Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2009, Article ID 290123, 16 pages
doi:10.1155/2009/290123

Research Article

Harvesting of a Single-Species System Incorporating Stage Structure and Toxicity

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Received 25 June 2009; Accepted 8 November 2009

Recommended by Leonid Berezansky

A single species stage-structured model incorporating both toxicant and harvesting is proposed and studied. It is shown that toxicant has no influence on the persistent property of the system. The existence of the bionomic equilibrium is also studied. After that, we consider the system with variable harvest effect; sufficient conditions are obtained for the global stability of bionomic equilibrium by constructing a suitable Lyapunov function. The optimal policy is also investigated by using Pontryagin's maximal principle. Some numeric simulations are carried out to illustrate the feasibility of the main results. We end this paper by a brief discussion.

1. Introduction

As the development of industry, the influence of toxicant becomes more and more serious; toxicant which was produced by water pollution, air pollution, heavy metal pollution and organisms themselves, and so on, has great effects on the ecological communities.

Mathematical models which concerned with the influence of toxicant were first studied by Hallam and his colleagues [1–3]. After that, Freedman and Shukla [4] studied the single-species and predator-prey model; Chattopadhyay [5] and many scholars paid attention to the competition model [6–10]; Ma et al. [11], Das et al. [12], and Saha and Bandyopadhyay [13] laid emphasis on the predator-prey models. However, seldom did scholars investigated the stage-structured models with toxicant effects; to the best of authors' knowledge, only Xiao and Chen [14] explored a single-species model with stage-structured and toxicant substance. It is well known that many species in the natural world have a lifetime going through many stages, and in different stages, they have different reactions to the environment. For example, the immature may be more susceptible to the toxicant than the mature. Although there are many works on the stage-structured model (see [15–19] and the references cited therein), seldom did scholars consider the influence of the toxicant substance on the immature species.

In this paper, we study the single-species model with simplified toxicant effect, and we also take the commercially exploit into account. Since many species can be resources as human food, harvesting has a great influence both on the species population and on the economic revenue. There are many papers that deal with the effects of harvesting [10, 12, 20–22]; such topics as the optimal harvesting policy and the bionomic equilibrium are well studied by them. However, only recently scholars considered the ecosystem with both harvesting and toxicant effects (see [10, 12]), while no scholar investigated the stage structure population dynamics with both harvesting and toxicant effect.

We will study the following singe species stage structure ecosystem with both toxicant effect and harvesting:

$$x_1'(t) = ax_2 - d_1x_1 - d_2x_1^2 - \beta x_1 - r_1x_1^3,$$
$$x_2'(t) = \beta x_1 - b_1x_2 - c_2Ex_2, \tag{1.1}$$

where $x_1(t), x_2(t)$ represent the population density of the immature and the mature at time t, respectively, $r_1x_1^3$ is the effects of toxicant on the immature, E is the harvesting effort, c_2 is the catchability coefficient. We assume that the immature is density restriction, toxicant affects the immature population and only harvesting the mature species.

The paper is arranged as follows The stability property of equilibria is studied in the next section, and the existence of the bionomic equilibrium is explored in Section 3. In order to investigate the stability of the bionomic equilibrium and discuss how the population will be changed according to the the variable harvest effects, we assume that the E' is proportion to the economic revenue [23], that is,

$$E'(t) = kE(p_2c_2x_2 - c). \tag{1.2}$$

Sufficient condition which ensures the global stability of bionomic equilibrium is then investigated in Section 4. The optimal harvesting policy is studied in Section 5 and some numeric simulations are carried out in Section 6 to illustrate the feasibility of the main results. We end this paper by a briefly discussion.

2. The Steady States and Stability

It can be calculated that system (1.1) has two possible equilibriums:

(i) the trivial Equilibrium $E_0(0,0)$,

(ii) the equilibrium $E_*(x_1^*, x_2^*)$, where

$$ax_2^* - d_1x_1^* - d_2x_1^{*2} - \beta x_1^* - r_1x_1^{*3} = 0,$$
$$\beta x_1^* - b_1x_2^* - c_2Ex_2^* = 0. \tag{2.1}$$

By simple calculation we have

$$x_1^* = \frac{-d_2 + \sqrt{d_2^2 + 4r_1\left(a\beta/(b_1 + c_2E) - d_1 - \beta\right)}}{2r_1}, \qquad x_2^* = \frac{\beta}{b_1 + c_2E}x_1^*. \tag{2.2}$$

To ensure the positivity of the equilibrium $E_*(x_1^*, x_2^*)$, we assume that

$$a\beta > (b_1 + c_2E)(d_1 + \beta) \tag{2.3}$$

holds. We can see that x_1^*, x_2^* decrease as r_1 increases.

