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*Research Article*

# Impulsive Exponential Stabilization of Functional Differential Systems with Infinite Delay

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By using the Razumikhin technique and Lyapunov functions, we investigated the impulsive exponential stabilization of functional differential systems with infinite delay. A new result on the exponential stabilization by impulses is gained. Our result shows that impulses can make unstable systems stable. A numerical example is given to illustrate the feasibility of the result.



## 1. Introduction

Recently, there are many results of impulsive stability for delay systems as impulses can make unstable systems stable and stable systems unstable after impulse effects; see [1–21] and references therein. The problem of stabilizing the solutions by imposing proper impulsive control for delayed system now attracts more and more authors' attentions; see [22–26]. For example, in [22, 26], the authors have investigated impulsive stabilization of second-order differential equations with finite delay. The main tools used are Lyapunov functionals, stability theory, and control by impulses. In [23], by employing the Razumikhin technique and Lyapunov functions, several global exponential stability criteria are established for general impulsive differential equations with finite delay. However, not much has been developed in the direction of the stabilization theory of impulsive functional differential systems, especially for infinite delays of impulsive functional differential systems. This is due to some theoretical and technical difficulties; see [14, 16–21, 24]. In [24], Luo and Shen studied impulsive stabilization of functional differential equations with infinite delay. By using Lyapunov functions and Razumikhin techniques, some Razumikhin type theorems on uniform asymptotical stability are obtained. However, to the best of the authors' knowledge, there is little work on the impulsive exponential stabilization of functional differential systems with infinite delay.

The aim of this work is to establish a criterion on the impulsive exponential stabilization of functional differential systems with infinite delay by using Lyapunov functions and the Razumikhin technique. Our result shows that functional differential equations with infinite delay may be exponentially stabilized by impulses. Moreover, to some degree, the result we obtained is less conservative and more feasible than that given in [23].

This paper is organized as follows. In Section 2, we introduce some notations and definitions. Section 3 is devoted to the main results, and a numerical example is given to demonstrate the effectiveness of our result. In the last section, concluding remarks are given in Section 4.

## 2. Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, and $\mathbb{R}^n$ the $n$-dimensional real space equipped with the Euclidean norm $\|\cdot\|$. For any $t \geq t_0 \geq 0 > -\alpha \geq -\infty$, let $f(t, x(s))$ where $s \in [t-\alpha, t]$ or $f(t, x(\cdot))$ be a Volterra type functional. In the case when $\alpha = +\infty$, the interval $[t-\alpha, t]$ is understood to be replaced by $(-\infty, t]$.

Consider the following impulsive functional differential systems:

$$\begin{aligned} x'(t) &= f(t, x(\cdot)), \quad t \geq t_0,\ t \neq t_k, \\ \Delta x|_{t=t_k} &= I_k\left(t_k, x_{t_k^-}\right), \quad k = 1, 2 \ldots, \end{aligned} \tag{2.1}$$

where the impulse times $t_k$ satisfy $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$, $\lim_{k\to+\infty} t_k = +\infty$, and $x'$ denotes the right-hand derivative of $x$. $f \in C([t_{k-1}, t_k) \times \mathbb{C}, \mathbb{R}^n)$, $f(t, 0) = 0$. $\mathbb{C}$ is an open set in $PC([-\alpha, 0], \mathbb{R}^n)$, where $PC([-\alpha, 0], \mathbb{R}^n) = \{\psi : [-\alpha, 0] \to \mathbb{R}^n$ is continuous everywhere except at finite number of points $t_k$, at which $\psi(t_k^+)$ and $\psi(t_k^-)$ exist and $\psi(t_k^+) = \psi(t_k)\}$. Define $PCB(t) = \{x \in \mathbb{C} : x \text{ is bounded}\}$. For $\psi \in PCB(t)$, the norm of $\psi$ is defined by $\|\psi\| = \sup_{-\alpha\leq\theta\leq 0}|\psi(\theta)|$. For any $\sigma \geq 0$, let $PCB_\delta(\sigma) = \{\psi \in PCB(\sigma) : \|\psi\| < \delta\}$. Let $K_1 = \{a \in C(\mathbb{R}_+, \mathbb{R}_+) \mid a(0) = 0$ and $a(t) > 0$ for $t > 0$ and $a$ is strictly increasing in $t\}$.

