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Research Article

Global Stability for a Delayed Predator-Prey System with Stage Structure for the Predator

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A delayed predator-prey system with stage structure for the predator is investigated. By analyzing the corresponding characteristic equations, the local stability of equilibria of the system is discussed. The existence of Hopf bifurcation at the positive equilibrium is established. By using an iteration technique and comparison argument, respectively, sufficient conditions are derived for the global stability of the positive equilibrium and two boundary equilibria of the system. Numerical simulations are carried out to illustrate the theoretical results.

1. Introduction

Stage-structure is a natural phenomenon and represents, for example, the division of a population into immature and mature individuals. As is common, the dynamics-eating habits, susceptibility to predators, and so forth. are often quite different in these two subpopulations. Hence, it is of ecological importance to investigate the effects of such a subdivision on the interaction of species. In [1], Chen et al. introduced the following stage-structured single-species population model:

$$\begin{aligned} \dot{N}_i(t) &= B(t) - D_i(t) - W(t), \\ \dot{N}_m(t) &= \alpha W(t) - D_m(t), \end{aligned} \tag{1.1}$$

where $N_i(t)$ and $N_m(t)$ denote the immature and mature population densities at time t, respectively; $B(t)$ is the birth rate of the immature population at time t; $D_i(t)$ and $D_m(t)$ are the death rates of the immature and mature at time t; $W(t)$ represents the transformation rate of the immature into the mature; α is the probability of the successful transformation of the immature into the mature. If the birth rate of model (1.1) obeys the Malthus rule, that is, $B(t) = aN_m$, the death rates of the immature and mature populations are logistic, that is,

$$D_i(t) = r_i N_i(t) + b_i N_i^2(t), \qquad D_m(t) = r_m N_m(t) + b_m N_m^2(t), \tag{1.2}$$

and the transformation rate of the immature into mature is proportional to the immature population, that is, $W(t) = bN_i(t)$, then model (1.1) becomes

$$\begin{aligned} \dot{N}_i(t) &= aN_m - r_i N_i(t) - b_i N_i^2(t) - bN_i(t), \\ \dot{N}_m(t) &= \alpha b N_i(t) - r_m N_m(t) - b_m N_m^2(t). \end{aligned} \tag{1.3}$$

Based on the idea above, many authors studied different kinds of stage-structured models, and a significant body of work has been carried out (see, for example, [2–8]).

In [3], Gao et al. considered the following predator-prey model with stage structure:

$$\begin{aligned} \dot{x}(t) &= x(t)\left(r - a_{11}x(t) - a_{12}y_1(t-\tau_1)\right), \\ \dot{y}_1(t) &= y_1(t)\left[-r_1 + a_{21}x(t-\tau_2) - a_{22}y_1(t)\right] + \theta y_2(t), \\ \dot{y}_2(t) &= \alpha b y_1(t) - (\theta + r_2) y_2(t), \end{aligned} \tag{1.4}$$

where $x(t)$ represents the density of the prey at time t; $y_1(t)$ and $y_2(t)$ represent the densities of the mature and the immature predator at time t, respectively. The parameters r, r_1, r_2, a_{11}, a_{12}, a_{21}, a_{22}, b, θ are positive constants in which r is the intrinsic growth of the prey, r_1 is the death rate of the mature predator population, r_2 is the death rate of the immature predator population, a_{11} is the intra-specific competition rate of the prey population, a_{12} is the capturing rate of the predator population, a_{21}/a_{12} is the conversion rate of nutrients into the reproduction of the predator, a_{22} is the intra-specific competition rate of the mature predator, b is the birth rate of the immature predator, θ is the transformation rate from the immature predator individuals to mature predator individuals. The predation decreases the average growth rate of prey linearly with a certain time delay $\tau_1 \geq 0$, this assumption corresponds to the fact that predators cannot hunt prey when the predators are infant; predators have to mature for a duration of τ_1 units of time before they are capable of decreasing the average growth rate of the prey species; $\tau_2 \geq 0$ is the time delay due to gestation, the delay in time for prey biomass to increase predator number. In [3], Gao et al. studied the global stability of the positive equilibrium and boundary equilibria of model (1.4) by constructing Liapunov functionals and comparison argument, respectively.

We note that most of the predator-prey models with time delays studied in the literature are all of the Kolmogorov-type. In [9], Wangersky and Cunningham proposed delayed predator-prey models that are not of the Kolmogorov-type. They considered the following delayed system:

$$\begin{aligned} \dot{x}(t) &= x(t)\left(r - a_{11}x(t) - a_{12}y(t)\right), \\ \dot{y}(t) &= a_{21}x(t-\tau)y(t-\tau) - r_1 y(t), \end{aligned} \tag{1.5}$$

where the delay τ is a constant based on the assumption that the change rate of predators depends on the number of both the prey and the predators present at some previous time.