Next, we use the Jacobian matrix to determine the locally stability of the equilibriums. By simple calculation, we see that the Jacobian matrix of system (1.1) is

$$\begin{bmatrix} -d_1 - \beta - 2d_2x_1 - 3r_1x_1^2 & a \\ \beta & -b_1 - c_2E \end{bmatrix}. \tag{2.4}$$

For $E_0(0,0)$, the characteristic equation is

$$\lambda^2 + (d_1 + \beta + b_1 + c_2E)\lambda + (d_1 + \beta)(b_1 + c_2E) - a\beta = 0. \tag{2.5}$$

It is not hard to see that when $a\beta < (d_1 + \beta)(b_1 + c_2E)$, (2.5) has two negative roots or two complex roots with negative real parts; thus $E_0(0,0)$ is locally asymptotically stable; when $a\beta > (d_1 + \beta)(b_1 + c_2E)$, $E_0(0,0)$ is a saddle point.

For $E_*(x_1^*, x_2^*)$, the characteristic equation is

$$\lambda^2 + \left(d_1 + \beta + b_1 + c_2E + 2d_2x_1^* + 3r_1x_1^{*2}\right)\lambda + \left(d_1 + \beta + 2d_2x_1^* + 3r_1x_1^{*2}\right)(b_1 + c_2E) - a\beta = 0. \tag{2.6}$$

By applying (2.1), we have

$$\left(d_1 + \beta + 2d_2x_1^* + 3r_1x_1^{*2}\right)(b_1 + c_2E) - a\beta = (b_1 + c_2E)\left(d_2x_1^* + 2r_1x_1^{*2}\right) > 0. \tag{2.7}$$

Therefore, the characteristic equation of $E_*(x_1^*, x_2^*)$ has two negative roots or two complex roots with negative real parts; thus $E_*(x_1^*, x_2^*)$ is locally asymptotically stable.

Following we will take the idea and method of Xiao and Chen [14] to investigate the globally asymptotically stability property of the equilibriums, and we need to determine the existence or nonexistence of the limit cycle in the first quadrant.

For $E_*(x_1^*, x_2^*)$, it exists if $a\beta > (b_1 + c_2E)(d_1 + \beta)$; in this case $E_0(0,0)$ is a saddle point; thus, $E_*(x_1^*, x_2^*)$ is the unique stable equilibrium in the first quadrant if it exists. Let AB be the line segment of $L_1 : x_1 = p$ and BC the line segment of $L_2 : x_2 = q$, where $A(p,0), B(p,q), C(0,q)$, and p, q are positive constants which satisfy $p > x_1^*$, and

$$\frac{\beta p}{b_1 + c_2E} < q < \frac{p\left(d_1 + \beta + d_2p + r_1p^2\right)}{a}. \tag{2.8}$$

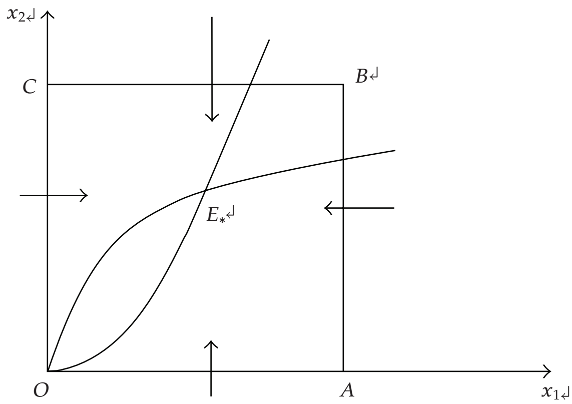

Figure 1: Trajectories enter rectangle $OABCO$ from exterior to interior.

By simple calculation, we have

$$\begin{aligned} \dot{x}_1|_{AB} &= ax_2 - d_1p - d_2p^2 - \beta p - r_1p^3|_{0\le x_2\le q} < 0, \\ \dot{x}_2|_{BC} &= \beta x_1 - (b_1 + c_2E)q|_{0\le x_1\le p} < 0. \end{aligned} \tag{2.9}$$

Thus AB, BC are the transversals of system (1.1). It is no hard to check that OA, OC are the transversals of system (1.1), and any trajectory enters region $OABCO$ from its exterior to interior (see Figure 1).

Denote

$$\begin{aligned} x_1'(t) &= ax_2 - d_1x_1 - d_2x_1^2 - \beta x_1 - r_1x_1^3 - c_1Ex_1 = P(x_1, x_2), \\ x_2'(t) &= \beta x_1 - b_1x_2 - c_2Ex_2 = Q(x_1, x_2). \end{aligned} \tag{2.10}$$

It is easy to see that

$$\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} = -d_1 - \beta - 2d_2x_1 - 3r_1x_1^2 - b_1 - c_2E < 0. \tag{2.11}$$

By Poincare-Bendixson theorem, there are no limit cycles in the first quadrant; thus $E_*(x_1^*, x_2^*)$ is globally asymptotically stable if it exists.