For each $k = 1, 2 \ldots$, $I_k(t, x) \in C([t_0, \infty) \times \mathbb{R}^n, \mathbb{R}^n)$, $I_k(t, 0) = 0$, and for any $\rho > 0$, there exists a $\rho_1 > 0$ $(0 < \rho_1 < \rho)$ such that $x \in S(\rho_1)$ implies that $x + I_k(t_k, x) \in S(\rho)$, where $S(\rho) = \{x : \|x\| < \rho, x \in \mathbb{R}^n\}$.

For any given $\sigma \geq t_0$, system (2.1) is supplemented with initial conditions of the form

$$x_\sigma = \phi, \tag{2.2}$$

where $\phi \in PC([-\alpha, 0], \mathbb{R}^n)$.

In this paper, we assume that the solution for the initial problem (2.1)-(2.2) does exist and is unique which will be written in the form $x(t, \sigma, \phi)$; see [1, 4, 13]. Since $f(t, 0) = 0, I(t, 0) = 0, k = 1, 2, \ldots$, then $x(t) = 0$ is a solution of (2.1)-(2.2), which is called the trivial solution. In this paper, we always assume that the solution $x(t, \sigma, \phi)$ of (2.1)-(2.2) can be continued to $\infty$ from the right of $\sigma$.

We introduce some definitions as follows.

*Definition 2.1* (see [1]). The function $V : [-\alpha,\infty) \times \mathbb{C} \to \mathbb{R}_+$ belongs to class $v_0$ if

(i) $V$ is continuous on each of the sets $[t_{k-1}, t_k) \times \mathbb{C}$ and $\lim_{(t,\varphi)\to(t_k^-,\varphi)} V(t,\varphi) = V(t_k^-,\varphi)$ exists;

(ii) $V(t,x)$ is locally Lipschitzian in $x$ and $V(t,0) \equiv 0$.

*Definition 2.2* (see [1]). Let $V \in v_0$, for any $(t,\varphi) \in [t_{k-1}, t_k) \times \mathbb{C}$, the upper right-hand Dini derivative of $V(t,x)$ along the solution of (2.1)-(2.2) is defined by

$$D^+V(t,\varphi(0)) = \limsup_{h\to 0^+} \frac{\{V(t+h,\varphi(0)+hf(t,\varphi)) - V(t,\varphi(0))\}}{h}. \tag{2.3}$$

*Definition 2.3* (see [1]). Assume that $x(t) = x(t,\sigma,\phi)$ is the solution of (2.1)-(2.2) through $(\sigma,\phi)$. Then the trivial solution of (2.1)-(2.2) is said to be exponentially stable if for any $\varepsilon > 0$, $\sigma \geq t_0$, there exist constants $\lambda > 0$ and $\delta = \delta(\varepsilon) > 0$ such that $\phi \in PCB_\delta(\sigma)$ implies $\|x(t)\| < \varepsilon \cdot e^{-\lambda(t-\sigma)}$, $t \geq \sigma$.

## 3. Main Results

In this section, we shall develop Lyapunov-Razumikhin methods and establish some theorems which provide sufficient conditions for exponential stability of the trivial solution of (2.1)-(2.2).

**Theorem 3.1.** *Assume that there exist functions $V \in v_0$, $w \in K_1$ and positive constants $p, c_1, \eta, \lambda, \beta_k \geq 0$, $k \in \mathbb{Z}_+$ and $\gamma \in (1, M^\star]$, $M^\star > 1$ such that the following conditions hold:*

(i) $c_1\|x\|^\eta \leq V(t,x) \leq w(\|x\|)$, $(t,x) \in [-\alpha,\infty) \times S(\rho)$,

(ii) *for any $\sigma \geq t_0$ and $\varphi \in PC([-\alpha,0], S(\rho))$, if $\gamma e^{\lambda(t-\sigma)} V(t,\varphi(0)) \geq V(t+\theta,\varphi(\theta))$, $-\alpha \leq \theta \leq 0$, $t \neq t_k$, $k \in \mathbb{Z}_+$, then*

$$D^+V(t,\varphi(0)) \leq pV(t,\varphi(0)), \tag{3.1}$$

(iii) *for all $(t_k,\varphi) \in R_+ \times PC([-\alpha,0], S(\rho_1))$, $V(t_k, \varphi(0) + I_k(t_k,\varphi)) \leq \beta_k V(t_k^-, \varphi(0))$, $k \in \mathbb{Z}_+$, where $\beta_k$ satisfies $\prod_{k=1}^{\infty} \max\{\beta_k M^\star, 1\} < \infty$,*