Motivated by the work of Gao et al. [3] and Wangersky and Cunningham [9], in the present paper, we consider the following predator-prey model with stage structure and time delay:

$$\begin{aligned}\dot{x}(t) &= x(t)\big(r - a_{11}x(t) - a_{12}y_1(t)\big),\\ \dot{y}_1(t) &= a_{21}x(t-\tau)y_1(t-\tau) - r_1y_1(t) - a_{22}y_1^2(t) + \theta y_2(t),\\ \dot{y}_2(t) &= by_1(t) - (\theta + r_2)y_2(t).\end{aligned} \tag{1.6}$$

The meanings of the positive parameters $r,\ r_1,\ r_2,\ a_{11},\ a_{12},\ a_{21},\ a_{22},\ b,\ \theta$ are the same as those in system (1.4). The meaning of time delay $\tau \geq 0$ is the same as in system (1.5).

The initial conditions for system (1.6) take the form

$$\begin{gathered} x(\theta) = \phi(\theta), \qquad y_1(\theta) = \psi_1(\theta), \qquad y_2(\theta) = \psi_2(\theta),\\ \phi(\theta) \geq 0, \qquad \psi_1(\theta) \geq 0, \qquad \psi_2(\theta) \geq 0, \quad \theta \in [-\tau, 0],\\ \phi(0) > 0, \qquad \psi_1(0) > 0, \qquad \psi_2(0) > 0,\end{gathered} \tag{1.7}$$

where $(\phi_1(\theta), \psi_1(\theta), \psi_2(\theta)) \in C([-\tau, 0], R^3_{+0})$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^3_{+0}, where $R^3_{+0} = \{(x_1, x_2, x_3) : x_i \geq 0,\ i = 1, 2, 3\}$.

This paper is organized as follows. In the next section, we introduce some notations and state several lemmas which will be essential to our proofs. In Section 3, we discuss the local stability of a positive equilibrium and boundary equilibria of system (1.6). The existence of Hopf bifurcation is studied. In Section 4, by means of an iterative technique and comparison argument, sufficient conditions are derived for the global stability of the positive equilibrium and boundary equilibria of system (1.6). Some numerical examples are given to illustrate the results above. A brief discussion is given in Section 6 to end this work.

2. Preliminaries

In this section, we introduce some notations and state several results which will be useful in next section. Let $\mathbb{R}^n_+$ be the cone of nonnegative vectors in $\mathbb{R}^n$. If $x, y \in \mathbb{R}^n$, we write $x \leq y$ $(x < y)$ if $x_i \leq y_i$ $(x_i < y_i)$ for $1 \leq i \leq n$. Let $e_1, e_2, \ldots, e_n$ denote the standard basis in $\mathbb{R}^n$. Suppose $r \geq 0$ and let $C = C([-r, 0], \mathbb{R}^n)$ be the Banach space of continuous functions mapping the interval $[-r, 0]$ into $\mathbb{R}^n$ with supremum norm. If $\phi, \psi \in C$, we write $\phi \leq \psi$ $(\phi < \psi)$ when the indicated inequality holds at each point of $[-r, 0]$. Let $C^+ = \{\phi \in C : \phi \geq 0\}$ and let denote the inclusion $\mathbb{R}^n \to C([-r, 0], \mathbb{R}^n)$ by $x \to \widehat{x},\ \widehat{x}(\theta) = x,\ \theta \in [-r, 0]$. Denote the space of functions of bounded variation on $[-r, 0]$ by $\mathrm{BV}[-r, 0]$. If $t_0 \in \mathbb{R}^n$, $A \geq 0$, and $x \in C([-t_0 - r, t_0 + A], \mathbb{R}^n)$, then for any $t \in [t_0, t_0 + A]$, we let $x_t \in C$ be defined by $x_t(\theta) = x(t + \theta)$, $-r \leq \theta \leq 0$.

We now consider

$$\dot{x}(t) = f(t, x_t). \tag{2.1}$$

We assume throughout this section that $f : \mathbb{R} \times C \to \mathbb{R}^n$ is continuous; $f(t, \phi)$ is continuously differentiable in ϕ; $f(t + T, \phi) = f(t, \phi)$ for all $(t, \phi) \in \mathbb{R} \times C^+$, and some $T > 0$. Then by [10], there exists a unique solution of (2.1) through (t_0, ϕ) for $t_0 \in \mathbb{R}, \phi \in C^+$. This solution will be denoted by $x(t,t_0,\phi)$ if we consider the solution in $\mathbb{R}^n$, or by $x_t(t_0,\phi)$ if we work in the space C. Again by [10], $x(t,t_0,\phi)(x_t(t_0,\phi))$ is continuously differentiable in ϕ. In the following, the notation $x_{t0}=\phi$ will be used as the condition of the initial data of (2.1), by which we mean that we consider the solution $x(t)$ of (2.1) which satisfies $x(t_0+\theta)=\phi(\theta),\ \theta\in[-r,0]$.