For $E_0(0,0)$, it is a unique equilibrium which is locally asymptotical stable if $a\beta < (b_1 + c_2E)(d_1 + \beta)$. Similarly to the above analysis we can show that $E_0(0,0)$ is globally asymptotically stable if $a\beta < (b_1 + c_2E)(d_1 + \beta)$ holds.

Therefore, we have the following.

(i) If $a\beta < (d_1 + \beta)(b_1 + c_2E)$, the trivial equilibrium $E_0(0,0)$ is globally asymptotically stable.

(ii) If $a\beta > (d_1 + \beta)(b_1 + c_2E)$, the positive equilibrium $E_*(x_1^*, x_2^*)$ is globally asymptotically stable.

We mention here that since condition (2.3) is independent of the toxicant of the system, thus, the globally asymptotically stability of the systems is independent of the intensities of toxicant, but from the expression of positive equilibrium we know that the density of both the immature and the mature species decreases while the toxicant increases; specially, the density of species will tend to indefinitely small if the toxicant substance is large enough.

3. Bionomic Equilibrium

For simplicity, we assume that the harvesting cost is a constant. Let c be the constant fishing cost per unit effort, and let p_2 be the constant price per unit biomass of the mature. The net revenue of harvesting at any time is given by:

$$P(x_1, x_2, E) = p_2 c_2 E x_2 - cE. \tag{3.1}$$

A bionomic equilibrium is both a biological equilibrium and a economic equilibrium, the biological equilibrium is given by $x_1'(t) = x_2'(t) = 0$, and the economic equilibrium occurs when the economic rent is $P = 0$, thus the bionomic equilibrium $\overline{E}(x_{1\infty}, x_{2\infty}, E_\infty)$ satisfying

$$ax_{2\infty} - d_1 x_{1\infty} - d_2 x_{1\infty}^2 - \beta x_{1\infty} - r_1 x_{1\infty}^3 = 0, \tag{3.2}$$

$$\beta x_{1\infty} - b_1 x_{2\infty} - c_2 x_{2\infty} E_\infty = 0, \tag{3.3}$$

$$p_2 c_2 x_{2\infty} - c = 0. \tag{3.4}$$

From (3.4) we get $x_{2\infty} = c/p_2 c_2$. Combining (3.4) and (3.2) we can obtain that $x_{1\infty}$ is one of the roots of the following equation:

$$r_1 x_1^3 + d_2 x_1^2 + (d_1 + \beta) x_1 - \frac{ac}{p_2 c_2} = 0. \tag{3.5}$$

Denoting $f(x) = r_1 x^3 + d_2 x^2 + (d_1 + \beta)x - \dfrac{ac}{p_2 c_2}$, we have

$$f(0) = -\frac{ac}{p_2 c_2} < 0, \quad f(+\infty) = +\infty, \quad f'(x) > 0 \quad (x \in [0, \infty)). \tag{3.6}$$

Hence, by the continuity of $f(x)$, there exists exactly one root in $(0, +\infty)$. From (3.3) and (3.4), to ensure the positivity of E_∞, one needs

$$x_{1\infty} > \frac{b_1 c}{\beta p_2 c_2}, \tag{3.7}$$

Thus we need to find a solution of $f(x)$ in $(b_1c/\beta p_2c_2, +\infty)$. Since (3.6) always holds, we only need

$$f\left(\frac{b_1c}{\beta p_2c_2}\right) < 0. \tag{3.8}$$

Thus, there exists a unique bionomic equilibrium if inequality (3.8) holds.

The existence of the bionomic equilibrium means that (i) Harvesting efforts $E > E_\infty$ cannot be maintained all the time, it will decrease because the total cost of harvesting exceed the total revenues; (ii) $E < E_\infty$ cannot be maintained indefinitely, harvesting is profitable in this occasion, and it will make the harvesting effort increases. Hence, the harvesting effort is always oscillating around E_∞. However, there is no answer about whether it will become stable or not because of the complex changing of E.

4. Globally Stability of the Bionomic Equilibrium

In this section, we study system (1.1) with variable harvest effects; sufficient condition for the globally asymptotically stability of the bionomic equilibrium will be derived. We assume that $E'(t) = kE(p_2c_2x_2 - c)$; then system (1.1) becomes

$$\begin{aligned}
x_1'(t) &= ax_2 - d_1x_1 - d_2x_1^2 - \beta x_1 - r_1x_1^3, \\
x_2'(t) &= \beta x_1 - b_1x_2 - c_2Ex_2, \\
E'(t) &= kE\left(p_2c_2x_2 - c\right).
\end{aligned} \tag{4.1}$$

System (4.1) has three possible equilibrium:

(i) the trivial equilibrium $V_0(0,0,0)$,

(ii) equilibrium in the absence of harvesting $V_1(\tilde{x}_1, \tilde{x}_2, 0)$, where

$$\tilde{x}_1 = \frac{-d_2 + \sqrt{d_2^2 + 4r_1(\beta a/b_1 - d_1 - \beta)}}{2r_1}, \qquad \tilde{x}_2 = \frac{\beta}{b_1}\tilde{x}_1, \tag{4.2}$$

and for the positiveness of $\tilde{x}_1$, $\tilde{x}_2$, we need

$$\beta a > (d_1 + \beta)b_1, \tag{4.3}$$

(iii) the interior equilibrium $\overline{E}(x_{1\infty}, x_{2\infty}, E_\infty)$, which is the bionomic equilibrium in Section 3; it exists if (3.8) holds.