(iv) $t_k - t_{k-1} < \ln\gamma/(p+\lambda)$, $k \in \mathbb{Z}_+$,

*then the trivial solution of* (2.1)-(2.2) *is exponentially stable with the approximate exponential convergence rate $\lambda/\eta$.*

*Proof.* From condition (iii), there exists constant $M > 0$ such that

$$\prod_{k=1}^{\infty} \max\{\beta_k M^\star, 1\} \leq M. \tag{3.2}$$

For any $\varepsilon \in (0,\rho_1)$, we choose $\delta = \delta(\varepsilon) > 0$ such that $w(\delta) \le (M^\star M)^{-1}\min\{c_1\varepsilon^\eta, \varepsilon\}$. Then, for any $\sigma \ge t_0$, let $x(t) = x(t,\sigma,\phi)$ be a solution of (2.1)-(2.2) through $(\sigma,\phi)$. We shall show for any $\phi \in PCB_\delta(\sigma)$

$$\|x(t)\| < \varepsilon \cdot e^{-(\lambda/\eta)(t-\sigma)}, \quad t \ge \sigma. \tag{3.3}$$

For convenience, suppose $V(t) = V(t,x(t))$ and

$$\widetilde{V}(\sigma) = \max\left\{ \sup_{-\alpha \le \theta \le 0} V(\sigma+\theta, \varphi(\theta)), M^\star V(\sigma) \right\}. \tag{3.4}$$

Since $\phi \in PCB_\delta(\sigma)$, we have $\widetilde{V}(\sigma) \le M^\star w(\delta)$.

Suppose that $\sigma \in [t_{l-1}, t_l),\ l \in \mathbb{Z}_+$. Next we prove for $t \in [\sigma, t_l) \cup [t_k, t_{k+1}),\ k \ge l$,

$$V(t) \le \widetilde{V}(\sigma)\left(\prod_{s=l}^{k} \max\{\beta_s M^\star, 1\}\right) e^{-\lambda(t-\sigma)}, \quad t \ge \sigma. \tag{3.5}$$

First, it is clear that for $t \in [\sigma-\alpha, \sigma]$

$$c_1\|x(t)\|^\eta \le V(t) \le \widetilde{V}(\sigma) \le M^\star w(\delta) \le M^{-1}\min\{c_1\varepsilon^\eta, \varepsilon\} < c_1\varepsilon^\eta, \tag{3.6}$$

which implies $\|x(t)\| < \varepsilon < \rho_1,\ t \in [\sigma-\alpha, \sigma]$.

We next claim that (3.5) holds for all $t \in [\sigma, t_l)$, that is,

$$V(t) \le \widetilde{V}(\sigma) e^{-\lambda(t-\sigma)}, \quad t \in [\sigma, t_l). \tag{3.7}$$

In order to do this, let

$$\Gamma(t) = \begin{cases} V(t)e^{\lambda(t-\sigma)}, & t \ge \sigma, \\ V(t), & \sigma-\alpha \le t \le \sigma, \end{cases} \tag{3.8}$$

then it is obvious that $\Gamma(t) \ge V(t)$ for all $t \ge \sigma-\alpha$.

Next we prove $\Gamma(t) \le \widetilde{V}(\sigma)$ for $t \in [\sigma, t_l)$, which implies that (3.5) holds for $t \in [\sigma, t_l)$. Suppose that this assertion is false, then there exists some $t \in [\sigma, t_l)$ such that $\Gamma(t) > \widetilde{V}(\sigma)$.

Let

$$t^\star = \inf\left\{ t \in [\sigma, t_l) \mid \Gamma(t) \ge \widetilde{V}(\sigma) \right\}, \tag{3.9}$$

then in view of $\Gamma(\sigma) = V(\sigma) \le M^{\star -1}\widetilde{V}(\sigma) < \widetilde{V}(\sigma)$, we get

$$t^\star \in (\sigma, t_l), \quad \Gamma(t^\star) = \widetilde{V}(\sigma), \quad \Gamma(t) < \widetilde{V}(\sigma), \quad t \in [\sigma, t^\star). \tag{3.10}$$

Considering (3.6), we also obtain

$$V(t) \le \Gamma(t) < \widetilde{V}(\sigma) \quad \forall t \in [\sigma-\alpha, t^\star). \tag{3.11}$$