To proceed further, we need the following results from [11, 12]. Let $r=(r_1,r_2,\dots,r_n)\in\mathbb{R}^n,\ |r|=\max_i\{r_i\}$, and define

$$C_r=\prod_{i=1}^{n}C([-r_i,0],\mathbb{R}). \tag{2.2}$$

We write $\phi=(\phi_1,\phi_2,\dots,\phi_n)$ for a generic point of C_r. Let $C_r^+=\{\phi\in C_r:\phi\geq 0\}$. Due to the ecological applications, we choose C_r^+ as the state space of (2.1) in the following discussions.

Fix $\phi_0\in C_r^+$ arbitrarily. Then we set $L(t,\cdot)=D_{\phi_0}f(t,\phi_0)$, and $D_{\phi_0}f(t,\phi_0)$ denotes the Frechet derivation of f with respect to ϕ_0. It is convenient to have the standard representation of $L=(L_1,L_2,\dots,L_n)$ as

$$L_i(t,\phi)=\sum_{j=1}^{n}\int_{-r_j}^{0}\phi_j(\theta)d_\theta\eta_{ij}(\theta,t)\quad(1\leq i\leq n), \tag{2.3}$$

in which $\eta_{ij}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies

$$\begin{gathered}\eta_{ij}(\theta,t)=\eta_{ij}(0,t),\quad\theta\leq -r_j,\\ \eta_{ij}(\theta,t)=0,\quad\theta\geq 0,\\ \eta_{ij}(\cdot,t)\in\mathrm{BV}[-r_j,0],\end{gathered} \tag{2.4}$$

where $\eta_{ij}(\cdot,t)$ is continuous from the left in $(-r_j,0)$.

We make the following assumptions for (2.1).

(h0) If $\phi,\psi\in C^+,\ \phi\leq\varphi$, and $\phi_i(0)=\varphi_i(0)$ for some i, then $f_i(t,\phi)\leq f_i(t,\varphi)$.

(h1) For all $\phi\in C_r^+$ with $\phi_i(0)=0,\ L_i(t,\phi)\geq 0$ for $t\in\mathbb{R}$.

(h2) The matrix $A(t)$ defined by

$$A(t)=\mathrm{col}(L(t,\hat{e}_1),L(t,\hat{e}_2),\dots,L(t,\hat{e}_n))=(\eta_{ij}(0,t)), \tag{2.5}$$

is irreducible for each $t\in\mathbb{R}$.

(h3) For each j, for which $r_j>0$, there exist i such that for all $t\in\mathbb{R}$ and for positive constant ε sufficiently small, $\eta_{ij}(-r_j+\varepsilon,t)>0$.

(h4) If $\phi=0$, then $x(t,t_0,\phi)\equiv 0$ for all $t\geq t_0$.

The following result was established by Wang et al. [12].

Lemma 2.1. *Let (h1)–(h4) hold. Then the hypothesis (h0) is valid and the following.*

(i) *If ϕ and ψ are distinct elements of C_r^+ with $\phi \leq \psi$ and $[t_0, t_0 + \sigma)$ with $n|r| < \sigma \leq \infty$ is the intersection of the maximal intervals of existence of $x(t, t_0, \phi)$ and $x(t, t_0, \psi)$, then*

$$\begin{gathered} 0 \leq x(t, t_0, \phi) \leq x(t, t_0, \psi) \quad \text{for } t_0 \leq t < t_0 + \sigma, \\ 0 \leq x(t, t_0, \phi) < x(t, t_0, \psi) \quad \text{for } t_0 + n|r| \leq t < t_0 + \sigma. \end{gathered} \tag{2.6}$$

(ii) *If $\phi \in C_r^+$, $\phi \neq 0$, $t_0 \in \mathbb{R}$ and $x(t, t_0, \phi)$ is defined on $[t_0, t_0 + \sigma)$ with $\sigma > n|r|$, then*

$$0 < x(t, t_0, \phi) \quad \text{for } t_0 + n|r| \leq t < t_0 + \sigma. \tag{2.7}$$

Lemma 2.1 shows that if (h1)–(h4) hold, then the positivity of solutions of (2.1) follows.

The following definitions and results are useful in proving our lemma.

Definition 2.2. System (2.1) is cooperative if $\partial f_i / \partial x_j \geq 0$ whenever $i \neq j$.

Definition 2.3. A square matrix A is said to be a reducible matrix if and only if for some permutation matrix P the matrix $P^T AP$ is block upper triangular. If a square matrix is not reducible, it is said to be an irreducible matrix. System (2.1) is called irreducible if the Jacobian matrix $((\partial f_i / \partial x_j))$ is irreducible.