For $V_0(0,0,0)$, the characteristic equation is given by

$$(\lambda + kc)\left((\lambda + d_1 + \beta)(\lambda + b_1) - \beta a\right) = 0. \tag{4.4}$$

It is easy to see that all of the roots of (4.4) are negative if $\beta a < b_1(d_1+\beta)$ holds; thus $V_0(0,0,0)$ is locally asymptotically stable if $\beta a < b_1(d_1+\beta)$, and unstable if $\beta a > b_1(d_1+\beta)$.

For $V_1(\tilde{x}_1, \tilde{x}_2, 0)$, the characteristic equation is given by

$$(\lambda - k(p_2c_2\tilde{x}_2 - c))\Big(\big(\lambda + d_1 + \beta + 2d_2\tilde{x}_1 + 3r_1\tilde{x}_1^2\big)(\lambda + b_1) - a\beta\Big) = 0. \tag{4.5}$$

It is no hard to see that $V_1(\tilde{x}_1, \tilde{x}_2, 0)$ is locally asymptotically stable if $p_2c_2\tilde{x}_2 - c < 0$, and unstable if $p_2c_2\tilde{x}_2 - c > 0$.

From the condition for the stability of V_0, V_1, we can see that low birth rate can make the population be driven to extinction, high harvesting cost has negative effect on fishing effort, and it can make the harvesting effect approach zero.

For $\overline{E}(x_{1\infty}, x_{2\infty}, E_\infty)$, the characteristic equation is

$$\lambda^3 + U\lambda^2 + V\lambda + L = 0, \tag{4.6}$$

where

$$\begin{aligned}
U &= b_1 + c_2E_\infty + d_1 + \beta + 2d_2x_{1\infty} + 3r_1x_{1\infty}^2 > 0,\\
V &= (b_1 + c_2E_\infty)\left(d_1 + \beta + 2d_2x_{1\infty} + 3r_1x_{1\infty}^2\right) + c_2^2kp_2x_{2\infty}E_\infty - a\beta\\
&= (b_1 + c_2E_\infty)\left(d_2x_{1\infty} + 2r_1x_{1\infty}^2\right) + c_2^2kp_2x_{2\infty}E_\infty > 0,\\
L &= c_2^2kp_2x_{2\infty}E_\infty\left(d_1 + \beta + 2d_2x_{1\infty} + 3r_1x_{1\infty}^2\right) > 0.
\end{aligned} \tag{4.7}$$

By Routh-Hurwitz criterion, all roots of (4.6) have negative real parts if and only if

$$U > 0, \quad L > 0, \quad UV > L. \tag{4.8}$$

By simple calculation, we know that condition (4.8) holds always, Thus, $\overline{E}(x_{1\infty}, x_{2\infty}, E_\infty)$ is locally asymptotically stable.

For the global stability of $\overline{E}(x_{1\infty}, x_{2\infty}, E_\infty)$, we construct the following Lyapunov function:

$$V = x_1 - x_{1\infty} - x_{1\infty}\ln\frac{x_1}{x_{1\infty}} + \left(x_2 - x_{2\infty} - x_{2\infty}\ln\frac{x_2}{x_{2\infty}}\right) + n\left(E - E_\infty - \ln\frac{E}{E_\infty}\right). \tag{4.9}$$

The time derivative of V along the positive solution of system (4.1) is

$$\begin{aligned}
\dot{V} &= \frac{x_1 - x_{1\infty}}{x_1} x_1'(t) + \frac{x_2 - x_{2\infty}}{x_2} x_2'(t) + n\frac{E - E_\infty}{E} E'(t) \\
&= \frac{x_1 - x_{1\infty}}{x_1}\left\{ a(x_2 - x_{2\infty}) - (d_1 + \beta)(x_1 - x_{1\infty}) - d_2\left(x_1^2 - x_{1\infty}^2\right) - r_1\left(x_1^3 - x_{1\infty}^3\right)\right\} \\
&\quad + \frac{x_2 - x_{2\infty}}{x_2}\{\beta(x_1 - x_{1\infty}) - b_1(x_2 - x_{2\infty}) - c_2(Ex_2 - E_\infty x_{2\infty})\} \\
&\quad + nk\frac{E - E_\infty}{E} E\{p_2 c_2(x_2 - x_{2\infty})\} \\
&= -\frac{(x_1 - x_{1\infty})^2}{x_1}\left\{ d_1 + \beta + d_2(x_1 + x_{1\infty}) + r_1\left(x_1^2 + x_1 x_{1\infty} + x_{1\infty}^2\right)\right\} \\
&\quad - \frac{(x_2 - x_{2\infty})^2}{x_2}(b_1 + c_2 E_\infty) + \left(\frac{a}{x_1} + \frac{\beta}{x_2}\right)(x_1 - x_{1\infty})(x_2 - x_{2\infty}) \\
&\quad + (-c_2 + nkp_2c_2)(x_2 - x_{2\infty})(E - E_\infty).
\end{aligned} \tag{4.10}$$

