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Research Article

Stability Results of a Class of Hybrid Systems under Switched Continuous-Time and Discrete-Time Control

M. De la Sen[1] and A. Ibeas[2]

[1] *Department of Electricity and Electronics, Faculty of Science and Technology, University of the Basque Country, Campus of Leioa (Bizkaia), Aptdo, 644-Bilbao, Spain*
[2] *Department of Telecommunication and Systems Engineering, Engineering School, Autonomous University of Barcelona, Cerdanyola del Vallés, Bellaterra, 08193 Barcelona, Spain*

Correspondence should be addressed to M. De la Sen, manuel.delasen@ehu.es

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This paper investigates the stability properties of a class of switched systems possessing several linear time-invariant parameterizations (or configurations) which are governed by a switching law. It is assumed that the parameterizations are stabilized individually via an appropriate linear state or output feedback stabilizing controller whose existence is first discussed. A main novelty with respect to previous research is that the various individual parameterizations might be continuous-time, discrete-time, or mixed so that the whole switched system is a hybrid continuous/discrete dynamic system. The switching rule governs the choice of the parameterization which is active at each time interval in the switched system. Global asymptotic stability of the switched system is guaranteed for the case when a common Lyapunov function exists for all the individual parameterizations and the sampling period of the eventual discretized parameterizations taking part of the switched system is small enough. Some extensions are also investigated for controlled systems under decentralized or mixed centralized/decentralized control laws which stabilize each individual active parameterization.

1. Introduction

The stabilization of dynamic systems is a very important question since it is the first requirement for most of applications. Powerful techniques for studying the stability of dynamic systems are Lyapunov stability theory and fixed point theory. Those techniques can be easily extended from the linear time-invariant case to the time-varying one as well as to functional differential equations as those arising, for instance, from the presence of internal delays, and to certain classes of nonlinear systems, [1, 2]. Dynamic systems which are of increasing interest are the so-called switched systems which consist of a set of individual parameterizations and a switching law which selects along time which parameterization is active. Switched systems are essentially time-varying by nature even if all the individual parameterizations are time-invariant. The interest of such systems arises from the fact that some existing systems in the real world modify their parameterizations to better adapt to their environments. Another important interest relies on the fact that changes of parameterizations through time can lead to benefits in certain applications [3–13]. The natural way of modelling these situations lies in the definition of appropriate switched dynamic systems. For instance, the asymptotic stability of Liénard-type equations with Markovian switching is investigated in [4, 5]. Also, time-delay dynamic systems are very important in the real life for appropriate modelling of certain biological and ecology systems and they are present in physical processes implying diffusion, transmission, teleoperation, population dynamics, war and peace models, and so forth (see, e.g., [1, 2, 12–18].)

A switched system can also be associated with the use of a multimodel scheme, a multicontroller scheme, a buffer system or a multiestimation scheme. For instance, a non exhaustive list of papers with deal with some of these questions related to switched systems follow.

(1) In [15], the problem of delay-dependent stabilization for singular systems with multiple internal and external incommensurate delays is focused on. Multiple memory-less state-feedback controls are designed so that the resulting closed-loop system is regular independent of delays, impulse-free, and asymptotically stable.

(2) In [19], the problem of the N-buffer switched flow networks is discussed based on a theorem on positive topological entropy.

(3) In [20], a multimodel scheme is used for the regulation of the transient regime occurring between stable operation points of a tunnel diode-based triggering circuit.

(4) In [21, 22], a parallel multiestimation scheme is derived to achieve close-loop stabilization in robotic manipulators whose parameters are not perfectly known. The multiestimation scheme allows the improvement of the transient regime compared to the use of a single estimation scheme while achieving at the same time closed-loop stability.

(5) In [23], a parallel multiestimation scheme allows the achievement of an order reduction of the system prior to the controller synthesis so that this one is of reduced-order (then less complex) while maintaining closed-loop stability.

(6) In [24], the stabilization of switched dynamic systems is discussed through topologic considerations via graph theory.

(7) The stability of different kinds of switched systems subject to delays has been investigated in [11–13, 17].

(8) The stability switch and Hopf bifurcation for a diffusive prey-predator system is discussed in [6] in the presence of delay.

(9) A general theory with discussed examples concerning dynamic switched systems is provided in [3].

The main objectives of this manuscript are as follows.

(a) To investigate the validity of Lyapunov functions of continuous-time systems after their discretization through zero-order holds under a sufficiently small-sampling period.

(b) To incorporate as individual parameterizations continuous-time and discrete-time parameterizations, and even discretized versions of certain continuous-time parameterizations, which remain as valid discrete-time parameterizations of the whole switched system. If all those parameterizations share a common Lyapunov function then the switched system is proved to be stable under arbitrary switching. It turns out that the switched system lies in the class of hybrid switched controlled systems due to the capability of mixing distinct continuous-time and discrete-time controlled parameterizations through time while guaranteeing closed-loop stability. In this context, the configuration of the general switched system is novel related to previous configurations on the subject dealt with in the literature.

(c) To investigate centralized/decentralized stabilizing control laws for a switched system of the above class.

The paper is organized as follows. Section 2 discusses the existence of a common Lyapunov function for a linear continuous-time parameterization and its discretized counterpart under sufficiently small-sampling period. It is then discussed in Section 3 the global asymptotic stability and stabilization for a switched system consisting of a set of continuous-time and discrete-time parameterizations which share a common Lyapunov function. The stabilizing controllers are assumed to be either of linear state-feedback type or output-feedback type. Firstly, their existence is discussed under standard assumptions of controllability and either observability or stabilizability for the individual parameterizations.

The switched system lies in a class of linear hybrid switched systems since the various individual parameterizations as well as their associate controllers may be either continuous-time or discrete-time or even mixed, and operating as potentially active, in the whole switched system. The feedback controller has the form $u(t) = K_{\sigma(t)} v(t)$ where the discrete map $\sigma : \mathbf{R}_{0+} \rightarrow \overline{N}$ governs the selection of the controller matrix values for the next time interval $[t_k, t_{k+1})$ where the current controller parameterization is active. If the whole controlled system is hybrid then $\mathbf{v} : \mathbf{R}_{0+} \rightarrow \mathbf{R}^{\mathbf{n}}$ is defined by $v(t) = z(t) \vee z(t_k)$ and $z(t)$ is either the sate or output vectors of the dynamic system. If the controller parameterization is the discretization of a continuous one then $t_{k+1} = t_k + h_k$, where h_k is a sufficiently small, in general time-varying, sampling period. The sampling period is required to be sufficiently small in order that the discretized parameterizations would be able to share a common Lyapunov function with the continuous-time ones according to previous mathematical proofs in Section 2. Section 4 extends the results of Sections 2 and 3 to the synthesis of decentralized controllers for a switched system composed of coupled linear subsystems. In general, the decentralized controller is not purely decentralized in the sense that some couplings between distinct subsystems are allowed to exist and the related information is used by both controllers to synthesize the control law. At the same time, the use of centralized controllers for some of the subsystems is also allowed. The case of synthesizing purely decentralized controllers is also discussed. Some examples are discussed in Section 5.

1.1. Notation

$\mathbf{C}$, $\mathbf{R}$, and $\mathbf{Z}$ are the sets of complex, real, and integer numbers, respectively.

$\mathbf{R}_+$ and $\mathbf{Z}_+$ are the sets of positive real and integer numbers, respectively, and $\mathbf{C}_+$ is the set of complex numbers with positive real part.

$\mathbf{C}_{0+} := \mathbf{C}_+ \bigcup \{\mathbf{i}\omega : \omega \in \mathbf{R}\}$, where $\mathbf{i}$ is the complex unity, $\mathbf{R}_{0+} := \mathbf{R}_+ \bigcup \{0\}$ and $\mathbf{Z}_{0+} := \mathbf{Z}_+ \bigcup \{0\}$.

$\mathbf{R}_-$ and $\mathbf{Z}_-$ are the sets of negative real and integer numbers, respectively, and $\mathbf{C}_-$ is the set of complex numbers with negative real part.

$\mathbf{C}_{0-} := \mathbf{C}_- \bigcup \{\mathbf{i}\omega : \omega \in \mathbf{R}\}$, where $\mathbf{i}$ is the complex unity, $\mathbf{R}_{0-} := \mathbf{R}_- \bigcup \{0\}$ and $\mathbf{Z}_{0-} := \mathbf{Z}_- \bigcup \{0\}$.

$\overline{N} := \{1, 2, \ldots, N\} \subset \mathbf{Z}_{0+}$, "$\vee$" is the logic disjunction, and "$\wedge$" is the logic conjunction. $[t/h]$ is the integer part of the rational quotient t/h and $S \setminus E := S - (S \bigcap E)$ for any given sets S and E.

$P > 0$, $P \geq 0$, $P < 0$, and $P \leq 0$ denote, respectively, that the square matrix P is positive definite, positive semidefinite, negative definite, and negative semidefinite.

$\sigma(M)$ denotes the spectrum of a real matrix M (i.e., its set of distinct eigenvalues) and $\overline{s}(M) = \sqrt{\lambda_{\max}(M^T M)}$ and $\underline{s}(M) = \overline{s}(M^{-1})$ if M^{-1} exists and $\underline{s}(M) = 0$, otherwise, are, respectively, the maximum and minimum singular values of M, where M^T is the conjugate transpose of M. M^T is replaced by the complex conjugate transpose M^* of M if M is complex. $\lambda_{\max}(M^T M)$ is the maximum eigenvalue of $M^T M$.

$\|M\|_p$ is the ℓ_p-norm of the real or complex matrix M and $\mu_p(M)$ denotes its ℓ_p-matrix measure.

I_n is the nth identity matrix.

$X \otimes Y := (x_{ij} Y)$ is the Kronecker product of the matrices $X = (x_{ij})$ of entries x_{ij} and Y. If $x_i (i \in \overline{n})$ is the ith column of the matrix $X^T := (x_1, x_2, \ldots, x_n)$ then the vectorization of the matrix X is defined by $\operatorname{vec}(X) := (x_1^T, x_2^T, \ldots, x_n^T)^T$.

ZOH is an acronym for a zero-order-hold device.

Assume that (A_i, B_i, C_i) are system, control and output matrices of N linear time-invariant systems. A switching law among the above systems is a piecewise constant function $\sigma : \mathbf{R}_{0+} \to \overline{N}$. The set of switching instants of the switching law $\sigma : \mathbf{R}_{0+} \to \overline{N}$ is a strictly ordered sequence $SI_\sigma := \{t_i\}_{i \in \mathbf{Z}_{0+}}$ which verifies $\sigma(t_i^+) \neq \sigma(t_i^-) \equiv \sigma(t_i)$ for the sake of notational abbreviation. The continuous-time switched system obtained via $\sigma : \mathbf{R}_{0+} \to \overline{N}$ from the parameterizations (A_i, B_i, C_i), $i \in \overline{N}$ is

$$\dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t); \quad y(t) = C_{\sigma(t)} x(t), \tag{1.1}$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$ and $y(t) \in \mathbf{R}^p$ are, respectively, the state, input and output vectors and $A_i \in \mathbf{R}^{n \times n}$, $B_i \in \mathbf{R}^{n \times m}$, $C_i \in \mathbf{R}^{p \times m}$ are the matrices of dynamics, control, and output for the ith parameterization; for all $i \in \overline{N}$ and $A_\sigma : \mathbf{R}_{0+} \to \{A_i : i \in \overline{N}\}$, $B_\sigma : \mathbf{R}_{0+} \to \{B_i : i \in \overline{N}\}$ and $C_\sigma : \mathbf{R}_{0+} \to \{C_i : i \in \overline{N}\}$ define the switched system (1.1) for the switching law $\sigma : \mathbf{R}_{0+} \to \overline{N}$.

2. Connecting a Class of Lyapunov Functions for Linear Continuous Time Systems with those of the Discretized Counterparts

The following result establishes that if a symmetric positive definite matrix P defines a Lyapunov function for a linear and time-invariant system then the same matrix defines a discrete Lyapunov sequence for the discrete counterpart of such a system.