Lemma 2.4 (Smith [11]). *If* (2.1) *is cooperative and irreducible in D, where D is an open subset of C, and the solution with positive initial data is bounded, then the trajectory of* (2.1) *tends to some single equilibrium.*

We now consider the following delay differential system:

$$\begin{gathered} \dot{y}_1(t) = -ay_1(t) + by_1(t - \tau) - cy_1^2(t) + \theta y_2(t), \\ \dot{y}_2(t) = dy_1(t) - (\theta + e)y_2(t), \end{gathered} \tag{2.8}$$

with initial conditions

$$y_i(s) = \varphi_i(s) \geq 0, \quad s \in [-\tau, 0), \; \varphi_i \in C([-\tau, 0), \mathbb{R}_+), \; \varphi_i(0) > 0, \; (i = 1, 2). \tag{2.9}$$

System (2.8) always has a trivial equilibrium $E_0(0, 0)$. If $a(\theta + e) < d\theta + b(\theta + e)$, then system (2.8) has a unique positive equilibrium $E_+(y_1^*, y_2^*)$, where

$$y_1^* = \frac{(-a + b)(\theta + e) + \theta d}{c(\theta + e)}, \qquad y_2^* = \frac{d(-a + b)(\theta + e) + \theta d^2}{c(\theta + e)^2}. \tag{2.10}$$

The characteristic equation of system (1.6) at the equilibrium E_0 is of the form

$$\lambda^2 + (a + e + \theta)\lambda + a(\theta + e) - d\theta - (b\lambda + b(\theta + e))e^{-\lambda\tau} = 0. \tag{2.11}$$

Let

$$f_1(\lambda) = \lambda^2 + (a+e+\theta)\lambda + a(\theta+e) - d\theta - (b\lambda + b(\theta+e))e^{-\lambda\tau}. \tag{2.12}$$

If $a(\theta+e) < d\theta + b(\theta+e)$, then it is easy to see that, for λ real,

$$f_1(0) = a(\theta+e) - d\theta - b(\theta+e) < 0, \quad \lim_{\lambda\to+\infty} f_1(\lambda) = +\infty. \tag{2.13}$$

Hence, $f_1(\lambda) = 0$ has a positive real root. Therefore, the equilibrium E_0 is unstable. If $a(\theta+e) > d\theta + b(\theta+e)$, when $\tau = 0$, it is easy to see that the equilibrium E_0 is stable. Therefore, if $a(\theta+e) - d\theta > b(\theta+e)$, by Kuang and So [13, Lemma B], we see that the equilibrium E_0 is locally stable for all $\tau > 0$.

If $a(\theta+e) < d\theta + b(\theta+e)$, the characteristic equation of the positive equilibrium E_+ takes the form

$$\lambda^2 + \left(a+e+\theta+2cy_1^*\right)\lambda + \left(a+2cy_1^*\right)(\theta+e) - d\theta - (b\lambda + b(\theta+e))e^{-\lambda\tau} = 0. \tag{2.14}$$

When $\tau = 0$, it is easy to see that the equilibrium E_+ is stable. Therefore, if $a(\theta+e) < d\theta + b(\theta+e)$, by Kuang and So [13, Lemma B], we see that the equilibrium E_+ is locally stable for all $\tau > 0$.

Lemma 2.5. *For system* (2.8), *one has the following.*

(i) *If $a(\theta+e) < d\theta + b(\theta+e)$, then the positive equilibrium E_+ of system* (2.8) *is globally stable.*

(ii) *If $a(\theta+e) > d\theta + b(\theta+e)$, the equilibrium E_0 of system* (2.8) *is globally stable.*

Proof. We represent the right-hand side of (2.8) by $f(t,x_t) = (f_1(t,x_t), f_2(t,x_t))$ and set $L(t,\cdot) = D_\phi f(t,\phi)$. By direct calculation we have

$$\begin{gathered} L_1(t,h) = -ah_1(0) + bh_1(-\tau) - 2c\phi_1(0)h_1(0) + \theta h_2(0), \\ L_2(t,h) = dh_1(0) - (\theta+e)h_2(0). \end{gathered} \tag{2.15}$$

We now claim that the hypotheses (h1)–(h4) hold for system (2.8). It is easily seen that (h1) and (h4) hold for system (2.8). We need only to verify that (h2) and (h3) hold.

The matrix $A(t)$ takes the form

$$\begin{pmatrix} -a+b-2c\phi_1(0) & \theta \\ d & -(\theta+e) \end{pmatrix}. \tag{2.16}$$

Clearly, the matrix $A(t)$ is irreducible for each $t \in \mathbb{R}$.