Let $nkp_1 = 1$, then we have

$$\begin{aligned}
\dot{V} = &-\frac{(x_1 - x_{1\infty})^2}{x_1}\left\{ d_1 + \beta + d_2(x_1 + x_{1\infty}) + r_1\left(x_1^2 + x_1 x_{1\infty} + x_{1\infty}^2\right)\right\} \\
&- \frac{(x_2 - x_{2\infty})^2}{x_2}(b_1 + c_2 E_\infty) + \left(\frac{a}{x_1} + \frac{\beta}{x_2}\right)(x_1 - x_{1\infty})(x_2 - x_{2\infty}).
\end{aligned} \tag{4.11}$$

If inequality

$$\frac{1}{x_1 x_2}\left(d_1 + \beta + d_2(x_1 + x_{1\infty}) + r_1\left(x_1^2 + x_1 x_{1\infty} + x_{1\infty}^2\right)\right)(b_1 + c_2 E) > \frac{1}{4}\left(\frac{a}{x_1} + \frac{\beta}{x_2}\right)^2 \tag{4.12}$$

holds, then $\dot{V}(t) < 0$ in set $\Omega = \{x_1 > 0,\ x_2 > 0\}$. Set

$$\begin{aligned}
g(x_1, x_2) = {}& x_1 x_2\left(d_1 + \beta + d_2(x_1 + x_{1\infty}) + r_1\left(x_1^2 + x_1 x_{1\infty} + x_{1\infty}^2\right)\right)(b_1 + c_2 E) \\
&- \frac{1}{4}(a x_2 + \beta x_1)^2,
\end{aligned} \tag{4.13}$$

then (4.12) holds in set Ω if $g(x_1, x_2) > 0$. By applying (3.2) and (3.3), we have

$$g(x_1, x_2) = \frac{1}{2}a\beta x_1 x_2 + x_1 x_2\left(d_1 x_1 + r_1 x_1^2 + r_1 x_1 x_{1\infty}\right)(b_1 + c_2 E) - \frac{1}{4}a^2 x_2^2 - \frac{1}{4}\beta^2 x_1^2. \tag{4.14}$$

If $x_1 \geq x_2$, then

$$g(x_1, x_2) \geq \frac{1}{2}a\beta x_2^2 + x_2^2\left(d_1 x_2 + r_1 x_2^2 + r_1 x_2 x_{1\infty}\right)(b_1 + c_2 E) - \frac{1}{4}\left(a^2 + \beta^2\right)x_1^2. \tag{4.15}$$

Thus, we can get that if

$$x_2 \leq x_1 < h_2(x_2) \tag{4.16}$$

holds, then $g(x_1, x_2) > 0$, where

$$h_2(x_2) = x_2\sqrt{\frac{2a\beta + 4(d_1 x_2 + r_1 x_2^2 + r_1 x_2 x_{1\infty})(b_1 + c_2 E_\infty)}{a^2 + \beta^2}}. \tag{4.17}$$

If $x_1 < x_2$, by the same way above, we can get the other sufficient condition for $g(x_1, x_2) > 0$, that is,

$$x_1 < x_2 < h_1(x_1), \tag{4.18}$$

where

$$h_1(x_1) = x_1\sqrt{\frac{2a\beta + 4(d_1 x_1 + r_1 x_1^2 + r_1 x_1 x_{1\infty})(b_1 + c_2 E_\infty)}{a^2 + \beta^2}}. \tag{4.19}$$

Therefore, if (4.16) or (4.18) holds, then $\dot{V}(t) < 0$ and the bionomic equilibrium is globally asymptotically stable.

The globally asymptotically stability of the bionomic equilibrium means that harvesting effect E which changes along (1.2) will make system (4.1) drive to the "bionomic equilibrium" and keep stable in the bionomic equilibrium.