On the other hand, considering $\gamma \in (1, M^\star]$, we get

$$\begin{aligned} \Gamma(t^\star) &= \widetilde{V}(\sigma) > \frac{1}{\gamma}\widetilde{V}(\sigma), \\ \Gamma(\sigma) &= V(\sigma) < \frac{1}{M^\star}\widetilde{V}(\sigma) \le \frac{1}{\gamma}\widetilde{V}(\sigma). \end{aligned} \tag{3.12}$$

So we can define

$$t^{\star\star} = \sup\left\{ t \in [\sigma, t^\star] \mid \Gamma(t) \le \frac{1}{\gamma}\widetilde{V}(\sigma) \right\}, \tag{3.13}$$

which implies that

$$t^{\star\star} \in [\sigma, t^\star), \quad \frac{1}{\gamma}\widetilde{V}(\sigma) = \Gamma(t^{\star\star}), \quad \frac{1}{\gamma}\widetilde{V}(\sigma) < \Gamma(t), \quad t \in (t^{\star\star}, t^\star]. \tag{3.14}$$

It follows from (3.11) that

$$\gamma V(t) e^{\lambda(t-\sigma)} = \gamma\Gamma(t) \ge \widetilde{V}(\sigma) \ge V(s), \quad \sigma-\alpha \le s \le t,\ t \in [t^{\star\star}, t^\star]. \tag{3.15}$$

Using condition (ii), the inequality $D^+V(t) \le pV(t)$ holds for all $t \in [t^{\star\star}, t^\star]$. Hence, we obtain for $t \in [t^{\star\star}, t^\star]$

$$\begin{aligned} D^+\Gamma(t) &= D^+V(t)e^{\lambda(t-\sigma)} + \lambda V(t)e^{\lambda(t-\sigma)} \\ &= e^{\lambda(t-\sigma)}\left(D^+V(t) + \lambda V(t)\right) \\ &\le e^{\lambda(t-\sigma)}\left(pV(t) + \lambda V(t)\right) \\ &= V(t)e^{\lambda(t-\sigma)}(p+\lambda) \\ &\le (p+\lambda)\Gamma(t). \end{aligned} \tag{3.16}$$

From (iv), we define $\tau \doteq \max_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\}$, then $\tau(p+\lambda) < \ln\gamma$.

Thus, we have

$$\int_{t^{\star\star}}^{t^\star} \frac{d\Gamma(s)}{\Gamma(s)} = \int_{\Gamma(t^{\star\star})}^{\Gamma(t^\star)} \frac{ds}{s} = \int_{(1/\gamma)\widetilde{V}(\sigma)}^{\widetilde{V}(\sigma)} \frac{ds}{s} \ge \ln\gamma > \tau p + \lambda\tau. \tag{3.17}$$

However,

$$\int_{\Gamma(t^{**})}^{\Gamma(t^*)} \frac{ds}{s} \leq \int_{t^{**}}^{t^*} (p+\lambda) ds \leq \int_{t^{**}}^{t^{**}+\tau} (p+\lambda) ds = \tau p + \lambda \tau. \tag{3.18}$$

This is a contradiction. Therefore, we obtain $\Gamma(t) \leq \widetilde{V}(\sigma),\ t \in [\sigma, t_l)$, which implies that (3.5) holds for all $t \in [\sigma, t_l)$.

Meanwhile, we get for $t \in [\sigma, t_l)$

$$c_1 \|x\|^{\eta} \leq V(t) \leq \widetilde{V}(\sigma) e^{-\lambda(t-\sigma)} \leq \widetilde{V}(\sigma) < M^{\star} w(\delta) \leq M^{-1} \min\{c_1 \varepsilon^{\eta}, \varepsilon\} < c_1 \varepsilon^{\eta}, \tag{3.19}$$

which implies that $x(t_l^-) \in S(\rho_1),\ x(t_l) \in S(\rho)$.

Furthermore, note that

$$V(t_l) \leq \beta_l V(t_l^-) \leq \beta_l \widetilde{V}(\sigma) e^{-\lambda(t_l-\sigma)}, \tag{3.20}$$

we next prove

$$V(t) \leq \max\{\beta_l M^{\star}, 1\} \widetilde{V}(\sigma) e^{-\lambda(t-\sigma)}, \quad t \in [t_l, t_{l+1}), \tag{3.21}$$

which is equal to prove

$$\Gamma(t) \leq \max\{\beta_l M^{\star}, 1\} \widetilde{V}(\sigma), \quad t \in [t_l, t_{l+1}). \tag{3.22}$$

Suppose that this assertion is not true, then there exists some $t \in [t_l, t_{l+1})$ such that $\Gamma(t) > \max\{\beta_l M^{\star}, 1\} \widetilde{V}(\sigma)$.