From the definition of $A(t)$ and η_{ij}, it is readily seen that $\eta_{12}(\theta,t) = \eta_{12}(0,t) = \theta, \eta_{21}(\theta,t) = \eta_{21}(0,t) = d$, for $\theta \geq 0$; and $\eta_{ij}(\theta,t) = 0,\ i \neq j$ for $\theta \leq -\tau$; and $\eta_{ij}(\cdot,t) \in \mathrm{BV}[-\tau,0]$, where η_{ij} is a positive Borel measure on $[-\tau,0]$. Therefore, $\eta_{ij}(\cdot,t) > 0$. Thus, for each j, there is $i \neq j$ such that $\eta_{ij}(-r_j+\varepsilon,t) = \eta_{ij}(-\tau+\varepsilon,t) > 0$ for all $t \in \mathbb{R}$ and $\varepsilon > 0$ sufficiently small, $i = 1,2$. Hence, (h3) holds.

Thus, the conditions of Lemma 2.1 are satisfied. Therefore, the positivity of solutions to system (2.8) follows. It is easy to see that system (2.8) is cooperative. By Lemma 2.4 we see that any solution starting from $D = C_\tau^+$ converges to some single equilibrium. However, system (2.8) has only two equilibria: E_0 and E_+. Note that if $a(\theta+e) < d\theta + b(\theta+e)$, the equilibrium E_+ is locally stable. Hence, any solution starting from D converges to E^+. Using a similar argument one can show the global stability of the equilibrium E_0 when $a(\theta+e) > d\theta + b(\theta+e)$. This completes the proof. □

By a similar argument one can show that all solutions of system (1.6) with initial conditions (1.7) are defined on $[0,+\infty)$ and remain positive for all $t \geq 0$.

3. Local Stability

In this section, we discuss the local stability of each equilibria and the existence of Hopf bifurcation of system (1.6).

It is easy to show that system (1.6) always has two equilibria $E_0(0,0,0)$ and $E_1(r/a_{11},0,0)$. If the following holds:

(H1) $\theta b > r_1(\theta + r_2)$,

then system (1.6) has another boundary equilibrium $E_2(0,\tilde{y}_1,\tilde{y}_2)$, where

$$\tilde{y}_1 = \frac{1}{a_{22}}\left(\frac{\theta b}{\theta + r_2} - r_1\right), \qquad \tilde{y}_2 = \frac{b}{a_{22}(\theta + r_2)}\left(\frac{\theta b}{\theta + r_2} - r_1\right). \tag{3.1}$$

Further, if the following holds:

(H2) $r(\theta + r_2)(a_{12}a_{21} + a_{11}a_{22}) > a_{12}[\theta b a_{11} + (a_{21}r - a_{11}r_1)(\theta + r_2)] > 0$,

then system (1.6) has a unique positive equilibrium $E^* = (x^*, y_1^*, y_2^*)$, where

$$\begin{aligned} x^* &= \frac{r}{a_{11}} - \frac{a_{12}[\theta b a_{11} + (\theta + r_2)(a_{21}r - a_{11}r_1)]}{(\theta + r_2)a_{11}(a_{12}a_{21} + a_{11}a_{22})}, \\ y_1^* &= \frac{\theta b a_{11} + (\theta + r_2)(a_{21}r - a_{11}r_1)}{(\theta + r_2)(a_{12}a_{21} + a_{11}a_{22})}, \\ y_2^* &= \frac{b[\theta b a_{11} + (\theta + r_2)(a_{21}r - a_{11}r_1)]}{(\theta + r_2)^2(a_{12}a_{21} + a_{11}a_{22})}. \end{aligned} \tag{3.2}$$

We now study the local stability of each of the nonnegative equilibrium of system (1.6).

Let $\widehat{E} = (\widehat{x}, \widehat{y}_1, \widehat{y}_2)$ be any arbitrary equilibrium. Then the characteristic equation of system (1.6) at the equilibrium $\widehat{E}$ is given by

$$\begin{vmatrix} r - 2a_{11}\widehat{x} - a_{12}\widehat{y}_1 - \lambda & -a_{12}\widehat{x} & 0 \\ a_{21}\widehat{y}_1 e^{-\lambda\tau} & -r_1 - 2a_{22}\widehat{y}_1 + a_{21}\widehat{x}e^{-\lambda\tau} - \lambda & \theta \\ 0 & b & -\theta - r_2 - \lambda \end{vmatrix} = 0. \tag{3.3}$$

The characteristic equation of system (1.6) at the equilibrium $E_0(0,0,0)$ reduces

$$(\lambda - r)[(r_1 + \lambda)(\theta + r_2 + \lambda) - b\theta] = 0. \tag{3.4}$$

Clearly, $\lambda = r$ is a positive real root. Hence, $E_0(0,0,0)$ is always unstable.