Assertion 2.1. Assume that A is a stability matrix so that $V_c(t) = x^T(t)Px(t)$ is a Lyapunov function of the nth continuous-time linear time-invariant differential system $\dot{x}(t) = Ax(t)$ or any given real matrix $P = P^T > 0$ satisfying the Lyapunov matrix equation $A^TP + PA = -Q$ for any given real matrix $Q = Q^T > 0$. Then, $\{V_k\}_0^\infty$ defined with $V_k := x_k^T P x_k$ is a discrete-Lyapunov sequence for the discrete system $x_{k+1} := x[(k+1)h] = e^{Ah}x_k$ for any finite sampling period $h > 0$.

Proof. Direct calculus yields

$$\Delta V_k := V_{k+1} - V_k = x_{k+1}^T P x_{k+1} - x_k^T P x_k = x_k^T\left(e^{A^T h} P e^{Ah} - P\right)x_k = -x_k^T Q_d x_k. \tag{2.1}$$

Note that since A is a stability matrix then

$$\begin{aligned} P &= \int_0^\infty e^{A^T\tau} Q e^{A\tau} d\tau \\ &\Rightarrow e^{A^T h} P e^{Ah} = \int_0^\infty e^{A^T(\tau+h)} Q e^{A(\tau+h)} d\tau = \int_h^\infty e^{A^T\tau} Q e^{A\tau} d\tau \\ &= P - \int_0^h e^{A^T\tau} Q e^{A\tau} d\tau \\ &\Rightarrow e^{A^T h} P e^{Ah} - P = -Q_d = -Q_d^T = -\int_0^h e^{A^T\tau} Q e^{A\tau} d\tau < 0. \end{aligned} \tag{2.2}$$

Then, $\Delta V_k < 0$ for all nonzero x_k so that $\{V_k\}_0^\infty$ is a discrete Lyapunov sequence. □

Remark 2.2. The matrix P satisfying uniquely the Lyapunov matrix equation $A^TP + PA = -Q$ may be equivalently calculated through its entries as

$$\operatorname{vec}(P) = -\left(A^T \otimes I_n + I_n \otimes A^T\right)^{-1} \operatorname{vec}(Q). \tag{2.3}$$

Note that since A is a stability matrix, it is nonsingular implying that the n^2-square matrix $(A^T \otimes I_n + I_n \otimes A^T)$ is also nonsingular. Furthermore, this also implies that (2.3) is uniquely solvable as expected. Equivalently the matrix P can be also calculated from (2.2) for a sufficiently small-sampling period $h > 0$ as $\operatorname{vec}(P) = -(e^{A^T h} \otimes e^{A^T h} - I_{2n})^{-1} \operatorname{vec}(Q_d)$ since $e^{A^T t} = (e^{At})^T$.

The subsequent result establishes that if the linear and time invariant system $\dot{x}(t) = Ax(t) + Bu(t)$ is, so-called stabilizable, namely, it is globally asymptotically stabilized by some state-feedback linear control law $u(t) = Kx(t)$ then it is also globally asymptotically stabilized by the piecewise constant law $u(t) = Kx_k = Kx(kh)$; for all $t \in [kh, (k+1)h)$, for all $k \in \mathbf{Z}_{0+}$ for sufficiently small-sampling period $h > 0$. In other words, the discretized closed-loop system obtained through a Zero-Order-Hold (ZOH) is also globally asymptotically stable under the same controller gain for sufficiently small-sampling period.

Assertion 2.3. Assume that the pair (A, B) is stabilizable. Then, for each stabilizing controller matrix K (i.e., $A + BK$ is a stability matrix, equivalently all its eigenvalues are in $\mathbf{C}_-$) the following properties hold.

(i) There exists a sufficiently small maximum sampling period $h^* > 0$ such that the discrete pair $(e^{Ah}, \int_0^h e^{A(h-\tau)} B d\tau)$ is stabilizable under the same controller matrix K (i.e., $\Phi(h) := e^{Ah}[I_n + (\int_0^h e^{-A\tau} d\tau)BK]$ is a convergent matrix, for all $h \in (0, h^*)$: that is, $\Phi^k(h) \to 0$ as $\mathbf{Z}_{0+} \ni k \to \infty$ and, equivalently, all its eigenvalues lie in the open unit disk $C^o(0,1) := \{\mathbf{C} \ni z : |z| < 1\}$.

(ii) Assume that the pair (A, B) is stabilizable. Then, $\Phi(h) := e^{Ah}[I_n + (\int_0^h e^{-A\tau} d\tau)BK]$ is convergent for any controller gain matrix $K \in \mathbf{R}^{n\times n}$, such that $A_{c\ell} := A + BK$ is a stability matrix, and any sampling period $h > 0$ which satisfies $1 > \|e^{-A_{c\ell}h}\Phi(h) - I_n\|_2$. Furthermore, it always exists a first sampling period convergence interval $(0, h^*)$ for $\Phi(h)$ in the sense that $\Phi(h)$ is a convergent matrix, for all $h \in (0, h^*)$ with $h^* := \min(h \in \mathbf{R}_+ : 1 = \|e^{-A_{c\ell}h}\Phi(h) - I_n\|_2)$ provided that the set $\{h \in \mathbf{R}_+ : 1 \le \|e^{-A_{c\ell}h}\Phi(h) - I_n\|_2\} \neq \varnothing$. If $\{h \in \mathbf{R}_+ : 1 \le \|e^{-A_{c\ell}h}\Phi(h) - I_n\|_2\} = \varnothing$ then $h^* = \infty$.

Proof. (i) Direct calculation yields

$$e^{Ah}\left[I_n + \left(\int_0^h e^{-A\tau} d\tau\right)BK\right] = (I_n + Ah + M_1 o(h))\left[I_n + \left(hI_n - Ah^2/2 + M_2 o\left(h^2\right)\right)BK\right]$$
$$= I_n + (A + BK)h + M_3 o(h), \tag{2.4}$$

where M_i $(i = 1, 2)$, depending on the matrices A^k, $k \ge 2$, and M_3, depending on matrices B, K, and A^k, $k \ge 1$, are both bounded n-real matrices and "o" stands for Landau's "small-o" defined in standard way as follows. Given a real function f then $f = o(h)$ if $f = O(h)$, that is f is Landau's "big-O" of h (i.e., $|f| \le k_1 h + k_2$ for some real constants $k_i \in \mathbf{R}_{0+}$ $(i = 1, 2)$ and, furthermore, $\exists \lim_{h\to 0}(f/h) = 0$. For $h = 0$, the matrix (2.4) is identity so that it is critically stable. However, for $h > 0$, one gets from (2.4) the following inequalities by using the matrix measure $\vartheta := \mu_2(A + BK) = 1/2 \max\left(\operatorname{Re}\lambda(A + BK + (A + BK)^T) : \lambda(M) \in \sigma(M)\right)$:

$$\begin{aligned}1 - h\|A + BK\|_2 + |o(h)|\|M_3\|_2 &\le \mu_2\left(e^{Ah}\left[I_n + \left(\int_0^h e^{-A\tau} d\tau\right)BK\right]\right)\\ &= \mu_2(I_n + (A + BK)h + M_3 o(h))\\ &\le 1 + h\mu_2(A + BK) + o(h)\mu_2(M_3)\\ &\le 1 - h|\vartheta| + |o(h)|\|M_3\|_2\\ &\le 1 + h\|A + BK\|_2 + |o(h)|\|M_3\|_2,\end{aligned} \tag{2.5}$$

since $(A + BK)$ being a stability matrix and the properties of the matrix measure imply

$$-\|A + BK\|_2 \le \vartheta := \mu_2(A + BK) \le \|A + BK\|_2. \tag{2.6}$$

Now, for any given real constant $\varepsilon^* \in (0,1)$, it exists $h^* = h^*(\varepsilon^*) \in \mathbf{R}_+$ such that

$$|o(h)|\|M_3\|_2 \leq |o(h^*)|\|M_3\|_2 \leq \varepsilon^*; \quad \forall h \in (0,h^*), \tag{2.7a}$$

$$0 < 1 - h|\vartheta| + \varepsilon^* < 1, \qquad -1 < 1 - h\|A + BK\|_2 - \varepsilon^* < 0; \quad \forall h \in (0,h^*). \tag{2.7b}$$

Then, one gets from (2.5), (2.7a), and (2.7b) that $h^* \in (\min(\varepsilon^*/|\vartheta|, (1-\varepsilon^*)/\|A+BK\|_2), (1-\varepsilon^*)/|\vartheta|)$, and $\Phi(h) := e^{Ah}[I_n + (\int_0^h e^{-A_\tau} d_\tau)BK]$ is a convergent matrix; for all $h \in (0,h^*)$ for some sufficiently small $h^* > 0$ as it is $e^{(A+BK)h}$ for all $h \in \mathbf{R}_+$, since $(A + BK)$ is a stability matrix. Property (i) has been proved.

(ii) Since $A_{c\ell} := A + BK$ is a stability matrix, $e^{(A+BK)h}$ is a convergent matrix and then nonsingular, for all $h \in (0,h^*)$ with eigenvalues within the open unity disk. By direct construction, $\Phi(h) := e^{A_{c\ell}h}(I_n + e^{-A_{c\ell}h}(\Phi(h) - e^{A_{c\ell}h}))$. Thus, $\Phi(h)$ is nonsingular and then convergent for sufficiently small h since $e^{A_{c\ell}h}$ is convergent for sufficiently small-sampling period h that satisfies $1 > g(h) := \|e^{-A_{c\ell}h}(\Phi(h) - e^{A_{c\ell}h})\|_2$ from Banach Perturbation lemma. Since the real function g is a continuous function of the sampling period h which satisfies the above constraint for $h = 0$ since $g(0) = 0$, it exists $h^* := \min\ (h \in \mathbf{R}_+ : 1 = \|e^{-A_{c\ell}h}\Phi(h) - I_n\|_2)$, such that $\Phi(h)$ is convergent, for all $h \in (0,h^*)$ with $h^* < \infty$, provided that $\{h \in \mathbf{R}_+ : 1 \leq \|e^{-A_{c\ell}h}\Phi(h) - I_n\|_2\} \neq \varnothing$. If $\{h \in \mathbf{R}_+ : 1 \leq \|e^{-A_{c\ell}h}\Phi(h) - I_n\|_2\} = \varnothing$ then $h^* = \infty$. Property (ii) has been proved. □

Note that the first property of Assertion 2.3 is a local result about the existence of a minimum sampling period such that a continuous-time system which is closed-loop stable remains stable if it is discretized under a sufficiently small-sampling period with the same stabilizing controller. Assertion 2.3(ii) relies on a constraint which is numerically testable very easily and is useful to calculate a first admissibility interval for the sampling period such that the stability of the system still holds when using the same continuous-time stabilizing controller. Assertion 2.3 is now interpreted in practical terms of achievement of closed-loop stability of a system discretized through a ZOH sampling and hold device and the same controller gain matrix, as that of the continuous-time counterpart, provided that the sampling period is sufficiently small.

Assertion 2.4. Assume that the pair (A,B) is stabilizable. Then, there exists a sufficiently small-sampling period $h^* > 0$ such that all the solutions of $\dot{x}(t) = Ax(t) + Bx(kh)$; for all $t \in [kh,(k+1)h)$; for all $h \in [0,h^*)$; for all $k \in \mathbf{Z}_{0+}$ are globally asymptotically stable for each stabilizing controller matrix K.