Let

$$t^* = \inf\left\{t \in [t_l, t_{l+1}) \mid \Gamma(t) \geq \max\{\beta_l M^{\star}, 1\} \widetilde{V}(\sigma)\right\}. \tag{3.23}$$

Then we know

$$t^* \in (t_l, t_{l+1}), \quad \Gamma(t^*) = \max\{\beta_l M^{\star}, 1\} \widetilde{V}(\sigma), \quad \Gamma(t) < \max\{\beta_l M^{\star}, 1\} \widetilde{V}(\sigma), \quad t \in [t_l, t^*). \tag{3.24}$$

Also, in view of the fact $\Gamma(t) < \widetilde{V}(\sigma)$ for $t \in [\sigma - \alpha, t_l)$, we obtain

$$\Gamma(t) < \max\{\beta_l M^{\star}, 1\} \widetilde{V}(\sigma) \quad \forall t \in [\sigma - \alpha, t^*). \tag{3.25}$$

Note that $\gamma \in (1, M^{\star}]$, then we have

$$\Gamma(t^*) = \max\{\beta_l M^{\star}, 1\} \widetilde{V}(\sigma) > \frac{1}{\gamma} \max\{\beta_l M^{\star}, 1\} \widetilde{V}(\sigma), \tag{3.26}$$

$$\Gamma(t_l) \leq \beta_l \widetilde{V}(\sigma) e^{-\lambda(t_l-\sigma)} < \beta_l \widetilde{V}(\sigma) \leq \frac{1}{\gamma} \beta_l M^{\star} \widetilde{V}(\sigma) \leq \frac{1}{\gamma} \max\{\beta_l M^{\star}, 1\} \widetilde{V}(\sigma). \tag{3.27}$$

So we can define

$$t^{**} = \sup\left\{ t \in [t_l, t^*] \mid \Gamma(t) \le \frac{1}{\gamma} \max\{\beta_l M^\star, 1\} \widetilde{V}(\sigma) \right\}. \tag{3.28}$$

Then, we obtain

$$\begin{gathered} t^{**} \in [t_l, t^*), \quad \Gamma(t^{**}) = \frac{1}{\gamma} \max\{\beta_l M^\star, 1\} \widetilde{V}(\sigma), \\ \Gamma(t) > \frac{1}{\gamma} \max\{\beta_l M^\star, 1\} \widetilde{V}(\sigma), \quad t \in (t^{**}, t^*]. \end{gathered} \tag{3.29}$$

Thus, combining (3.25), we get

$$\begin{aligned} \gamma V(t) e^{\lambda(t-\sigma)} &= \gamma \Gamma(t) \\ &\ge \max\{\beta_l M^\star, 1\} \widetilde{V}(\sigma) \\ &\ge \Gamma(s) \\ &\ge V(s), \quad \sigma - \alpha \le s \le t, \ t \in [t^{**}, t^*]. \end{aligned} \tag{3.30}$$

Similarly, by assumptions (ii), (iv), as the proof of (3.16), we can obtain

$$D^+ \Gamma(t) \le (p + \lambda) \Gamma(t), \quad t \in [t^{**}, t^*]. \tag{3.31}$$

Consequently, we have

$$\int_{t^{**}}^{t^*} \frac{d\Gamma(s)}{\Gamma(s)} = \int_{\Gamma(t^{**})}^{\Gamma(t^*)} \frac{ds}{s} = \int_{(1/\gamma) \max\{\beta_l M^\star, 1\} \widetilde{V}(\sigma)}^{\max\{\beta_l M^\star, 1\} \widetilde{V}(\sigma)} \frac{ds}{s} \ge \ln \gamma > p\tau + \lambda\tau, \tag{3.32}$$

where $\tau \doteq \max_{k \in \mathbb{Z}_+} \{t_k - t_{k-1}\}$.

However, we note

$$\int_{t^{**}}^{t^*} \frac{d\Gamma(s)}{\Gamma(s)} = \int_{\Gamma(t^{**})}^{\Gamma(t^*)} \frac{ds}{s} \le \int_{t^{**}}^{t^{**}+\tau} (p + \lambda) ds = p\tau + \lambda\tau, \tag{3.33}$$

which is a contradiction. So (3.21) holds.