5. Optimal Harvesting Policy

In this section, we study the optimal harvesting policy of system (1.1), and we consider the following present value J of a continuous time-stream:

$$J = \int_0^\infty P(x_1, x_2, E, t)e^{-\delta t}dt, \tag{5.1}$$

where P is the net revenue given by $P(x_1, x_2, E, t) = p_2 c_2 E x_2 - cE$, and δ denotes the instantaneous annual rate of discount; the aim of this section is to maximize J subjected to state equation (1.1). Firstly we construct the following Hamiltonian function:

$$H = (p_2 c_2 x_2 - c)Ee^{-\delta t} + \lambda_1\left(a x_2 - d_1 x_1 - d_2 x_1^2 - \beta x_1 - r_1 x_1^3\right) + \lambda_2(\beta x_1 - b_1 x_2 - c_2 E x_2), \tag{5.2}$$

where $\lambda_1(t)$, $\lambda_2(t)$ are the adjoint variables, E is the control variable satisfying the constraints $0 \le E \le E_{\max}$, and $\phi(t) = e^{-\delta t}(p_2c_2x_2 - c) - \lambda_2c_2x_2$ is called the switching function [23]. We aim to find an optimal equilibrium $(x_{1\delta}, x_{2\delta}, E_\delta)$ to maximize Hamiltonian H; since Hamiltonian H is linear in the control variable E, the optimal control can be the extreme controls or the singular controls; thus, we have

$$\begin{aligned} E = E_{\max}, &\quad \text{when } \phi(t) > 0, \text{ that is, when } \lambda_2(t)e^{\delta t} < p_2 - \frac{c}{c_2x_2}; \\ E = 0, &\quad \text{when } \phi(t) < 0, \text{ that is, when } \lambda_2(t)e^{\delta t} > p_2 - \frac{c}{c_2x_2}. \end{aligned} \tag{5.3}$$

When $\phi(t) = 0$, that is,

$$\lambda_2(t)e^{\delta t} = p_2 - \frac{c}{c_2x_2}, \quad \text{or} \quad \frac{\partial H}{\partial E} = 0. \tag{5.4}$$

In this case, the optimal control is called the singular control [23], and (5.4) is the necessary condition for the maximization of Hamiltonian H. By Pontrayagin's maximal principle, the adjoint equations are

$$\begin{aligned} \frac{d\lambda_1}{dt} &= -\frac{\partial H}{\partial x_1} = \lambda_1\left(d_1 + 2d_2x_1 + \beta + 3r_1x_1^2\right) - \lambda_2\beta, \\ \frac{d\lambda_2}{dt} &= -\frac{\partial H}{\partial x_2} = -p_2c_2Ee^{-\delta t} + \lambda_2(b_1 + c_2E) - \lambda_1 a. \end{aligned} \tag{5.5}$$

From (5.4) and (5.5), we have

$$\frac{d\lambda_1}{dt} - B\lambda_1 = Ae^{-\delta t}, \tag{5.6}$$

where $B = d_1 + 2d_2x_1 + \beta + 3r_1x_1^2,\ A = \beta(c/c_2x_2 - p_2)$. We can calculate that

$$\lambda_1 = -\frac{A}{B+\delta}e^{-\delta t}. \tag{5.7}$$

Substituting (5.7) into the second equation of (5.5), we get

$$\frac{d\lambda_2}{dt} - G\lambda_2 = De^{-\delta t}, \tag{5.8}$$

where $G = b_1 + c_2E,\ D = -p_2c_2E + A/(B+\delta)$. Therefore, we have

$$\lambda_2 = -\frac{D}{G+\delta}e^{-\delta t}. \tag{5.9}$$

It is obviously that $\lambda_1(t),\ \lambda_2(t)$ are bounded as $t \to \infty$.

Substituting (5.9) into (5.4), we obtain

$$p_2 - \frac{c}{c_2 x_2} = -\frac{D}{G+\delta}. \tag{5.10}$$

Our purpose is to find an optimal equilibrium solution; so we have

$$x_{1\delta} = x_1^* = \frac{-d_2 + \sqrt{d_2^2 + 4r_1\left(a\beta/(b_1+c_2E) - d_1 - \beta\right)}}{2r_1}, \qquad x_{2\delta} = x_2^* = \frac{\beta}{b_1+c_2E}x_1^*. \tag{5.11}$$

By (5.10) and (5.11), we can get $x_{1\delta}$, $x_{2\delta}$, and E_δ. Thus, the optimal policy is

$$E = \begin{cases} E_{\max}, & \text{when } \lambda_2(t)e^{\delta t} < p_2 - \dfrac{c}{c_2x_2}, \\ E_\delta, & \text{when } \lambda_2(t)e^{\delta t} = p_2 - \dfrac{c}{c_2x_2}, \\ 0, & \text{when } \lambda_2(t)e^{\delta t} > p_2 - \dfrac{c}{c_2x_2}. \end{cases} \tag{5.12}$$

Again, from (5.10) we have

$$P = (p_2c_2x_2 - c)E = -\frac{Dc_2x_2}{G+\delta}E. \tag{5.13}$$

When $\delta \to \infty$, $P \sim o(\delta^{-1})$. Therefore, $\delta = 0$ leads to the maximization of P.

6. Number Simulations

In the following examples, we take the parameters values as $a = 2$, $d_1 = 0.1$, $d_2 = 0.1$, $c_2 = 0.2$, $b_1 = 0.1$, and $\beta = 0.2$. We will see how the system behavior is while the toxicant effect changes.