Proof. The state trajectory solution at sampling instants $t = kh$ (for all $k \in \mathbf{Z}_{0+}$) is given by

$$x[(k+1)h] = e^{Ah}\left(I_n + \left(\int_0^h e^{-A\tau} d\tau\right)BK\right)x(kh), \tag{2.8}$$

for each bounded initial conditions $x(0)$. The pair $(e^{Ah}, \int_0^h e^{A(h-\tau)} B d\tau)$ is stabilizable if and only if it exists K such that $\Phi(h) = e^{Ah}(I_n + (\int_0^h e^{-A\tau} d\tau)BK)$ is a convergent matrix for some K. Since (A,B) is stabilizable, it exists K such that $(A+BK)$ is a stability matrix and, equivalently,

$e^{(A+BK)h}$ is a convergent matrix, for all $h \in \mathbf{R}_+$. For the same matrix K, some real constants $\tau_{ij} \in (0, h)$; for all $i, j \in \overline{n}$, for all $h \in [0, h^*)$, one gets

$$\begin{aligned} &\left\| e^{Ah}\left(I_n + \left(\int_0^h e^{-A\tau} d\tau\right) BK\right) - e^{(A+BK)h} \right\|_2 \\ &\quad = \left\| I_n + Ah + h\left(e_{ij}^{A\tau_{ij}}\right) BK - I_n - (A+BK)h + o(h) \right\|_2 \\ &\quad = \left\| \left[\left(e_{ij}^{A\tau_{ij}}\right) - I_n\right] BKh + o(h) \right\|_2 = \left\| (A_{ij}\tau_{ij} + o(h)) BKh + o(h) \right\|_2 \\ &\quad \le \left(\left\| A_{ij}\tau_{ij} \right\|_2 + |o(h)| \right) \|BK\|_2 h + |o(h)| \le k_a \left\| A_{ij}\tau_{ij} \right\|_2 \|BK\|_2 h^*. \end{aligned} \tag{2.9}$$

Since the convergence abscissa of $e^{(A+BK)h}$ is in $(0, 1)$, for all $h \in \mathbf{R}_+$ (since $(A+BK)$ is a stability matrix), $\mu_2(e^{(A+BK)h}) \le -\rho \in \mathbf{R}_-$. Also, the convergence abscissa of $e^{Ah}(I_n + (\int_0^h e^{-A\tau} d\tau)BK)$ is also in $(0, 1)$ for sufficiently small $h^* \in \mathbf{R}_+$ and for all $h \in [0, h^*)$. Then, $e^{Ah}(I_n + (\int_0^h e^{-A\tau} d\tau)BK)$ is convergent, so that $(e^{Ah}, \int_0^h e^{A(h-\tau)} B d\tau)$ is stabilizable, and the discrete-time system is globally asymptotically Lyapunov stable. □

Time varying sampling periods may be considered as strictly ordered sequences $\{h_k\}_{k \in \mathbf{Z}_{0+}}$ of positive real numbers. The sampling instants are real sequences of positive real numbers $\{t_k\}_{k \in \mathbf{Z}_{0+}}$ defined by $t_0 = 0$, $t_k = \sum_{i=0}^{k-1} h_i$; for all $k \in \mathbf{Z}_+$. A special case arises when each sampling period is generated as some real function of preceding sampled states and preceding sampling instants $f : \mathbf{R}_{0+} \times \mathbf{Z}_{0+} \to \mathbf{R}_+$; that is, $h_{k+1} = f(t_i, x(t_i) : i \in \overline{k})$. Assertion 2.3 extends to time-varying sampling periods as follows.

Assertions 2.5. Assume that the pair (A, B) is stabilizable. Then, for each stabilizing controller matrix K

(i) there exists a sufficiently small maximum sampling period $h^* > 0$ such that for any time-varying sampling period $h_k \in (0, h^*)$; for all $k \in \mathbf{Z}_{0+}$, the time-varying discrete pair $(e^{Ah_k}, \int_0^{h_k} e^{A(h_k-\tau)} B d\tau)$ is stabilizable under the same controller matrix K (i.e., is a convergent matrix so that $\Phi(h_k) \to 0$ as $\mathbf{Z}_{0+} \ni k \to \infty$ and, equivalently, all its eigenvalues are in $C^o(0, 1) := \{\mathbf{C} \ni z : |z| < 1\}$);

(ii) for any sampling instant $t_k = \sum_{i=0}^{k-1} h_i$; for all $k \in \mathbf{Z}_+$, $\Phi(\sum_{i=0}^{k-1} h_i) := e^{A\sum_{i=0}^{k-1} h_i}\Big[I_n + \Big(\int_0^{\sum_{i=0}^{k-1} h_i} e^{-A\tau} d\tau\Big) BK\Big]$ is convergent, that is, $\Phi^\ell(\sum_{i=0}^{k-1} h_i) \to 0$ as $\mathbf{Z}_{0+} \ni \ell \to \infty$, for all $k \in \mathbf{Z}_{0+}$ and, equivalently, all the eigenvalues of $\Phi(\sum_{i=0}^{k-1} h_i)$ are in $C^o(0, 1) := \{\mathbf{C} \ni z : |z| < 1\}$.

The above results rely on the well-known fact that although $e^{(A+BK)h}$ is a convergent matrix for any real $h > 0$ if $A + BK$ is a stability matrix, the property does not hold for any h if the system is discretized through a ZOH and the same controller gain is used within each intersample period.

3. Stability of a Class of Hybrid Switched Systems

This section considers switched hybrid systems which consist of mixed continuous-time and discrete-time (through ZOH devices and sufficiently small-sampling period) switched controls for potentially distinct continuous-time parameterizations. The stability of the hybrid switched systems is investigated by using results of the above section. The potential application of mixed continuous-time discrete-time control laws is that when discrete controls are operating, the controller is not required to acquire data at all time but only at the rate of the sampling period so that the computational costs are reduced. Another advantage might be that in the case of data validity failure, sampled data at previous time instants can replace the incorrect data for the controller operation. If hybrid continuous-time/discrete-time controlled switched systems are designed then a common Lyapunov function exists for all (continuous-time and discrete-time) parameterizations for sufficiently small-sampling period. A common Lyapunov function exists if either the matrices of dynamics of the individual parameterization commute or if the pair-wise commutators are sufficiently close to zero in terms of norms, [3, 13, 17, 25–27]. If there is no common Lyapunov function, then a minimum residence time is requested at each individual active parameterization to guarantee closed-loop stability provided that all such parameterizations are stable, [3, 17, 20–22, 25–27]. Thus, the existence of a common Lyapunov function is a hypothesis commonly used for arbitrary switching in the sense that the parameterizations switch arbitrarily (i.e., at any switching instants). However, if such an assumption does not hold, then the switched system is still globally asymptotically stable if a minimum residence time is kept at each parameterization before the next switching occurs.

Theorem 3.1. *Assume that the pairs* (A_i, B_i), $i \in \overline{N}$ *are all stabilizable through linear state-feedback control gains* K_i *of appropriate orders so that all the closed-loop systems* $\dot{x}_i(t) = (A_i + B_i K_i) x_i(t)$, $i \in \overline{N}$ *possess a common Lyapunov function. Then, the time-varying control law* $u(t) = K_{\sigma(t)} v(t)$ *with* $K_{\sigma(t)} : \mathbf{R}_{0+} \to \{K_i : i \in \overline{N}\}$ *where the discrete map* $\sigma : \mathbf{R}_{0+} \to \overline{N}$ *is arbitrary and* $v : \mathbf{R}_{0+} \to \mathbf{R}^n$ *is defined by* $v(t) = x(t) \vee x(t_k)$ *for any* $t \in \mathbf{R}_{0+}$, *subject to* $t_0 = 0$ *or* $t_k = \sum_{i=0}^{k-1} h_i \le t < t_{k+1}$; *for all* $k \in \mathbf{Z}_+$, *globally asymptotically stabilizes the switched system:*

$$\dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t) \tag{3.1}$$

for any sampling period sequence $\{h_k\}_{k \in \mathbf{Z}_{0+}}$ *with* $h_k \in (0, h^*)$, *for all* $k \in \mathbf{Z}_{0+}$ *and some sufficiently small* $h^* > 0$ *with piecewise constant matrix functions* $A_{\sigma(t)} : \mathbf{R}_{0+} \to \{A_i : i \in \overline{N}\}$ *and* $B_{\sigma(t)} : \mathbf{R}_{0+} \to \{B_i : i \in \overline{N}\}$ *for* $\sigma : \mathbf{R}_{0+} \to \overline{N}$.

Proof. Note that for each individual of the N parameterizations.

(a) The continuous-time system and its discrete counterpart may be stabilized by a continuous-time controller or a discrete-time one under a ZOH for any (constant or time-varying) sufficiently small-sampling period by using the same controller gain (Assertions 2.3 and 2.4), since

$$x(t) = \Psi(t, t_i) x(t_i) = \Psi(t, t_i) \Psi(t_i, t_{i-1}) \Psi(t_{i-1}, \tau) x(\tau), \qquad \Psi(t_i, 0) = \Phi\left(\sum_{i=0}^{k-1} h_i\right). \tag{3.2}$$

(b) The continuous time Lyapunov function is also a discrete Lyapunov sequence with the same matrix P for the continuous-time system for any sampling period (Assertion 2.1).

(c) There is a common Lyapunov function for all the parameterizations so that the controller gain may be updated with arbitrary switching. That is, it may be any of the gains at any time $t \in \mathbf{R}_{0+}$ acting on the current state vector or during a sufficiently small-sampling period acting on the last sampled value of the state vector at a sampling instant for a sufficiently small, in general time-varying, sampling period $h(t)$ with the current time $t \in \mathbf{R}_{+}$ fulfilling $t_k = \sum_{i=0}^{k-1} h_i \le t < t_{k+1}$; for all $k \in \mathbf{Z}_{+}$. □

Remark 3.2. Theorem 3.1 relies on the fact that arbitrary switching related to the gain updating rule for a switched system may be done with nonconstant sampling periods for the discrete-time parameterizations. The gain updating process may be made at any time and the controller might take the current state vector as regressor. Instead the regressor may be the last sampled value of the state vector for sufficiently small (in general) time-varying sampling period. Both controller actions, namely, arbitrary controller updating (associated with arbitrary switching) and either continuous-time or discrete-time (but at sufficiently small-sampling rate) control actions may be interchanged through time. The closed-loop stability is preserved.

Theorem 3.1 applies if $(A_i + B_i K_i)$; for all $i \in \overline{N}$ share a common Lyapunov function. The subsequent result gives sufficient related conditions.