Note that

$$V(t_{l+1}) \le \beta_{l+1} V(t_{l+1}^-) \le \beta_{l+1} \max\{\beta_l M^\star, 1\} \widetilde{V}(\sigma) e^{-\lambda(t_{l+1} - \sigma)}. \tag{3.34}$$

Similarly, we can prove

$$V(t) \le \max\{\beta_{l+1} M^\star \cdot \max\{\beta_l M^\star, 1\}, \max\{\beta_l M^\star, 1\}\} \widetilde{V}(\sigma) e^{-\lambda(t-\sigma)}, \quad t \in [t_{l+1}, t_{l+2}), \tag{3.35}$$

that is,

$$V(t) \le \max\{\beta_l M^\star, 1\} \max\{\beta_{l+1} M^\star, 1\} \widetilde{V}(\sigma) e^{-\lambda(t-\sigma)}, \quad t \in [t_{l+1}, t_{l+2}). \tag{3.36}$$

By simple induction hypothesis, we may prove that for $t \in [\sigma, t_l) \cup [t_k, t_{k+1}),\ k \ge l$,

$$V(t) \le \widetilde{V}(\sigma)\left(\prod_{s=l}^{k} \max\{\beta_s M^\star, 1\}\right) e^{-\lambda(t-\sigma)}, \quad t \ge \sigma. \tag{3.37}$$

Hence, (3.5) holds.

Under the help of conditions (i), (iii), and the definition of $\widetilde{V}(\sigma)$, we arrive at

$$\begin{aligned} c_1\|x\|^\eta &\le V(t) \\ &\le \widetilde{V}(\sigma)\left(\prod_{s=l}^{k} \max\{\beta_s M^\star, 1\}\right) e^{-\lambda(t-\sigma)} \\ &\le w(\delta) M^\star \left(\prod_{s=l}^{k} \max\{\beta_s M^\star, 1\}\right) e^{-\lambda(t-\sigma)} \\ &\le w(\delta) M^\star M e^{-\lambda(t-\sigma)} \\ &\le \min\{c_1\varepsilon^\eta, \varepsilon\} e^{-\lambda(t-\sigma)} \\ &\le c_1\varepsilon^\eta e^{-\lambda(t-\sigma)}, \quad t \ge \sigma, \end{aligned} \tag{3.38}$$

which implies that

$$\|x(t)\| < \varepsilon \cdot e^{-(\lambda/\eta)(t-\sigma)}, \quad t \ge \sigma. \tag{3.39}$$

Therefore, (3.3) holds. The proof of Theorem 3.1 is therefore complete. □

*Remark 3.2.* In Theorem 3.1, the impulsive condition is not straightforward to be verified. To simplify the result, we introduce some more testable conditions.

**Corollary 3.3.** *Assume that there exist functions $V \in v_0$, $w \in K_1$, and positive constants $M^\star, p, c_1, \eta, \lambda, \beta_k \ge 0,\ k \in \mathbb{Z}_+$ and $\gamma \in (1, e^{M^\star}]$ such that conditions (i), (ii), (iv) in Theorem 3.1 hold, moreover, suppose that*

(iii′) *for all $(t_k, \varphi) \in R_+ \times PC([-\alpha, 0], S(\rho_1))$, $V(t_k, \varphi(0) + I_k(t_k, \varphi)) \le \beta_k V(t_k^-, \varphi(0))$, $k \in \mathbb{Z}_+$, where $\beta_k \ge e^{-M^\star}$ and there exists a constant $M > 0$ such that*

$$\sum_{k=l}^{n} \ln \beta_k + (n - l + 1) M^\star < M \quad \forall n \ge l,\ l \in \mathbb{Z}_+, \tag{3.40}$$

*then the trivial solution of* (2.1)-(2.2) *is exponentially stable with the approximate exponential convergence rate $\lambda/\eta$.*

Especially, let $\gamma = M^\star$, $\beta_k = 1/\gamma > 0$ in Theorem 3.1, then we can obtain the following results whose proof is similar and thus omitted.