Example 6.1. $E = 1$; in this case, $a\beta = 0.4 > 0.09 = (d_1+\beta)(b_1+c_2E)$. From the results in Section 2, we know that for a given r_1, the system admits a unique global stable positive equilibrium. Indeed, considering system (1.1) and the initial conditions $(6,2)$, $(5,10)$, and $(1,5)$, respectively, we can see that

(i) $r_1 = 0$, $E^*(10.33, 6.89)$ is global stable;

(ii) $r_1 = 0.01$, $E^*(6.33, 4.22)$ is global stable (Figure 2);

(iii) $r_1 = 1$, $E^*(0.97, 0.65)$ is global stable (see Figure 3);

(iv) $r_1 = 100$, $E^*(0.01, 0.07)$ is global stable (Figure 4).

Example 6.2. $k = 0.1, p_2 = 2, c = 0.2, \delta = 0.01$, and $E'(t) = 0.1E(0.4x_2 - 0.2)$. Considering system (4.1) with initial condition $(2,3,3)$, $(4,5,6)$, and $(1,1,1)$, we have the following.

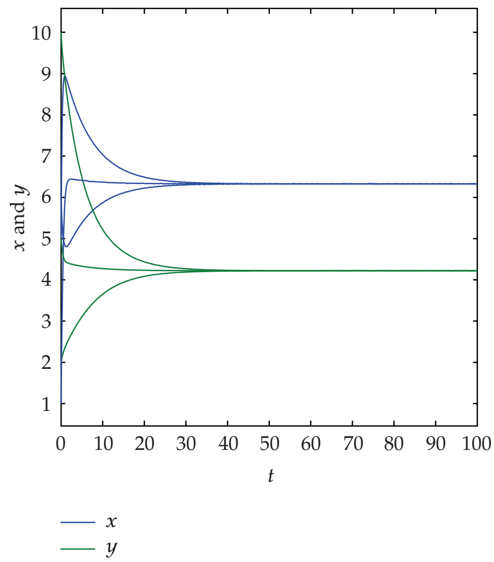

Figure 2: Solution curves of system (1.1) with the parameters given by Example 6.1 when $r_1 = 0.01$.

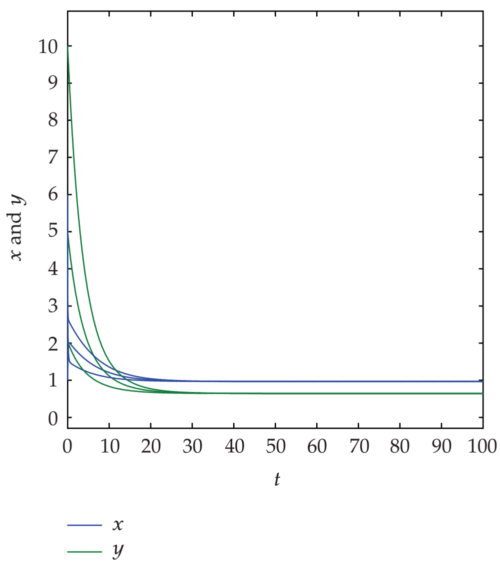

Figure 3: Solution curves of system (1.1) with the parameters given by Example 6.1 when $r_1 = 1$.

(i) $r_1 = 0$; the bionomic equilibrium $\overline{E}(2, 0.5, 3.5)$ is globally stable (Figure 5). The optimal equilibrium $(10.32, 6.87, 1)$ is far away from the bionomic equilibrium.

(ii) $r_1 = 1$; the bionomic equilibrium $\overline{E}(0.87, 0.5, 1.24)$ is globally stable (Figure 6). The optimal equilibrium is $(1.26, 1.28, 0.49)$.

(iii) $r_1 = 10$; the bionomic equilibrium $\overline{E}(0.44, 0.5, 0.38)$ is globally stable (Figure 7). The optimal equilibrium is $(0.51, 0.74, 0.18)$.

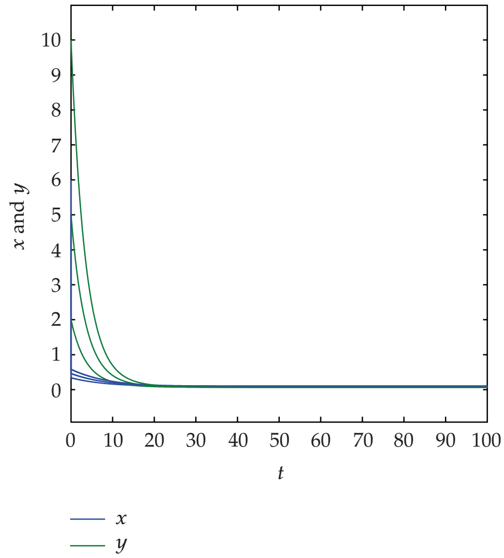

Figure 4: Solution curves of system (1.1) with the parameters given by Example 6.1 when $r_1 = 100$.