Theorem 3.3. *Assume that the pairs (A_i, B_i), $i \in \overline{N}$ are all stabilizable. Assume also that there exist a stability matrix $A^* \in \mathbf{R}^{n\times n}$ and matrices $B_i^* \in \mathbf{R}^{n\times m}$; for all $i \in \overline{N}$ such that the following constraints hold:*

(1) rank $(A_i - A^* + B_i K_i) =$ rank $(A_i - A^* + B_i K_i, B_i^*)$; *for all $i \in \overline{N}$;*

(2) $A^{*^T} P + P A^* + P B_i^* K_i^* + K_i^{*^T} B_i^{*^T} P < 0$; *for all $i \in \overline{N}$ for some $P = P^T > 0$ where $K_i^* \in \mathbf{R}^{m\times n}$ is any solution satisfying $A_i + B_i K_i = A^* + B_i^* K_i^*$.*

Thus, the closed-loop systems $\dot{x}_i(t) = (A_i + B_i K_i) x_i(t)$, $i \in \overline{N}$ are all globally asymptotically stable and share a common Lyapunov function $V(t) = x^T(t) P x(t)$. Thus, the arbitrary switching law of Theorem 3.1 with either $v(t) = x(t)$, or with $v(t) = x(t) \vee x(t_k)$ with the sampling period h_k being sufficiently small for all $k \in \mathbf{Z}_{0+}$, globally asymptotically stabilizes the open-loop switched system (3.1)*. The result also holds if and $A_i + B_i K_i$, for all $i \in \overline{N}$ are all stability matrices and pair-wise commute.*

Proof. If condition (1) holds then controller gains K_i^* (being unique if the rank is n); for all $i \in \overline{N}$ exist from Kronecker-Capelli theorem such that $A_i + B_i K_i = A^* + B_i^* K_i^*$; for all $i \in \overline{N}$. If a Lyapunov function candidate $V(t) = x^T(t) P x(t)$ is used, then $\dot{V}(t) = x^T(t)(A^{*^T} P + P A^* + P B_i^* K_i^* + K_i^{*^T} B_i^{*^T} P) x(t) < 0$ for $x(t) \neq 0$ are condition (2) holds provided that control law $u(t) = K_{\sigma(t)} x(t)$ where the discrete map $\sigma : \mathbf{R}_{0+} \to \overline{N}$ is arbitrary. If $v : \mathbf{R}_{0+} \to \mathbf{R}^n$ is defined by $v(t) = x(t) \vee x(t_k)$, then any two consecutive switching instants, either

$$\begin{aligned} \Delta V(t_k) &= V(t_{k+1}) - V(t_k) \\ &\le (t_{k+1} - t_k) \min_{t_k \le t \le t_{k+1}} x^T(t) \left(A^{*^T} P + P A^* + P B_{\sigma(t)}^* K_{\sigma(t)}^* + K_{\sigma(t)}^{*^T} B_{\sigma(t)}^{*^T} P \right) x(t) < 0 \end{aligned} \tag{3.3}$$

for $\sigma(t) = \sigma(t_k) \in \overline{N}$; for all $t \in [t_k, t_k + h_k)$ if $v(t) = x(t) \neq 0$, or

$$\Delta V(t_k) = V(t_{k+1}) - V(t_k) \leq h_k x^T(t_k) \times \left(e^{A^{*T} h_k} P + P e^{A^* h_k} + P \left(\int_0^{h_k} e^{A^*(h_k - \tau)} B^*_{\sigma(t)} d\tau \right) K^*_{\sigma(t)} + K^{*T}_{\sigma(t)} \left(\int_0^{h_k} e^{A^*(h_k - \tau)} B^*_{\sigma(t)} d\tau \right)^T P \right) \times x(t_k) < 0, \tag{3.4}$$

if $v(t) = x(t_k) \neq 0$ provided that $h_k \in (0, h^*]$ for any $k \in \mathbf{Z}_{0+}$ and h^* is sufficiently small from Assertions 2.3 and 2.4(i). The first part of the result has been proved. The second one is direct since linear time-invariant systems whose matrices of dynamics are stability matrices which commute share a common Lyapunov function, [3]. □

The subsequent technical result is directly applicable for the case when the closed-loop switched system is generated by a switched linear output-feedback controller acting on a linear and time-invariant open-loop system.

Theorem 3.4. *Consider the linear time-invariant open-loop system:*

$$\dot{x}(t) = Ax(t) + Bu(t); \qquad y(t) = Cx(t), \tag{3.5}$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, and $y(t) \in \mathbf{R}^p$, are the state, output, and control vectors, and A, B, C are matrices of compatible orders with the dimensions of such vectors. Consider also a switched linear output-feedback control law:

$$u(t) = K_{\sigma(t)} y(t) = K_{\sigma(t)} C x(t), \tag{3.6}$$

with a switched controller $K_\sigma : \mathbf{R}_{0+} \to \{K_1, K_2, \ldots, K_N\}$, $K_i \in \mathbf{R}^{m \times p}$; for all $i \in \overline{N}$. This leads, after injection of (3.6), *to the switched closed-loop system:*

$$\dot{x}(t) = \left(A + BK_{\sigma(t)} C\right) x(t) \tag{3.7}$$

for then, the following properties hold.

(i) *The open-loop system* (3.5) *is output-stabilizable by a control law* (3.6) *if* $\operatorname{rank}(B \otimes C^T) = \operatorname{rank}(B \otimes C^T, \operatorname{vec}(A^* - A))$ *for some stability matrix* $A^* \in \mathbf{R}^{n \times n}$.

(ii) *The open-loop system* (3.5) *is output-stabilizable by a control law* (3.6) *if* $\max(m, p) \geq n$, $\operatorname{rank}(B) = m$, *and* $\operatorname{rank}(C) = p$ *and the triple* (A, B, C) *is controllable and observable.*

Proof. (i) If the given rank condition holds with a stability matrix A^* then there is a solution K to the matrix identity $A + BKC = A^* \Leftrightarrow (B \otimes C^T) \operatorname{vec}(K) = \operatorname{vec}(A^* - A)$ from Kronecker-Capelli theorem. Then, the resulting time-invariant closed-loop system is globally asymptotically stable. Property (i) is proved since there is a constant controller which asymptotically stabilized the closed-loop system. (ii) If $\operatorname{rank}(B) = m$ and $\operatorname{rank}(C) = p$ and the triple (A, B, C)

is controllable and observable then there the spectrum of $(A+BKC)$ may be fixed as closely as desired to any given set of (possibly repeated) complex numbers in $\mathbf{C}_{0-}$. □

A system is said to be output-stabilizable if there is a linear output-feedback law which makes the closed-loop system to be globally asymptotically stable. An extension of Theorem 3.4 for the case of linear output feedback under a set of switching controllers follows.

Theorem 3.5. *Consider the linear time-invariant switched open-loop system:*

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \qquad y(t) = C_{\sigma(t)}x(t), \tag{3.8}$$

where $x(t) \in \mathbf{R}^n$, $y(t) \in \mathbf{R}^p$, $u(t) \in \mathbf{R}^m$ are the state, output and control vectors, and $\sigma : \mathbf{R}_{0+} \to \overline{N}$ is a switching law and $A_\sigma : \mathbf{R}_{0+} \to \{A_i \in \mathbf{R}^{n\times n}$; for all $i \in \overline{N}\}$, $B_\sigma : \mathbf{R}_{0+} \to \{B_i \in \mathbf{R}^{n\times m}$ for all $i \in \overline{N}\}$, $C_\sigma : \mathbf{R}_{0+} \to \{C_i \in \mathbf{R}^{p\times n}$; for all $i \in \overline{N}\}$ are piecewise constant function matrices of compatible orders with the dimensions of such vectors. Consider also a switched linear output-feedback control law:

$$u(t) = K_{\sigma(t)}v(t), \qquad v(t) = y(t) \vee y(t_k) \quad \text{for } t_k \in SI_\sigma \tag{3.9}$$

with $K_\sigma : \mathbf{R}_{0+} \to \{K_i \in \mathbf{R}^{m\times p}$; for all $i \in \overline{N}\}$ yielding after injection of (3.9) into (3.8), the switched closed-loop system:

$$\dot{x}(t) = \left(A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t)}C_{\sigma(t)}\right)x(t). \tag{3.10}$$

Then, the following properties hold.

(i) *Assume that there exists a set $A_i^* \in \mathbf{R}^{n\times n}$; for all $i \in \overline{N}$ of stability matrices such that:*

(1) $\operatorname{rank}(B_i \otimes C_i^T) = \operatorname{rank}(B_i \otimes C_i^T, \operatorname{vec}(A_i^* - A_i))$; *for all* $i \in \overline{N}$;

(2) $\exists P = P^T > 0$ *such that* $A_i^{*^T} P + P A_i^* < 0$; *for all* $i \in \overline{N}$.

Thus, the time-invariant closed-loop systems $\dot{x}_i(t) = (A_i + B_iK_iC_i)x_i(t)$, $i \in \overline{N}$ are all globally asymptotically stable provided that each K_i is selected such that $\operatorname{vec}(K_i)$ is a solution to the compatible linear algebraic system:

$$\left(B_i \otimes C_i^T\right)\operatorname{vec}(K_i) = \left(B_i \otimes C_i^T, \operatorname{vec}\left(A_i^* - A_i\right)\right), \qquad \forall i \in \overline{N}. \tag{3.11}$$

Furthermore, all those systems share a common Lyapunov function $V(t) = x^T(t)Px(t)$. As a result, the arbitrary switching law $\sigma : \mathbf{R}_{0+} \to \overline{N}$ of Theorem 3.1 with a control law (3.9) with the (potentially time-varying sampling periods) h_k being sufficiently small for all $k \in \mathbf{Z}_{0+}$, globally asymptotically stabilizes the open-loop switched system (3.1).

(ii) *The open-loop system (3.8) is output-stabilizable by a control law (3.9) if $\max(m,p) \ge n$, $\operatorname{rank}(B_i) = m$ and $\operatorname{rank}(C_i) = p$ and the triples$(A_i, B_i, C_i))$, for all $i \in \overline{N}$ are controllable and observable for any switching law $\sigma : \mathbf{R}_{0+} \to \overline{N}$ subject to a minimum sufficiently large residence time at each active parameterization $i \in \overline{N}$.*

Outline of Proof

(i) It is a direct extension of Theorem 3.1 by using Assertions 2.3 and 2.4 since firstly the controller gains being solutions to (3.11) imply $A_i^* = A_i + B_i K_i C_i$; for all $i \in \overline{N}$. Furthermore, $A_i^{*^T} P + P A_i^* < 0$; for all $i \in \overline{N}$ implies that $V(t) = x^T(t) P x(t)$ is a common Lyapunov function for the N parameterizations so that the closed-loop system obtained from any arbitrary switching law (3.9) is globally asymptotically stable under arbitrary switching.

(ii) From Theorem 3.3(ii), the closed-loop linear time-invariant systems $\dot{x}_i(t) = (A_i + B_i K_i C_i) x_i(t)$; $i \in \overline{N}$ are all stable and with eigenvalues closed to prescribed sets (even if $A_i + B_i K_i C_i$ does not matchs a prescribed A_i^*; for all $i \in \overline{N}$) for some existing controller gains K_i; for all $i \in \overline{N}$. Thus, the switched closed-loop system is globally asymptotically stable for any switching laws subject to a minimum sufficiently large residence time at each active parameterization $i \in \overline{N}$.

4. Stability of Centralized and Decentralized Switched Linear Systems

The results of Section 3 are extended to the following class of composite open-loop switched systems:

$$\dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t); \qquad y(t) = C_{\sigma(t)} x(t), \tag{4.1}$$

where $x(t)$, $u(t)$, and $y(t)$ are the n-state, m-input, and p-output vectors, $\sigma : \mathbf{R}_{0+} \to \overline{N}$ and

$$A_{\sigma(t)} \in \left\{ A_k \in \mathbf{R}^{n\times n} : k \in \overline{N} \right\}, \quad B_{\sigma(t)} \in \left\{ B_k \in \mathbf{R}^{n\times m} : k \in \overline{N} \right\}, \quad C_{\sigma(t)} \in \left\{ C_k \in \mathbf{R}^{p\times n} : k \in \overline{N} \right\}, \tag{4.2}$$

in such a way that (4.1) and (4.2) consist of q interconnected composite systems of respective state, input and output $x_i(t)$, $u_i(t)$, and $y_i(t)$, that is

$$\dot{x}_i(t) = \sum_{j=1}^{q} A_{\sigma(t)}^{(ij)} x_j(t) + B_{\sigma(t)}^{(i)} u_i(t); \quad y_i(t) = C_{\sigma(t)}^{(i)} x_i(t); \quad \forall i \in \overline{q}, \tag{4.3}$$

with

$$\begin{aligned}
x(t) &= \left(x_1^T(t), x_2^T(t), \ldots, x_q^T(t)\right)^T, \\
u(t) &= \left(u_1^T(t), u_2^T(t), \ldots, u_q^T(t)\right)^T, \quad y(t) = \left(y_1^T(t), y_2^T(t), \ldots, y_q^T(t)\right)^T, \\
A_k &:= \text{Block matrix}\left(A_k^{(ij)} \in \mathbf{R}^{n_i\times n_j} : i,j \in \overline{q}\right) \in \mathbf{R}^{n\times n}; \quad \forall k \in \overline{N} \\
B_k &:= \text{Block diag}\left(B_k^{(i)} \in \mathbf{R}^{n_i\times m_i} : i \in \overline{q}\right) \in \mathbf{R}^{m\times n}, \\
C_k &:= \text{Block row matrix}\left(C_k^{(i)} \in \mathbf{R}^{p\times n_i} : i \in \overline{q}\right) \in \mathbf{R}^{p\times n}; \quad \forall k \in \overline{N}.
\end{aligned} \tag{4.4}$$