**Corollary 3.4.** *Assume that there exist functions $V \in v_0$, $w \in K_1$ and constants $p, c_1, \eta, \gamma > 1$, $\lambda > 0$ such that the following conditions hold:*

(i) $c_1\|x\|^\eta \le V(t,x) \le w(\|x\|)$, $(t,x) \in [-\alpha,\infty) \times S(\rho)$;

(ii) *for any $\sigma \ge t_0$ and $\psi \in PC([-\alpha,0], S(\rho))$, if $\gamma e^{\lambda(t-\sigma)}V(t,\psi(0)) \ge V(t+\theta,\psi(\theta))$, $-\alpha \le \theta \le 0$, $t \ne t_k$, $k \in \mathbb{Z}_+$, then $D^+V(t,\psi(0)) \le pV(t,\psi(0))$;*

(iii) *for all $(t_k,\psi) \in R_+ \times PC([-\alpha,0], S(\rho_1))$, $V(t_k,\psi(0)+I_k(t_k,\psi)) \le (1/\gamma)V(t_k^-,\psi(0))$, $k \in \mathbb{Z}_+$;*

(iv) $t_k - t_{k-1} < \ln\gamma/(p+\lambda)$, $k \in \mathbb{Z}_+$,

*then the trivial solution of* (2.1)-(2.2) *is exponentially stable with the approximate exponential convergence rate $\lambda/\eta$.*

*Remark 3.5. If $\alpha < +\infty$,* then the exponential stability of system (2.1)-(2.2) has been investigated extensively in [23] under the following assumptions (where the definition of $p, \alpha^\star, c_1, c, \lambda, \tau, d_k$, see [23]):

(i) $c_1\|x\|^p \le V(t,x) \le c_2\|x\|^p$, for any $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$;

(ii) $D^+V(t,\varphi(0)) \le cV(t,\varphi(0))$, for all $t \in [t_{k-1},t_k)$, $k \in \mathbb{Z}_+$, whenever $qV(t,\varphi(0)) \le V(t+s,\varphi(s))$ for $s \in [-\tau,0]$, where $q \ge \max\{e^{\alpha^\star c}, e^{2\lambda\alpha^\star}\}$ is a constant;

(iii) $V(t_k,\varphi(0)+I_k(t_k,\varphi)) \le d_k V(t_k,\varphi(0))$, where $d_k > 0, \forall k \in \mathbb{Z}_+$, are constants;

(iv) $\tau \le t_k - t_{k-1} \le \alpha^\star$ and $\ln(d_k) + (\lambda+c)\alpha^\star < -\lambda(t_{k+1}-t_k)$, $k \in \mathbb{Z}_+$.

It is easily seen that these conditions are more restrictive than ones given in Theorem 3.1. For example, let $d_k = d$(constant), $t_{k+1} - t_k = 1$, then it is necessary that condition $\ln(d) < -(\lambda+c)\alpha^\star - \lambda$ holds (see [23]). Note in our Corollary 3.4, we only require that $\ln(d) < -\lambda - c$, where here $\gamma = 1/d$, $p = c$ in Theorem 3.1. Moreover, we see that condition $t_{k+1} - t_k > \tau$ is not necessary in our results, which are milder than the restrictions in [23].

*Remark 3.6.* To author's knowledge, there is little work on exponential stability of impulsive differential systems with infinite delay with $D^+V \ge 0$. Our result allows for significant increases in $V$ between impulses as long as the decreases of $V$ at impulse times balance it properly, which shows that differential equations with infinite delay may be exponentially stabilized by impulses.

In the following, an example is given to demonstrate the effectiveness of our result.

*Example 3.7.* Consider the following equations:

$$\begin{gathered} x'(t) = \left(\frac{1}{5} + \frac{1}{18}e^{-0.1t}\right)x(t) - \frac{1}{15}\int_{-\infty}^{0} e^{s-0.1t}x(t+s)ds, \quad t \ge 0,\ t \ne k, \\ x(k) = \frac{1}{\sqrt{e}}x(k^-), \quad k = 1,2\ldots, \\ x_\sigma = \phi. \end{gathered} \tag{3.41}$$