The characteristic equation of system (1.6) at the equilibrium $E_1(r/a_{11}, 0, 0)$ reduces

$$(\lambda + r)\left[\lambda^2 + (r_1 + r_2 + \theta)\lambda + r_1(\theta + r_2) - b\theta - \frac{a_{21}r}{a_{11}}(\lambda + \theta + r_2)e^{-\lambda\tau}\right] = 0. \tag{3.5}$$

Clearly, $\lambda = -r$ is a negative real root of (3.5). All other roots are give by the roots of equation

$$\lambda^2 + (r_1 + r_2 + \theta)\lambda + r_1(\theta + r_2) - b\theta - \frac{a_{21}r}{a_{11}}(\lambda + \theta + r_2)e^{-\lambda\tau} = 0. \tag{3.6}$$

Let

$$f(\lambda) = \lambda^2 + (r_1 + r_2 + \theta)\lambda + r_1(\theta + r_2) - b\theta - \frac{a_{21}r}{a_{11}}(\lambda + \theta + r_2)e^{-\lambda\tau}. \tag{3.7}$$

If $(\theta + r_2)(r_1 - a_{21}r/a_{11}) < b\theta$, then it is easy to see that for λ real,

$$f(0) = (\theta + r_2)\left(\frac{r_1 - a_{21}r}{a_{11}}\right) - b\theta < 0, \qquad \lim_{\lambda \to +\infty} f(\lambda) = +\infty. \tag{3.8}$$

Hence, $f(\lambda) = 0$ has a positive real root. Therefore, the equilibrium $E_1(r/a_{11}, 0, 0)$ is unstable. If $(\theta + r_2)(r_1 - a_{21}r/a_{11}) > b\theta$, when $\tau = 0$, it is easy to see that the equilibrium $E_1(r/a_{11}, 0, 0)$ is stable. Therefore, if $(\theta + r_2)(r_1 - a_{21}r/a_{11}) > b\theta$, by Kuang and So [13, Lemma B], we see that the equilibrium $E_1(r/a_{11}, 0, 0)$ is locally stable for all $\tau > 0$.

The characteristic equation of system (1.6) at the equilibria $E_2(0, \widetilde{y}_1, \widetilde{y}_2)$ reduces to

$$(\lambda - r + a_{12}\widetilde{y}_1)\left[\lambda^2 + (r_1 + 2a_{22}\widetilde{y}_1 + \theta + r_2)\lambda + (r_1 + 2a_{22}\widetilde{y}_1)(\theta + r_2) - b\theta\right] = 0. \tag{3.9}$$

Clearly, if $r > a_{12}\tilde{y}_1$, $\lambda = r - a_{12}\tilde{y}_1$ is a positive real root. Hence, if $r > a_{12}\tilde{y}_1$, then $E_2(0, \tilde{y}_1, \tilde{y}_2)$ is unstable. Noting that

$$(r_1 + 2a_{22}\tilde{y}_1)(\theta + r_2) - b\theta = b\theta - r_1(\theta + r_2) > 0, \tag{3.10}$$

if $r < a_{12}\tilde{y}_1$, then (3.9) only has negative real root, and $E_2(0, \tilde{y}_1, \tilde{y}_2)$ is stable.

The characteristic equation of system (1.6) at the positive equilibria $E^*(x^*, y_1^*, y_2^*)$ is

$$\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 + \left(q_2\lambda^2 + q_1\lambda + q_0\right)e^{-\lambda\tau} = 0, \tag{3.11}$$

where

$$\begin{gathered}
p_0 = a_{11}x^*(\theta + r_2)\left(a_{21}x^* + a_{22}y_1^*\right), \\
p_1 = a_{11}x^*\left(r_1 + 2a_{22}y_1^*\right) + (\theta + r_2)\left(a_{21}x^* + a_{22}y_1^* + a_{11}x^*\right), \\
p_2 = a_{11}x^* + 2a_{22}y_1^* + r_1 + r_2 + \theta, \\
q_0 = a_{21}(\theta + r_2)x^*\left(a_{12}y_1^* - a_{11}x^*\right), \\
q_1 = a_{21}x^*\left(a_{12}y_1^* - a_{11}x^* - \theta - r_2\right), \\
q_2 = -a_{21}x^*.
\end{gathered} \tag{3.12}$$

It is easy to show that

$$\begin{gathered}
p_0 + q_0 = (a_{11}a_{22} + a_{21}a_{12})(\theta + r_2)x^*y_1^* > 0, \\
p_1 + q_1 = (a_{11}a_{22} + a_{12}a_{21})x^*y_1^* + (\theta + r_2)\left(a_{22}y_1^* + a_{11}x^*\right) + \frac{b\theta a_{11}x^*}{(\theta + r_2)} > 0, \\
p_2 + q_2 = a_{11}x^* + a_{22}y_1^* + r_2 + \theta + \frac{b\theta}{(\theta + r_2)} > 0.
\end{gathered} \tag{3.13}$$

It is easy to see that $(p_2 + q_2)(p_1 + q_1) > p_0 + q_0$. Hence, by the Routh-Hurwitz theorem, when $\tau = 0$, the positive equilibrium E^* of system (1.6) is locally asymptotically stable.