A state (resp., output)-feedback control law for the system (4.1)–(4.4) is said to be decentralized if $u_i(t)$ has the ith state $x_i(t)$ (resp., output $y_i(t)$) available for measurement but not all $x_j(t)$ $(j \neq i)$ is available for the ith controller (or, at least, some components of the $x_j(t)$ substate are not available to the $i(\neq j)$th for controller). If the whole $x(t)$ (resp., $y(t)$) is available for each $u_i(t)$; for all $i \in \overline{q}$ then the control law is said to be a centralized control law. Decentralized state-feedback (resp., output feedback) control has important applications. The main reason is that it allows the control of several interconnected subsystems with a partial state (resp., output) information, [28, 29]. This is very useful when the individual systems are physically separated. A practical example is a group of separated electric power production stations operating as a tandem group located in the same river. Another application is the reduction of computational costs as a result of minimizing the shared information by a group of local controllers. A new open possibility of switched decentralized control is to eliminate crossed information for the individual controllers depending on each particular active parameterization. A linear state-feedback controller for the switched system (4.1)–(4.4) is

$$u(t) = K_{\sigma(t)} x(t), \tag{4.5}$$

where the controller gain is piecewise constant governed by the switching law $\sigma : \mathbf{R}_{0+} \to \overline{N}$

$$K_{\sigma(t)} \in \left\{ K_k := \text{Block matrix} \left(K_k^{(ij)} \in \mathbf{R}^{m_i \times n_j} : i, j \in \overline{q} \right) \in \mathbf{R}^{m \times n} : k \in \overline{N} \right\}. \tag{4.6}$$

A centralized control law is defined by matrices K_k $(k \in \overline{N})$ whose entries can be all nonzero. A decentralized control law is defined by those matrices in such a way that at least a nondiagonal entry per row is necessarily zero; that is, some component of the substate $x_j(t)$ is not available for the substate $x_i(t)$ for each $j(\neq i) \in \overline{q}$; for all $i \in \overline{q}$. To simplify the subsequent discussion, any decentralized control law is assumed to be constrained to have at least a nondiagonal block identically zero, that is, $K_k^{(ij_{ik})} = 0$ for at least a $j_{ik}(\neq i) \in \overline{q}$, depending in general on i and k; for all $i \in \overline{q}$, for all $k \in \overline{N}$. The substitution of (4.5)-(4.6) into (4.1)–(4.4) leads to the following closed-loop system:

$$\dot{x}(t) = \left(A_{\sigma(t)} + B_{\sigma(t)} K_{\sigma(t)} \right) x(t). \tag{4.7}$$

To distinguish centralized and decentralized control laws, subscripts "c" and "d" are used in (4.7) as follows:

$$\dot{x}(t) = \overline{A}_{c\sigma(t)} x(t); \qquad \overline{A}_{c\sigma(t)} := A_{\sigma(t)} + B_{\sigma(t)} K_{c\sigma(t)} \tag{4.8}$$

is the closed-loop system (4.7) under centralized control, and

$$\dot{x}(t) = \overline{A}_{d\sigma(t)} x(t); \qquad \overline{A}_{d\sigma(t)} := A_{\sigma(t)} + B_{\sigma(t)} K_{d\sigma(t)} \tag{4.9}$$

is the closed-loop system (4.7) under decentralized control, where

$$K_{d\sigma(t)} = K_{c\sigma(t)} - \Delta_{\sigma(t)}, \tag{4.10}$$

$$\begin{aligned} K_{c\sigma(t)} \in \Big\{ K_i := \text{ Block matrix} \Big(K_k^{(ij)} \in \mathbf{R}^{n_i \times m_j} : i, j \in \overline{q} \Big) \\ \in \mathbf{R}^{m \times n} : \Delta_k^{(ij_{ik})} = 0; \quad \forall i, j(\neq i) \in \overline{q},\ k \in \overline{N} \Big\}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} \Delta_{\sigma(t)} \in \Big\{ \Delta_k := \text{ Block matrix} \Big(\Delta_k^{(ij_{ik})} \in \mathbf{R}^{n_i \times m_j} : i, j \in \overline{q} \Big) \\ \in \mathbf{R}^{m \times n} : \Delta_k^{(ij_{ik})} = 0; \ \forall j_{ik}(\neq i) \in S_{ik} \subset (\overline{q} \setminus \{i\}),\ k \in \overline{N} \Big\}. \end{aligned} \tag{4.12}$$

Note from (4.9)-(4.10) that

$$\overline{A}_{d\sigma(t)} := A_{\sigma(t)} + B_{\sigma(t)} \big(K_{c\sigma(t)} - \Delta_{\sigma(t)} \big) = \overline{A}_{c\sigma(t)} - B_{\sigma(t)} \Delta_{\sigma(t)} \tag{4.13}$$

$$= \overline{A}_{c\sigma(t)} \big(I_n - \overline{A}_{c\sigma(t)}^{-1} B_{\sigma(t)} \Delta_{\sigma(t)} \big), \tag{4.14}$$

where (4.14) is well posed if $\overline{A}_{c\sigma(t)}^{-1}$ exists, and in particular, if it is a stability matrix for all $t \in \mathbf{R}_{0+}$. The small gain theorem [3] yields the following direct result from (4.13).

Proposition 4.1. *Assume that the matrices $\overline{A}_{ci} = A_i + B_i K_{ci}$ are all stability matrices, for all $i \in \overline{N}$. Thus, $\overline{A}_{di} = A_i + B_i K_{di}$ are stability matrices, for all $i \in \overline{N}$ if $\|\overline{A}_{ci}^{-1} B_i \Delta_i\|_2 < 1$; for all $i \in \overline{N}$ which is guaranteed if $\|\Delta_i\|_2 < 1/\|\overline{A}_{ci}^{-1} B_i\|_2$; for all $i \in \overline{N}$. The last condition is guaranteed in terms of singular values if $\overline{s}(\Delta_i) < \underline{s}(A_i)/\overline{s}(B_i)$; for all $i \in \overline{N}$.*

Proposition 4.1 guarantees the stability of all the individual parameterizations of the switched system under decentralized control. It is assumed that all of them are stable under centralized control and the deviation of the decentralized controller gains with respect to the stabilizing centralized ones are sufficiently small. Such a "smallness" is quantified in terms of testable conditions obtained by evaluating matrix norms. Note that the "a priori" necessary condition for the existence of matrices K_{ci} for stabilization of the ith open-loop system parameterization through some continuous-time centralized controller is that the pair (A_i, B_i) be stabilizable, namely, rank $(sI_n - A_i, B_i) = n$; for all $s \in \mathbf{C}_{0+}$ (Popov-Belevitch-Hautus rank stabilizability test). Equivalently, all the uncontrollable modes, if any, have to be stable. The following extension of Proposition 4.1 is direct.

Proposition 4.2. *Assume that the matrices $\overline{A}_{ci} = A_i + B_i K_{ci}$ are all stability matrices, for all $i \in \overline{N}$ with common guaranteed stability abscissa $(-\rho_c) \in \mathbf{R}_-$, that is, $\max_{i \in \overline{N}} (\mathrm{re}\, \lambda_{\max}(\overline{A}_{ci})) \leq -\rho_c^-$ (if the left-hand side corresponds to an eigenvalue with multiplicity one then $\rho_c := -\max_{i \in N} (\mathrm{re}\, \lambda_{\max}(\overline{A}_{ci}))$). Thus, $\overline{A}_{di} = A_i + B_i K_{di}$ are stability matrices, for all $i \in \overline{N}$ with common guaranteed stability abscissa $(-\rho_d) \in \mathbf{R}_-$, that is, $\max_{i \in \overline{N}} (\mathrm{re}\, \lambda_{\max}(\overline{A}_{di})) \leq -\rho_d$ if*

$$\|\Delta_i\|_2 < \frac{1 - \Big\| \big(\overline{A}_{ci} + \rho_c I_n \big)^{-1} B_i \Big\|_2 (\rho_c - \rho_d)}{\Big\| \big(\overline{A}_{ci} + \rho_c I_n \big)^{-1} B_i \Big\|_2}, \tag{4.15}$$

provided that

$$\rho_d \in \left(\rho_c - \frac{1}{\min_{i \in \overline{N}} \left\| \left(\overline{A}_{ci} + \rho_c I_n \right)^{-1} B_i \right\|_2}, \rho_c \right); \quad \forall i \in \overline{N}. \tag{4.16}$$

Proof. It has to be proved that $(\overline{A}_{di} + \rho_d I_n)$ is a stability matrix; for all $i \in \overline{N}$ provided that $(\overline{A}_{ci} + \rho_c I_n)$ is a stability matrix; for all $i \in \overline{N}$. Direct calculations yield

$$\overline{A}_{di} + \rho_d I_n = \left(\overline{A}_{ci} + \rho_c I_n \right) \left(I_n - \left(\overline{A}_{ci} + \rho_c I_n \right)^{-1} \left(B_i \Delta_i + (\rho_c - \rho_d) I_n \right) \right), \tag{4.17}$$

which is a stability matrix from the small gain theorem if $\overline{A}_{ci} + \rho_c I_n$ is a stability matrix and $\|(\overline{A}_{ci} + \rho_c I_n)^{-1} (B_i \Delta_i + (\rho_c - \rho_d) I_n)\|_2 < 1$; for all $i \in \overline{N}$ which is guaranteed by (4.15) and (4.16). □

The interpretations of Propositions 4.1-4.2 are directed as follows. Assume that the centralized control law stabilizes the ith parameterization. Then, a sufficiently small $\|\Delta_i\|_2$ satisfying (4.15) (or its counterpart in Proposition 4.1) and a tradeoff between stability abscissas, see (4.16), guarantee that the decentralized controller gain (4.10) stabilizes the ith parameterization of the switched system. However, it is required for the off-diagonal blocks of a stabilizing centralized controller gain K_{ci} to have sufficiently small norms. This allows their removal from the decentralized gain while guaranteeing closed-loop stabilization of each particular parameterization. An alternative useful approach in practice is to give conditions for decentralized stabilization and then either to remove the off-diagonal blocks of the centralized controller gain (then, the controller is a decentralized one) or to use some of them to increase the robustness against an increasing of the coupling effects between the various subsystems, evaluated in terms of increasing norms of the corresponding off-diagonal entries of the matrix of dynamics of the open-loop system. To formulate these issues more precisely, suppose a block diagonal structure for the decentralized controller gain

$$K_{di} := \text{Block diag} \left(K_1^{(jj)}, K_2^{(jj)}, \ldots, K_q^{(jj)} \right) \in \mathbf{R}^{m \times n}; \quad \forall i \in N_d \subset \overline{N}, \ j \in \overline{q}, \tag{4.18}$$

where $K_i^{(jj)} \in \mathbf{R}^{m_j \times n_j}$; for all $i \in N_d \subset \overline{N}$, for all $j \in \overline{q}$. It is now assumed that some of the N parameterizations of the switched system are stabilizable through a purely diagonal decentralized controller gain. The following technical assumption is then needed.

Assumption 4.3. The pairs $(A_i^{(jj)} + \rho_{ij} I_n, B_i^{(j)})$ are stabilizable; some $\rho_{ij} \in \mathbf{R}_{0+}$, for all $i \in N_d$, for all $j \in \overline{q}$.

The following result holds directly from the small gain theorem in a close way to that used in the proofs of Propositions 4.1 and 4.2.

