\mathcal{S}_d(\theta) \subseteq \mathcal{S}(\theta)$ is immediate using the property (iii) from Definition 4.8.

Let $x \in \mathcal{U}_d(\theta)$ and let $\varphi \in \mathcal{A}(\theta)$ with $\varphi(0) = x$ and $\sup_{j\in\mathbb{Z}_-} \|\varphi(j)\| < \infty$. We consider the function

$$\psi : \mathbb{R}_- \longrightarrow X, \qquad \psi(t) = \Phi(\sigma(\theta, [t]), t - [t])\varphi([t]). \tag{4.10}$$

It is easy to see that $\psi \in \mathcal{Q}(\theta)$ and $\sup_{s\leq 0}\|\psi(s)\| < \infty$. This implies that $x = \psi(0) \in \mathcal{U}(\theta)$, and so $\mathcal{U}(\theta) = \mathcal{U}_d(\theta)$. □

The connection between the uniform exponential dichotomy of a linear skew-product flow $\pi = (\Phi, \sigma)$ and the uniform exponential dichotomy of the variational difference equation (A_π) is given by the following theorem. For the proof we refer to [27, Theorem 3.1].

Theorem 4.12. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E}$ and let (A_π) be the linear system of variational difference equations associated with π. Then, π is uniformly exponentially dichotomic if and only if the system (A_π) is uniformly exponentially dichotomic.*

We denote by $\mathcal{V}$ the set of all continuous nondecreasing functions $V : \mathbb{R}_+ \to \mathbb{R}_+$ with $V(0) = 0$ and $V(t) > 0$, for all $t > 0$.

Corollary 4.13. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E}$. Then π is uniformly exponentially dichotomic if and only if $\mathcal{S}(\theta) + \mathcal{U}(\theta) = X$, for all $\theta \in \Theta$ and there are $V \in \mathcal{V}$ and $q > 0$ such that the following assertions hold:*

(i) $\sum_{n=0}^{\infty} V(\|\Phi(\theta, n)x\|) \leq qV(\|x\|)$, *for all $x \in \mathcal{S}(\theta)$ and all $\theta \in \Theta$;*

(ii) $V(\|\Phi(\theta, 1)x\|) \geq (1/q)V(\|x\|)$, *for all $x \in \mathcal{U}(\theta)$ and all $\theta \in \Theta$;*

(iii) $\sum_{n=0}^{\infty} V(1/\|\Phi(\theta, n)x\|) \leq qV(1/\|x\|)$, *for all $x \in \mathcal{U}(\theta) \setminus \{0\}$ and all $\theta \in \Theta$.*

Proof. This proof follows from Theorem 3.3, Proposition 4.11 and Theorem 4.12. □

Corollary 4.14. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E}$. Then π is uniformly exponentially dichotomic if and only if $\mathcal{S}(\theta) + \mathcal{U}(\theta) = X$, for all $\theta \in \Theta$ and there are $p \in [1, \infty)$ and $L > 0$ such that*

(i) $\sum_{n=0}^{\infty} \|\Phi(\theta, n)x\|^p \leq L\|x\|^p$, *for all $x \in \mathcal{S}(\theta)$ and all $\theta \in \Theta$;*

(ii) $\|\Phi(\theta, 1)x\|^p \geq (1/L)\|x\|^p$, *for all $x \in \mathcal{U}(\theta)$ and all $\theta \in \Theta$;*

(iii) $\sum_{n=0}^{\infty} 1/\|\Phi(\theta, n)x\|^p \leq L/\|x\|^p$, *for all $x \in \mathcal{U}(\theta) \setminus \{0\}$ and all $\theta \in \Theta$.*

Proof. This follows from Corollary 4.13 for $V(t) = t^p$. □

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