If $i\omega$ $(\omega > 0)$ is a root of (3.11), separating the real and imaginary parts, we obtain

$$\begin{gathered}
p_0 - p_2\omega^2 = q_2\omega^2\cos\omega\tau - q_1\omega\sin\omega\tau - q_0\cos\omega\tau, \\
-\omega^3 + p_1\omega = -q_2\omega^2\sin\omega\tau - q_1\omega\cos\omega\tau + q_0\sin\omega\tau.
\end{gathered} \tag{3.14}$$

Squaring and adding the two equations of (3.14), it follows that

$$\omega^6 + Q_1\omega^4 + Q_2\omega^2 + Q_3 = 0, \tag{3.15}$$

where

$$Q_1 = p_2^2 - 2p_1 - q_2^2, \qquad Q_2 = p_1^2 + 2q_2q_0 - 2p_2p_0 - q_1^2, \qquad Q_3 = p_0^2 - q_0^2. \tag{3.16}$$

It is easy to show that

$$\begin{aligned}
Q_1 &= p_2^2 - q_2^2 - 2p_1 \\
&= a_{11}^2 (x^*)^2 + (r_2+\theta)^2 + (r_1 + 2a_{22}y_1^* + a_{21}x^*)\left(a_{22}y_1^* + \frac{b\theta}{\theta + r_2}\right) + 2b\theta > 0, \\
Q_2 &= p_1^2 + 2q_2q_0 - 2p_2p_0 - q_1^2 \\
&= \left[(a_{11}a_{22}+a_{21}a_{12})x^*y_1^* + \frac{a_{11}b\theta x^*}{(r_2+\theta)}\right]\left[2a_{11}a_{21}(x^*)^2 - (a_{12}a_{21}-a_{11}a_{22})x^*y_1^* + \frac{a_{11}b\theta x^*}{(r_2+\theta)}\right] \\
&\quad + (\theta + r_2)^2\left[a_{22}^2(y_1^*)^2 + a_{11}^2(x^*)^2\right] + 2a_{21}a_{22}(\theta + r_2)^2 x^* + 2b\theta a_{11}^2 (x^*)^2.
\end{aligned} \tag{3.17}$$

Noting that if $p_0 > q_0$, $2a_{11}a_{21}x^* > (a_{12}a_{21} - a_{11}a_{22})y_1^*$, hence $Q_2 > 0$, $Q_3 > 0$. The positive equilibrium E^* of system (1.6) is locally asymptotically stable for all $\tau > 0$. If $p_0 < q_0$, we know that $Q_3 < 0$, (3.15) has a unique positive root ω_0. Define

$$\tau_j = \frac{1}{\omega_0}\arccos\frac{(q_1 - q_2p_2)\omega_0^4 + (p_0q_2 + p_2q_0 - p_1q_1)\omega_0^2 - q_0p_0}{q_2^2\omega_0^4 + (q_1^2 - 2q_0q_2)\omega_0^2 + q_0^2} + \frac{2j\pi}{\omega_0}, \quad j = 0, 1, \ldots. \tag{3.18}$$

Then (τ_j, ω_0) solves (3.11). This means that when $\tau = \tau_j$, (3.11) has a pair of purely imaginary roots $\pm i\omega_0$. Noting that the positive equilibrium E^* is locally stable for $\tau = 0$, by the general theory on characteristic equations of delay differential equations from [14, Theorem 4.1], E^* remains stable for $\tau < \tau_0$.

Let $p_0 < q_0$ and τ_0 be defined in (3.18). Denoting

$$\lambda(\tau) = \alpha(\tau) + i\omega(\tau), \tag{3.19}$$

the root of (3.11) is such that

$$\alpha(\tau_0) = 0, \qquad \omega(\tau_0) = \omega_0. \tag{3.20}$$

In the following we claim that

$$\left.\frac{d(\operatorname{Re}\lambda)}{d\tau}\right|_{\tau=\tau_0} > 0. \tag{3.21}$$