Proposition 4.4. *If Assumption 4.3 holds then there exists a purely decentralized controller, given by a block diagonal controller gain* (4.18)*, such that the closed-loop* i*th parameterization is globally* *asymptotically stable. The matrix of the closed-loop dynamics* $\overline{A}_i = A_i + B_i K_{di}$ *is a stability matrix satisfying* re $\lambda \le -\rho_i$*; for all* $\lambda \in \sigma(\overline{A}_i)$*; for all* $i \in N_d$*, provided that*

$$\left\|\widehat{A}_i\right\|_2 < \frac{1}{\left\|\left(A_{0i} + B_i K_{di} + \rho_i I_n\right)^{-1}\right\|_2}; \quad \forall i \in N_d, \tag{4.19}$$

where $\widehat{A}_i = A_i - A_{0i}$*,* $A_{0i} :=$ Block diag $(A_i^{(11)}, A_i^{(22)}, \ldots, A_i^{(qq)})$ *and* $\rho_i := \min_{j\in\overline{q}}(\rho_{ij})$*; for all* $i \in N_d$*.*

Note that the norm constraint in Proposition 4.4 refers to the sufficiently norm smallness of the off-diagonal dynamics in order to achieve decentralized stabilization. It is now assumed that the rest of the parameterizations are stabilizable under a centralized controller as follows.

Assumption 4.5. The pairs $(A_i + \rho_i I_n, B_i)$ are stabilizable, some $\rho_i \in \mathbf{R}_{0+}$, for all $i \in \overline{N} \setminus N_d$.

The following result follows from Assumption 4.5 and Proposition 4.2 with $\rho_i = \rho_c; \rho_d \to \rho_{di}$ (being parameterization-dependent); for all $i \in \overline{N} \setminus N_d$.

Proposition 4.6. *The following properties hold.*

(i) *If Assumption 4.5 holds then there exists a centralized controller of gain* $K_i :=$ Block matrix $(K_i^{(jk)} \in \mathbf{R}^{m_j \times n_k} : j, k \in \overline{q}) \in \mathbf{R}^{m \times n}$*; for all* $i \in \overline{N} \setminus N_d$ *such that the closed-loop ith parameterization is globally asymptotically stable. The matrix of the closed-loop dynamics* $\overline{A}_i = A_i + B_i K_i$ *is a stability matrix satisfying* re $\lambda \le -\rho_i$*; for all* $\lambda \in \sigma(\overline{A}_i)$*; for all* $i \in \overline{N} \setminus N_d$*,*

(ii) *Assume that the centralized controller gain* K_i *is replaced with a (at least partially) decentralized one* $K_{di} = K_i - \Delta_i$ *such* Δ_i *has some zero off-diagonal entries and*

$$\|\Delta_i\|_2 < \frac{1 - \left\|\left(\overline{A}_{ci} + \rho_i I_n\right)^{-1} B_i\right\|_2 (\rho_i - \rho_{di})}{\left\|\left(\overline{A}_{ci} + \rho_i I_n\right)^{-1} B_i\right\|_2}; \quad \forall i \in \overline{N} \setminus N_d. \tag{4.20}$$

The matrix of the closed-loop dynamics $\overline{A}_i = A_i + B_i K_{di}$ *is a stability matrix satisfying* re $\lambda \le -\rho_{di}$*; for all* $\lambda \in \sigma(\overline{A}_i)$*; for all* $i \in \overline{N} \setminus N_d$*.*

Controllability Assumptions 4.7 below being stronger than Assumptions 4.3 and 4.5, together with Propositions 4.4 and 4.6, then lead to the first main result of this section.

Assumptions 4.7. The pairs $(A_i^{(jj)}, B_i^{(j)})$ are controllable for all $i \in N_d$, for all $j \in \overline{q}$ and the pairs (A_i, B_i) are also controllable for all $i \in \overline{N} \setminus N_d$. In other words, $\exists A_i^{(jj)^*}$ such that

$$\operatorname{rank}\left(B_i^{(j)} \otimes I_{n_j}\right) = \operatorname{rank}\left(B_i^{(j)} \otimes I_{n_j}, \operatorname{vec}\left(A_i^{(jj)^*} - A_i^{(jj)}\right)\right), \tag{4.21}$$

where $A_i^{(jj)^*}$ is a stability matrix with arbitrary finite stability abscissa $-\rho_{ij} < 0$; for all $i \in N_d$, for all $j \in \overline{q}$; and

$$\exists A_i^* \text{ such that} \operatorname{rank}(B_i \otimes I_n) = \operatorname{rank}\left(B_i \otimes I_n, \operatorname{vec}\left(A_i^* - A_i\right)\right), \tag{4.22}$$

where A_i^* is a stability matrix with arbitrary stability abscissa $-\rho_i < 0$; for all $i \in \overline{N} \setminus N_d$.

Note that, since controllability is equivalent to arbitrary pole-placement via linear state feedback (and then implies stabilizability) Propositions 4.4–4.6 are also fulfilled under Assumptions 4.7. The first main result of this section follows.

Theorem 4.8. *The following properties hold.*

(i) *The closed-loop switched system obtained from the open-loop system* (4.1)–(4.4) *with the linear state-feedback controller* (4.5)-(4.6) *is globally asymptotically stable with stability abscissa equal to or less than (but then arbitrarily close to)* $(-\rho) := -\min(\min(\rho_{ij} : i \in N_d,\ j \in \overline{q}), \min(\rho_i : i \in \overline{N} \setminus N_d))$ *if*

(1) *Assumptions 4.7 hold,*

(2) *the switched controller gain* (4.6) *of the mixed decentralized/centralized switched system is calculated as a solution of the linear algebraic compatible systems below:*

$$\left(B_i^{(j)} \otimes I_{n_j}\right) \operatorname{vec}\left(K_i^{(jj)}\right) = \operatorname{vec}\left(A_i^{(jj)^*} - A_i^{(jj)}\right); \quad \forall i \in N_d,\ j \in \overline{q}, \tag{4.23}$$

$$(B_i \otimes I_n) \operatorname{vec}(K_i) = \operatorname{vec}\left(A_i^* - A_i\right); \quad \forall i \in \overline{N} \setminus N_d, \tag{4.24}$$

(3) *the switching law* $\sigma : \mathbf{R}_{0+} \to \overline{N}$ *generates a set of switching instants* $SI_\sigma := \{t_i\}_{i \in \mathbf{Z}_{0+}}$ *subject to a sufficiently large minimum residence time* $\min_{t_i \in SI_\sigma}(t_{i+1} - t_i) \geq T \in \mathbf{R}_+$ *at each current active parameterization* $\sigma(t) = k \in \overline{N}$*; for all* $t \in [t_i, t_{i+1})$*, for any two consecutive* $t_i, t_{i+1} \in SI_\sigma$.

If, furthermore, $m_\ell \geq n_\ell$*,* $\operatorname{rank}(B_k) = m$*, and* $\operatorname{rank}(B_i^{(j)}) = m_j$*; for all* $\ell, j \in \overline{q}$*, for all* $k \in \overline{N} \setminus N_d$*, for all* $i \in N_d$ *then Assumptions 4.7 hold with full column ranks in both rank identities so that the controller gains being solutions to* (4.23)-(4.24) *are then unique.*

(ii) *If Proposition 4.6(ii) holds then Property (i) is still valid with a purely decentralized controller for the parameterizations* $i \in N_d$ *and a partly decentralized one for the parameterizations* $i \in \overline{N} \setminus N_d$ *by replacing the algebraic system of equations* (4.24) *by*

$$(B_i \otimes I_n) \operatorname{vec}(K_{di}) = \operatorname{vec}\left(A_i^* - A_i\right); \quad \forall i \in \overline{N} \setminus N_d \tag{4.25}$$

such that such Δ_i *satisfies* (4.20) *while having some zero off-diagonal entry(entries); for all* $i \in \overline{N} \setminus N_d$ *with* (4.25) *being solved as a companion one to* (4.23).

Note that the switching control law used in Theorem is not arbitrary since a minimum residence time has to be kept at each active parameterization. Note also that the use of compatible algebraic systems to calculate the controller gain entries exceeds the requirement of closed-loop stabilization, under open-loop stabilizability, since in this case it is only requested to re-allocate the closed-loop poles in the stable region via state feedback. The statement of an algebraic system is made for illustration purposes with the topic of centralized/decentralized control for a system decomposed into subsystems. If the problem were focused for the whole open-loop system (without its decomposition into coupled subsystems) then the algebraic system (4.24) can be used for all parameterizations $i \in \overline{N}$ (in a centralized control context) as fully equivalent to the closed-loop stabilizability problem. In this context, assume that Theorem 4.8 is restricted to a centralized control law with both the open-loop matrices of dynamics and the controller gains being in companion form for the N parameterizations. Thus, the algebraic problem solved is equivalent to closed-loop stabilization with prescribed pole-placement for each parameterization of the switched system. The closed-loop modes of each parameterization match the eigenvalues of the corresponding closed-loop matrix of dynamics A_i^*; for all $i \in \overline{N}$. The second main result of this section is concerned with arbitrary switching under pair-wise commutation properties of all the matrices A_i^*; for all $i \in \overline{N}$ or its sufficient closeness in terms of sufficiently small norms of all the pair-wise error matrices.

Theorem 4.9. *The following properties hold.*

(i) *The controller gains calculated from the mixed decentralized/centralized control law* (4.23)-(4.24) *globally asymptotically stabilize the switched closed-loop system for any arbitrary switching law* $\sigma : \mathbf{R}_{0+} \to \overline{N}$ *(i.e., without requiring a minimum residence time at each active parameterization) provided that*

  (1) *the closed-loop matrices of dynamics of all the individual parameterizations are all stability matrices;*

  (2) *those matrices are sufficiently close to each other in terms of smallness of the corresponding norm errors.*

(ii) *Assume that* A_i^* *(for all* $i \in \overline{N}$*) are all stability matrices which commute pair-wise; that is, they have zero commutators* $[A_i^*, A_j^*] = A_i^* A_j^* - A_j^* A_i^* = 0$; *for all* $i, j \in \overline{N}$. *Thus, the closed-loop switched system is globally asymptotically Lyapunov's stable.*

Proof. (i) Consider a common Lyapunov function candidate $V(t) = x^T(t)Px(t)$ for some $\mathbf{R}^{n\times n} \ni P = P^T > 0$ for the closed-loop switched system (4.1)–(4.6) whose time derivative is

$$\dot{V}(t) = x^T(t)\Big(\big(A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t)}\big)^T P + P\big(A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t)}\big)\Big)x(t) = -x^T(t)Q_{\sigma(t)}x(t) < 0 \tag{4.26}$$

for all nonzero $x(t)$ if

$$\big(A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t)}\big)^T P + P\big(A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t)}\big) = -Q_{\sigma(t)} < 0 \tag{4.27}$$

for some matrix function $Q(= Q^T > 0) : \mathbf{R}_{0+} \to \mathbf{R}^{n\times n}$. Choose a piecewise constant $Q_{\sigma(t)} = Q_i = Q_i^T > 0$ if $\sigma(t) = i \in \overline{N}$. Define matrix increments:

$$\Delta A_{ji} := A_i - A_j; \quad \Delta B_{ji} := B_i - B_j; \quad \Delta K_{ji} := K_i - K_j; \quad \Delta Q_{ji} := Q_i - Q_j; \quad \forall i, j \in \overline{N}. \tag{4.28}$$