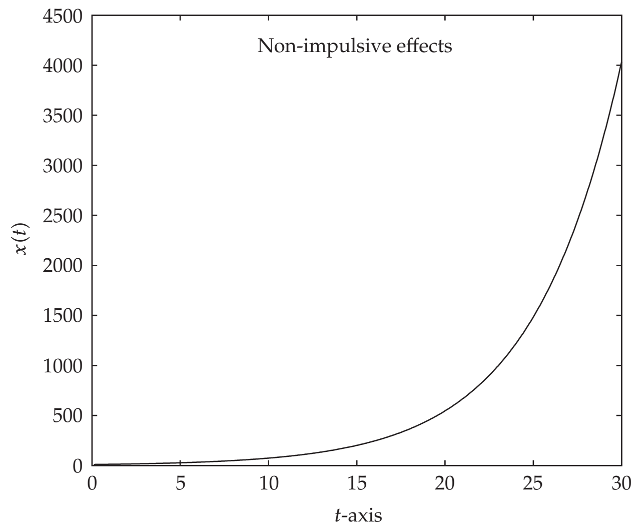


**Figure** 1

Choose $V(t) = x^2$, we easily observe $\eta = 2,\ c_1 = 1,\ w(s) = s^2,\ t_k - t_{k-1} \doteq \tau = 1$. Let $\gamma = e,\ \lambda = 0.2,\ p = 2/5 + 2/18 + (2\sqrt{e})/15$. It is easy to check that $\tau(p+\lambda) \approx 0.37 < 1 = \ln\gamma$.

Suppose that $\sigma = t_0 = 0$, then in view of Corollary 3.3, $\gamma e^{\lambda(t-\sigma)}V(t,\psi(0)) \geq V(t+\theta,\psi(\theta)),\ \theta \leq 0$, implies that $\sqrt{e}e^{0.1t}|x(t)| > |x(t+\theta)|,\ \theta \leq 0$.

Hence,

$$\begin{aligned} D^+V(t,x(\cdot)) &= 2x(t)\left\{\left(\frac{1}{5}+\frac{1}{18}e^{-0.1t}\right)x(t) - \frac{1}{15}\int_{-\infty}^{0} e^{s-0.1t}x(t+s)ds\right\} \\ &\leq 2x^2(t)\left\{\left(\frac{1}{5}+\frac{1}{18}e^{-0.1t}\right) + \frac{1}{15}\int_{-\infty}^{0} e^{s-0.1t}\sqrt{e}e^{0.1t}ds\right\} \\ &\leq 2x^2(t)\left\{\left(\frac{1}{5}+\frac{1}{18}e^{-0.1t}\right) + \frac{\sqrt{e}}{15}\int_{-\infty}^{0} e^{s}ds\right\} \\ &\leq 2x^2(t)\left\{\frac{1}{5}+\frac{1}{18}+\frac{\sqrt{e}}{15}\right\} \\ &\leq pV(t). \end{aligned} \tag{3.42}$$

Besides, $V(k) = x^2(k) = (1/e)x^2(k^-) = (1/\gamma)V(k^-)$.

By Corollary 3.3, the trivial solution of (3.41) is exponentially stable with the approximate exponential convergence rate 0.1. Taking initial values: $\phi = 10e^{0.2t}, t \leq 0$. The numerical simulations are illustrated in Figures 1 and 2.

*Remark 3.8.* From above example, we see that the trivial solution of system (3.41) without impulses is unstable. However, after impulsive control, the trivial solution becomes exponentially stable. This implies that differential systems with infinite delay may be exponentially stabilized by impulses and impulses can make unstable systems stable.

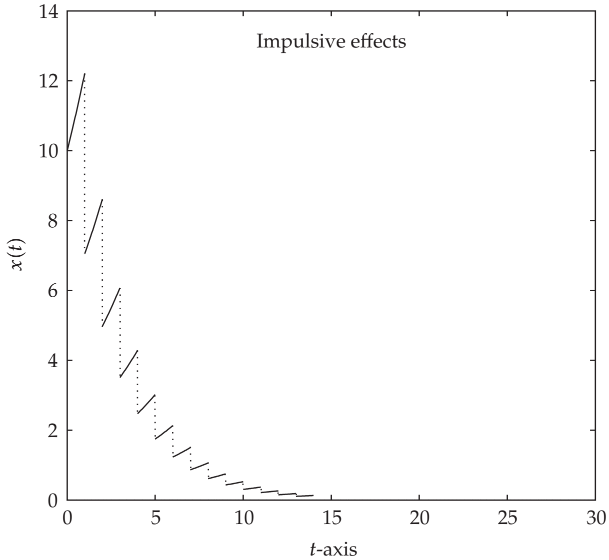


**Figure** 2

## 4. Conclusions

In this paper, we study exponential stability of impulsive differential systems with infinite delay. A new sufficient criterion ensuring exponential stability is gained by using the Razumikhintechnique and Lyapunov functions. Our result shows that differential equations with infinite delay may be exponentially stabilized by impulses. Also, the result here (with $\alpha < +\infty$) is discussed from the point of view of its comparison with the earlier result. An example is given to illustrate the feasibility of the result.

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