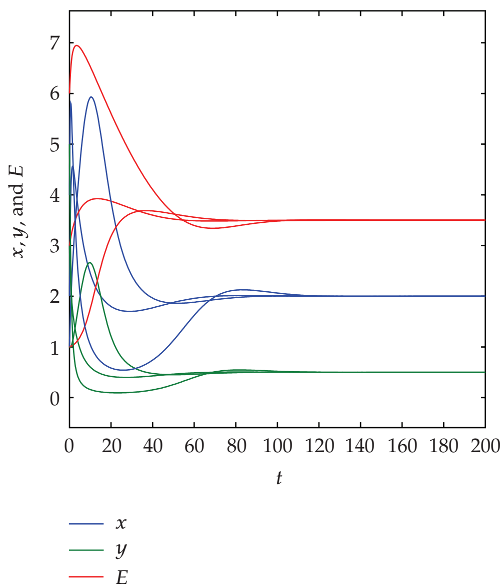

Figure 5: Solution curves of system (4.1) with the parameters given by Example 6.2 when $r_1 = 0$.

(iv) $r_1 = 100$; both the bionomic equilibrium $\overline{E}(0.2, 0.5, -0.08)$ and the optimal equilibrium $(0.20, 0.44, -0.046)$ are unfeasible.

From the above examples we can found the following phenomena:

(i) Increasing of toxicant will make the population of both mature and immature decrease.

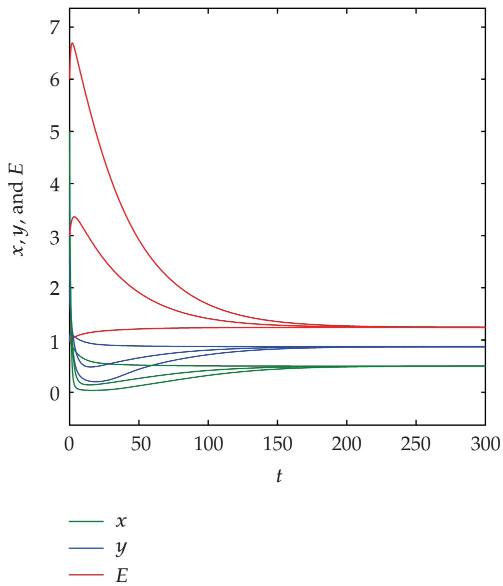

Figure 6: Solution curves of system (4.1) with the parameters given by Example 6.2 when $r_1 = 1$.

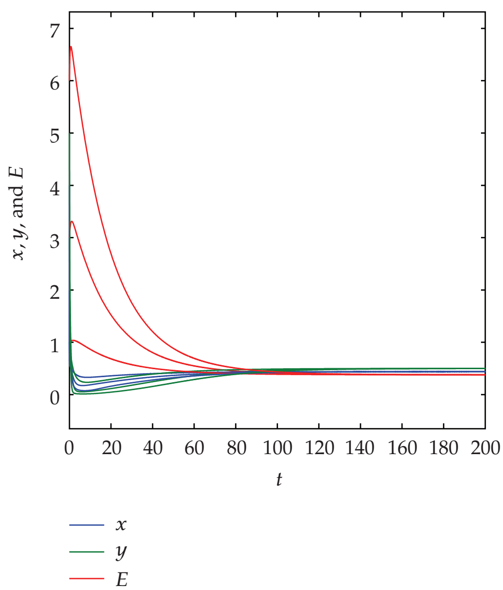

Figure 7: Solution curves of system (4.1) with the parameters given by Example 6.2 when $r_1 = 10$.

(ii) The bionomic equilibrium exists and globally stable both in the absence of toxicant and in the present of toxicant; however, with the increase of toxicant, the immature population $x_{1\infty}$ and the harvesting effect E decrease while the mature population $x_{2\infty}$ remains as the same.

(iii) The bionomic equilibrium and the optimal equilibrium will become unfeasible if the toxicant is large enough.

(iv) The immature, mature populations, and the harvesting effect in the optimal equilibrium are decreasing as the toxicant is increasing.

(v) The optimal equilibrium becomes more and more close to the bionomic equilibrium as the toxicant effect increases.

7. Discussion

In this paper, we consider the single-species stage structure model incorporating both toxicant and harvesting, and we assume that only the immature affected by the toxicant.

Firstly, we explore the local and global stability properties of the equilibria of the system. Next, we investigate the existence and stability properties of the bionomic equilibrium. Finally, the optimal harvesting is studied, and it is found that there exists two optimal equilibria when the toxicant varies in a certain set. Some numeric examples to illustrate how the equilibrium (include bionomic equilibrium and optimal equilibrium) changes with the toxicant are also given.

Nevertheless, as we know, the immature needs a certain time to develop to mature stage, the model incorporating time delay may be more reasonable and worth further study, and we leave this for future study.

Acknowledgment

This work was supported by the Program for New Century Excellent Talents in Fujian Province University (0330-003383).

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