Equation (4.27) holds if $Q_j > 0$ for any $j \in \overline{N}$ and $\Delta Q_{ji} > -Q_j$; for all $i(\neq j) \in \overline{N}$ since $(Q_j > 0 \wedge \Delta Q_{ji} > -Q_j$; for all $i(\neq j) \in \overline{N}) \Rightarrow Q_i > 0$; for all $i \in \overline{N}$. Thus, (4.27) holds, and then the switched system is globally asymptotically stable since it possesses a common Lyapunov function $V(t)$ for all its parameterizations from (4.26), if the constraints below hold:

$$Q_j = -\left(A_j + B_j K_j\right)^T P - P\left(A_j + B_j K_j\right) > 0 \quad \text{for some } j \in \overline{N}, \tag{4.29}$$

$$\begin{aligned} Q_j > -\Delta Q_{ji} = &-\left(\Delta A_{ji} + \Delta B_{ji} K_j + B_j \Delta K_{ji} + \Delta B_j \Delta K_{ji}\right)^T P \\ &- P\left(\Delta A_{ji} + \Delta B_{ji} K_j + B_j \Delta K_{ji} + \Delta B_j \Delta K_{ji}\right); \quad \forall i(\neq j) \in \overline{N} \end{aligned} \tag{4.30}$$

with $P = \int_0^\infty e^{A_j^{*T} t} Q_j e^{A_j^* t} dt P$; $j \in \overline{N}$ since $A_j^* = A_j + B_j K_j$ is a stability matrixv equations (4.29)-(4.30) imply that $A_i^* = A_i + B_i K_i$ are stability matrices; for all $i \in \overline{N}$. Equation (4.30) holds if

$$\begin{aligned} \lambda_{\min}(Q_j) &> \frac{K_j \lambda_{\max}(Q_j)}{\rho_j} \max_{i(\neq j)\in\overline{N}} \left(\left\| \Delta A_{ji} + \Delta B_{ji} K_j + B_j \Delta K_{ji} \right\|_2 \right) \\ &\geq 2\lambda_{\max}(P) \max_{i(\neq j)\in\overline{N}} \left(\left\| \Delta A_{ji} + \Delta B_{ji} K_j + B_j \Delta K_{ji} \right\|_2 \right), \end{aligned} \tag{4.31}$$

where $\lambda_{\max}(P) = \|P\|_2 \leq K_j \lambda_{\max}(Q_j)/2\rho_j$ for some $K_j, \rho_j \in \mathbf{R}_+$ such that the C_0-semigroup $e^{A_j^* t}$ of infinitesimal generator A_j^* is exponentially stable satisfying $\|e^{-A_j^* t}\|_2 \leq K e^{-\rho_j t}$; for all $t \in \mathbf{R}_{0+}$. Note from (4.23)-(4.24) that if $K_j^{(kk)}$ and K_j are first calculated for one of the parameterizations $j \in \overline{N}$ then the remaining gains are calculated from Theorem 4.8 as respective incremental gains from the subsequent nested algebraic sets of systems as follows:

$$\left(B_j^{(k)} \otimes I_{n_j}\right) \operatorname{vec}\left(K_j^{(kk)}\right) = \operatorname{vec}\left(A_j^{(kk)^*} - A_j^{(kk)}\right); \quad \text{some } j \in N_d,\ \forall k \in \overline{q}, \tag{4.32}$$

$$\begin{aligned} &\left(\left(\Delta B_{ji}^{(kk)} \otimes I_{n_j}\right) + \left(B_{jj}^{(kk)} \otimes I_{n_j}\right)\right) \operatorname{vec}\left(\Delta K_{ji}^{(kk)}\right) \\ &\quad = \operatorname{vec}\left(\Delta A j_{ji}^{(kk)^*} - \Delta A_{ji}^{(kk)}\right) - \left(\Delta B_{ji}^{(kk)} \otimes I_{n_j}\right) \operatorname{vec}\left(K_{jj}^{(kk)}\right) \end{aligned} \tag{4.33}$$

some $j \in N_d$, for all $i(\neq j) \in N_d$, for all $k \in \overline{q}$; and

$$\left(B_j \otimes I_n\right) \operatorname{vec}\left(K_j\right) = \operatorname{vec}\left(A_j^* - A_j\right); \quad \forall i \in \overline{N} \setminus N_d, \tag{4.34}$$

$$\begin{aligned} &\left(\left(\Delta B_{ji} \otimes I_n\right) + \left(B_{ji} \otimes I_n\right)\right) \operatorname{vec}\left(\Delta K_j\right) \\ &\quad = \operatorname{vec}\left(\Delta A_{ji}^* - \Delta A_{ji}\right) - \left(\Delta B_{ji} \otimes I_n\right) \operatorname{vec}\left(K_j\right); \quad \forall i \in \overline{N} \setminus N_d, \end{aligned} \tag{4.35}$$

provided that such algebraic systems are algebraically compatible, that is, the following rank conditions hold (Kronecker-Capelli theorem):

(1) $\operatorname{rank}(B_j^{(k)} \otimes I_{n_j}) = \operatorname{rank}(B_j^{(k)} \otimes I_{n_j} \operatorname{vec}(A_j^{(kk)^*} - A_j^{(kk)}))$ for some $j \in N_d$, for all $k \in \overline{q}$,

(2) $\operatorname{rank}((\Delta B_{ji}^{(kk)} \otimes I_{n_j}) + (B_{jj}^{(kk)} \otimes I_{n_j})) = \operatorname{rank}(\Delta B_{ji}^{(kk)} \otimes I_{n_j} + B_{jj}^{(kk)} \otimes I_{n_j} \operatorname{vec}(\Delta A j_{ji}^{(kk)^*} - \Delta A_{ji}^{(kk)}) - (\Delta B_{ji}^{(kk)} \otimes I_{n_j}) \operatorname{vec}(K_{jj}^{(kk)}))$ for some $j \in N_d$, for all $i(\neq j) \in N_d$, for all $k \in \overline{q}$,

(3) $\operatorname{rank}(B_j \otimes I_n) = \operatorname{rank}(B_j \otimes I_n \operatorname{vec}(A_j^* - A_j))$; for all $i \in \overline{N} \setminus N_d$,

(4) $\operatorname{rank}(\Delta B_{ji} \otimes I_n + B_{ji} \otimes I_n) = \operatorname{rank}(\Delta B_{ji} \otimes I_n + B_{ji} \otimes I_n \operatorname{vec}(\Delta A_{ji}^* - \Delta A_{ji}) - (\Delta B_{ji} \otimes I_n) \operatorname{vec}(K_j))$; for all $i \in \overline{N} \setminus N_d$.

From (4.32)–(4.34), the solution giving the incremental gains vary continuously with respect to the increments of the dynamics and control matrices of the open-loop system. Thus, for any arbitrary $\delta \in \mathbf{R}_+$, there exist constants $\varepsilon_a, \varepsilon_b \in \mathbf{R}_+$, depending on δ and which decrease monotonically with δ, such that $\|\Delta K_{ji}\|_2 \leq \varepsilon_K$; for all $i(\neq j) \in \overline{N}$, and $\max_i(\|\Delta A_{ji} + \Delta B_{ji} K_j + B_j \Delta K_{ji}\|_2) \leq \delta$ provided that $\|\Delta A_{ji}\|_2 \leq \varepsilon_a, \|\Delta B_{ji}\|_2 \leq \varepsilon_b$; for all $i(\neq j) \in \overline{N}$. Then, $V(t)$ is a Lyapunov function, since it is a common Lyapunov function for all its parameterizations, for δ is sufficiently small, so that $\varepsilon_a, \varepsilon_b$ are also sufficiently small, and $P = \int_0^\infty e^{A_j^{*T} t} Q_j e^{A_j^* t} dt$ is subject to $\lambda_{\max}(Q_j)/\lambda_{\min}(Q_j) < \rho_j/\delta K_j$ (then implying that $\lambda_{\max}(P) \leq K_j \lambda_{\max}(Q_j)/2\rho_j \leq \lambda_{\min}(Q_j)/2\rho_j$) for some chosen $j \in \overline{N}$ from (4.31). Property (i) has been proved. Property (ii) is a known result in the standard literature on switched systems (see, e.g., [3]). □

Remark 4.10. Note that if the deviation in norms of the corresponding matrices of the parameterizations are small enough the matrices of closed-loop dynamics of such parameterizations are almost commuting in the sense that the pair-wise commutators $[A_i^*, A_j^*]$ are matrices of sufficiently small norms.

Remark 4.11. A more general linear state-feedback controller than (4.5) allows hybrid control as follows:

$$u(t) = K_{\sigma(t)} v(t); \qquad v(t) = x(t) \vee x(t_k); \quad \text{for } t_k \in SI_{d\sigma} \subset SI_\sigma, \tag{4.36}$$

where $SI_\sigma := \{t_i\}_{i \in \mathbf{R}_{0+}}$ is the whole set of switching instants and $SI_{d\sigma}$ is some appropriate subset of SI_σ, where the controller switches to some discrete-time parameterization of sufficiently small-sampling period $h_k = t_{k+1} - t_k$ where it takes available fixed data to operate before the next switching. Theorems 4.8 and 4.9 can be extended to such an extended hybrid control law in a close way as it has been done in Sections 2 and 3. Also, direct extensions to the use of output-feedback control laws may be made by extending Theorems 4.8 and 4.9 by using the “ad hoc” tools of Sections 2 and 3.

5. Simulation Examples

This Section contains some simulation examples illustrating the theoretical results introduced in the previous Sections. In particular, simulation examples will comprise two different scenarios.

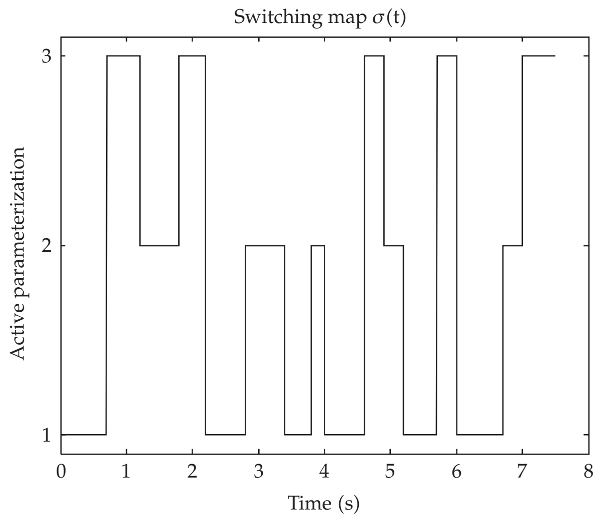

Figure 1: Switching map between different parameterizations.

5.1. State-Feedback Switched Hybrid Control

Consider the switched system given by $\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)$ with $\sigma : \mathbf{R}_{0+} \to \{1, 2, 3\}$, and

$$A_1 = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & -1.5 \\ 1 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} -1.5 & 0.5 \\ 1 & 0 \end{bmatrix}, \qquad B_1 = B_2 = B_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{5.1}$$

For each one of the above pairs (A_i, B_i), $i = 1, 2, 3$ a stabilizing state-feedback vector gain can be calculated to place the poles of the closed-loop system at −1 and −2. This can be performed since all pairs are controllable. This means that all the closed-loop matrices possess the same location for their poles. Indeed, all the closed-loop matrices turn into the same matrix under state-feedback. The controller is defined by $K_1 = [\,-5\ \ -1\,]$, $K_2 = [\,-2\ \ -0.5\,]$, $K_1 = [\,-1.5\ \ -2.5\,]$, and $A_i + B_iK_i = \left[\begin{smallmatrix} -3 & -2 \\ 1 & 0 \end{smallmatrix}\right]$ for $i = 1, 2, 3$.

The hybrid control law is calculated as in Theorem 3.1, $u(t) = K_{\sigma(t)}v(t)$ with $v(t) = x(t) \vee x(t_k)$ so that the control law is also switching between the continuous and the sampled state. While the discrete control law is used, it is not necessary to measure the value of the state at each time instant. Switching between different parameterizations is characterized by the arbitrary switching function depicted in the next Figure 1.

Furthermore, the switching between the continuous-time and the sampled control law is described by the sequence shown in Figure 2.

Thus, the system is hybrid while switching between parameterizations and between different control laws, continuous and discrete-time respectively. Also, note that since all the strictly stable closed-loop matrices are the same, they possess a common Lyapunov function. Then we are in conditions to apply Theorem 3.1 to guarantee the globally asymptotic stability of the above system. In particular, the evolution of the trajectories of (3.1) under (5.1) and the switching laws described by Figures 1 and 2 is given for sampling period $h = 0.1$ second by Figure 3.

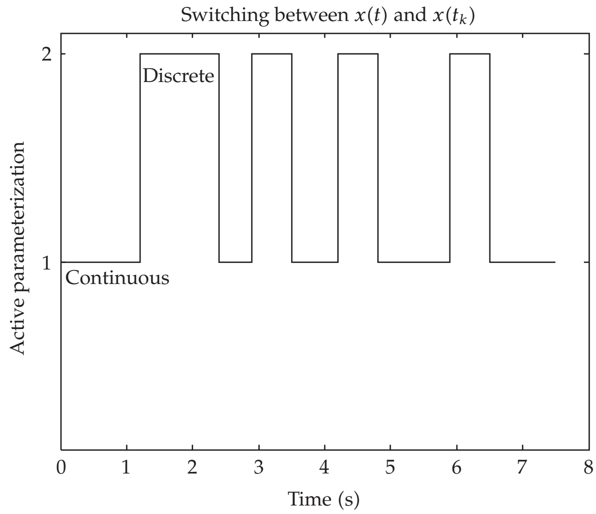

Figure 2: Type of state-feedback control law.

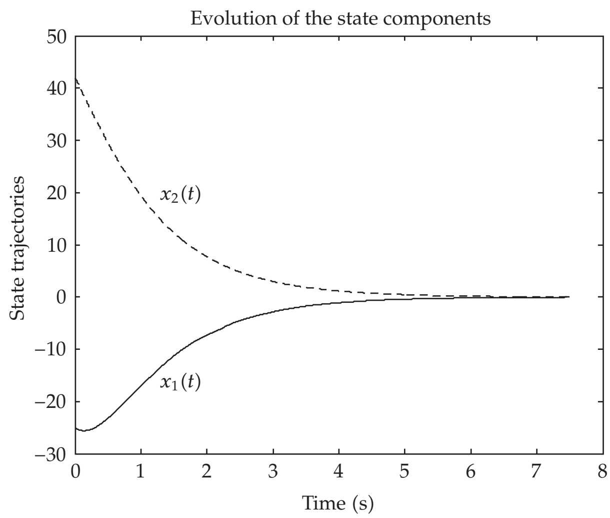

Figure 3: State trajectories for $h = 0.1$ second.

As it can be appreciated from Figure 3 all the state components tend to zero regardless the arbitrary switching in the state parameterizations and control law according to Theorem 3.1. Also, since all the closed-loop matrices have been selected to be same, the evolution of the closed-loop is free of jumps when switching between different parameterizations occurs. Moreover, as the sampling period increases, the behaviour of the system degrades as the following Figure 4 shows for a sampling period of 0.9 second.

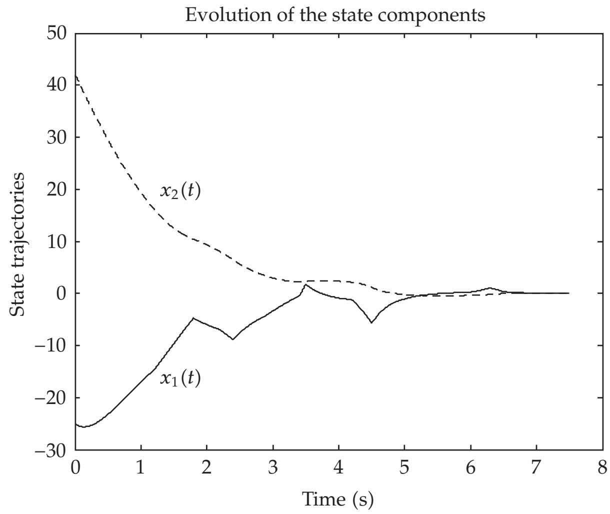

Figure 4: State trajectories for $h = 0.9$ second.

5.2. Decentralized Control for Hybrid Switching Coupled Systems

Consider a coupled system described by (4.1)–(4.4) in the form

$$\begin{aligned}\dot{x}(t) &= A^{(11)_{\sigma(t)}}x(t) + B^{(1)_{\sigma(t)}}u_1(t) + \varepsilon^{(12)_{\sigma(t)}}\begin{bmatrix}1 & 1\end{bmatrix}^T z(t),\\ \dot{z}(t) &= A^{(22)_{\sigma(t)}}z(t) + u_2(t) + \varepsilon^{(21)_{\sigma(t)}}\begin{bmatrix}1 & 1\end{bmatrix}x(t),\end{aligned} \tag{5.2}$$

with matrices $(A^{(11)_i}, B^{(1)_i})$, $i = 1, 2, 3$ being the same as in the example of Section 5.1 and $A^{(22)_{\sigma(t)}} \in \{2, 3, -2\}$, $\varepsilon^{(12)_{\sigma(t)}} \in \{0.5, 0.4, 0.4\}$ and $\varepsilon^{21_{\sigma(t)}} \in \{0.45, 0.5, 0.45\}$. Note that these latter coefficients describe the coupling between both subsystems. Thus, the complete dynamics matrices describing the system can be written as

$$A_{\sigma(t)} = \begin{bmatrix} A^{(11)_{\sigma(t)}} & \varepsilon^{(12)_{\sigma(t)}}\begin{bmatrix}1\\1\end{bmatrix} \\ \varepsilon^{(21)_{\sigma(t)}}\begin{bmatrix}1 & 1\end{bmatrix} & A^{(22)_{\sigma(t)}} \end{bmatrix}, \tag{5.3}$$

with state vector $x_a = [\, x^T \;\, z \,]^T$ while $A_{0\sigma(t)} = \begin{bmatrix} A^{(11)}_{\sigma(t)} & 0 \\ 0 & A^{(22)}_{\sigma(t)} \end{bmatrix}$ defines the dynamics matrices for the system without coupling. Our objective is to design a decentralized purely diagonal state-feedback hybrid control law to globally asymptotically stabilize the system. For the first subsystem, we design a state feedback control law to place the poles of each parameterization at −2.3 and −2.4 while for the second subsystem, the pole will be at −2.6 for all parameterizations. The purely diagonal control matrix reads $K = \text{Block Diag}\,(K_{1\sigma(t)}, K_{2\sigma(t)})$ with:

$$\begin{gathered} K_{11} = \begin{bmatrix} -2.6 & -3.28 \end{bmatrix}, \\ K_{12} = [-7.6 \;\; -3.28], \qquad K_{21} = -4.6, \\ K_{22} = -0.6, \qquad K_{23} = -5.6. \end{gathered} \tag{5.4}$$

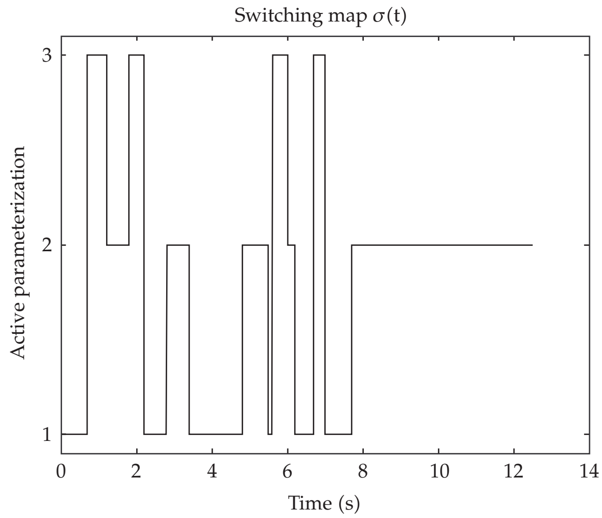

Figure 5: Switching map between different parameterizations.

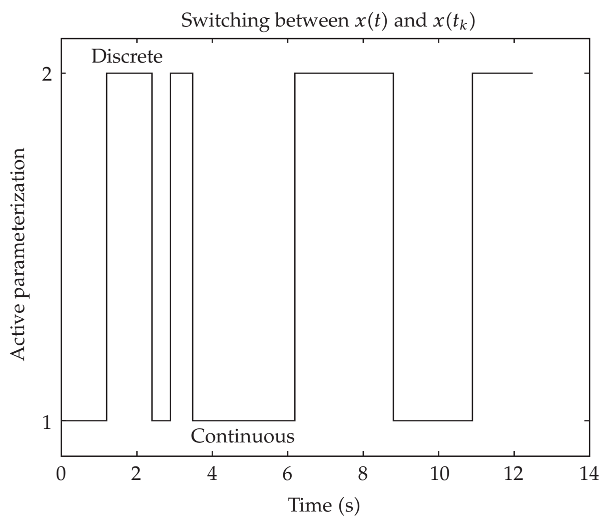

Figure 6: Switching between continuous-time and discrete control laws.

According to Proposition 4.4, the above block diagonal state-feedback gain stabilize the coupled system for $\rho_i = 0$, $i = 1, 2, 3$ since the mentioned small-gain-based condition becomes $0.707 < 0.741$, $0.7071 < 0.7464$, and $0.6364 < 0.7457$, respectively. Furthermore, all the matrices of the closed-loop dynamics are close to each other in terms of norms for the coupling effect associated with ε. As a result, the whole system is globally asymptotically stable for arbitrary switching between parameterizations and switching between continuous and discrete control laws for small-sampling period according to Theorem 4.9. In particular, consider the following switching between parameterizations and continuous and discrete control laws depicted in Figures 5 and 6.

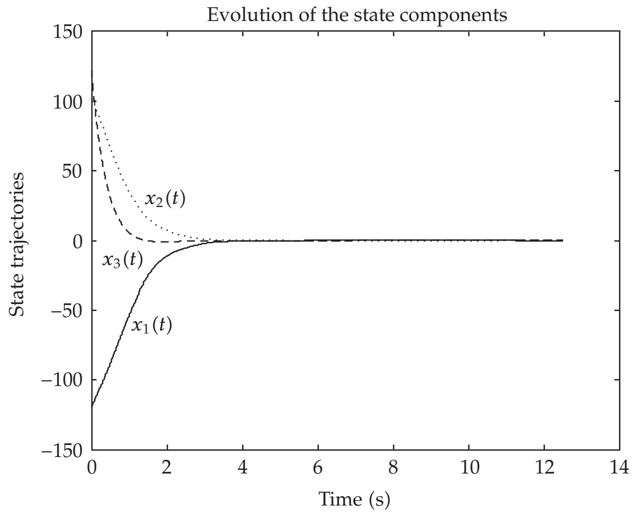

Figure 7: Evolution of the state-space components for $h = 0.1$ second.

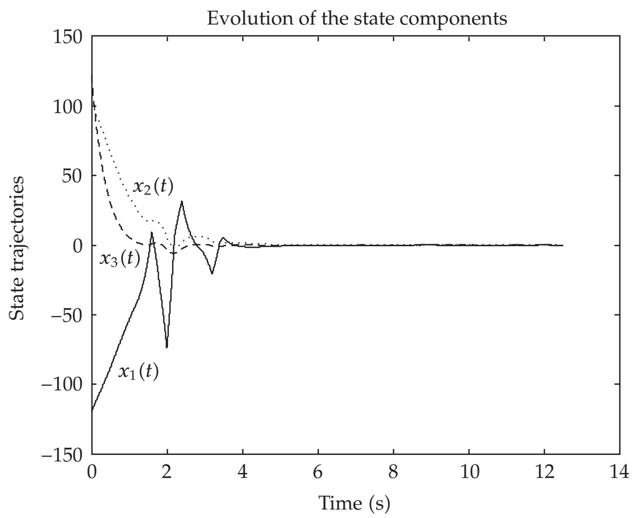

Figure 8: Evolution of the state-space components for $h = 0.4$ second.

As before, while the discrete control law is active, the state does not need to be measured at each time instant but only at sampling times. The evolution of the state-space trajectories when a sampling period of 0.1 second is used is described by Figure 7.

As depicted in Figure 7, the state-space trajectories tend to zero regardless the switching in both, parameterizations and type of control law, continuous or discrete. Also, there is no jump in the evolution of the components due to switching since the closed-loop remains unchanged due to an appropriate selection of the state feedback gains. As before, the behaviour of the state-space trajectories degrades as the sampling period increases as Figure 8 below shows for a sampling period of 0.4 second.

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