This will signify that there exists at least one eigenvalue with positive real part for $\tau > \tau_0$. Moreover, the conditions for the existence of a Hopf bifurcation [10] are then satisfied yielding a periodic solution. To this end, differentiating equation (3.11) with respect τ, we obtain that

$$\left(\frac{d\lambda(\tau)}{d\tau}\right)^{-1} = -\frac{3\lambda^2 + 2p_2\lambda + p_1}{\lambda(\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0)} + \frac{2q_2\lambda + q_1}{\lambda(q_2\lambda^2 + q_1\lambda + q_0)} - \frac{\tau}{\lambda}, \tag{3.22}$$

which leads to

$$\begin{aligned}\operatorname{sign}\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)\Big|_{\tau=\tau_j}\right\} &= \operatorname{sign}\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)\Big|_{\tau=\tau_j}^{-1}\right\} \\ &= \operatorname{sign}\left\{3\omega_0^4 + \left(2p_2^2 - 4p_1 - 2q_2^2\right)\omega_0^2 + p_1^2 + 2q_2q_0 - 2p_2p_0 - q_1^2\right\} \\ &= \operatorname{sign}\left\{3\omega_0^4 + 2Q_1\omega_0^2 + Q_2\right\}.\end{aligned} \tag{3.23}$$

If $Q_2 > 0$, $3\omega_0^4 + 2Q_1\omega_0^2 + Q_2 > 0$. Therefore, system (1.6) undergoes a Hopf bifurcation.

We therefore obtain the following results.

Theorem 3.1. *For system* (1.6), *let* τ_0 *be defined as in* (3.18), *one has the following.*

(i) *The positive equilibrium* E_0 *of system* (1.6) *is always unstable.*

(ii) *If* $(\theta + r_2)(r_1 - a_{21}r/a_{11}) > b\theta$, *the equilibrium* $E_1(r/a_{11}, 0, 0)$ *of system* (1.6) *is locally stable for all* $\tau > 0$*; and if* $(\theta + r_2)(r_1 - a_{21}r/a_{11}) < b\theta$, E_1 *is unstable for all* τ.

(iii) *Let* $(H1)$ *hold. If* $r > a_{12}\tilde{y}_1$, *the equilibrium* $E_2(0, \tilde{y}_1, \tilde{y}_2)$ *of system* (1.6) *is unstable; if* $r < a_{12}\tilde{y}_1$, E_2 *is stable for all* $\tau \geq 0$.

(iv) *Let (H2) hold. If* $p_0 > q_0$, *then the positive equilibrium* $E^*(x^*, y_1^*, y_2^*)$ *of system* (1.6) *is locally asymptotically stable for all* $\tau > 0$. *If* $p_0 < q_0$, *then* E^* *is locally stable for* $\tau < \tau_0$*; and* E^* *is unstable for* $\tau > \tau_0$*; if* $Q_2 > 0$ *system* (1.6) *undergoes a Hopf Bifurcation at the positive equilibrium* E^* *when* $\tau = \tau_0$.

4. Global Stability

In this section, we are concerned with the global stability of the equilibria E_1, E_2, E^* of system (1.6). The strategy of proofs is to use an iteration technique and comparison arguments, respectively.

Theorem 4.1. *Let (H2) hold. Then the positive equilibrium* E^* *of system* (1.6) *is globally asymptotically stable provided that*

(H3) $a_{11}a_{22} > a_{12}a_{21}$.

Proof. Let $(x(t), y_1(t), y_2(t))$ be any positive solution of system (1.6) with initial conditions (1.7). Let

$$\begin{aligned} U &= \limsup_{t\to+\infty} x(t), \qquad V = \liminf_{t\to+\infty} x(t), \\ U_i &= \limsup_{t\to+\infty} y_i(t), \qquad V_i = \liminf_{t\to+\infty} y_i(t) \quad (i = 1, 2). \end{aligned} \tag{4.1}$$

We now claim that $U = V = x^*$, $U_1 = V_1 = y_1^*$, $U_2 = V_2 = y_2^*$. The strategy of the proof is to use an iteration technique.

We derive from the first equation of system (1.6) that

$$\dot{x}(t) \leq x(t)(r - a_{11}x(t)). \tag{4.2}$$

A standard comparison argument shows that

$$U = \limsup_{t\to+\infty} x(t) \le \frac{r}{a_{11}} := M_1^x. \tag{4.3}$$

Hence, for $\varepsilon > 0$ sufficiently small there exists a $T_1 > 0$ such that if $t > T_1$, $x(t) \le M_1^x + \varepsilon$.

We derive from the second and the third equations of system (1.6) that for $t > T_1 + \tau$,

$$\begin{gathered} \dot{y}_1(t) \le -r_1 y_1(t) + a_{21}\left(M_1^x + \varepsilon\right) y_1(t-\tau) - a_{22} y_1^2(t) + \theta y_2(t), \\ \dot{y}_2(t) = b y_1(t) - (\theta + r_2) y_2(t). \end{gathered} \tag{4.4}$$

Consider the following auxiliary equations:

$$\begin{gathered} \dot{u}_1(t) = -r_1 u_1(t) + a_{21}\left(M_1^x + \varepsilon\right) u_1(t-\tau) - a_{22} u_1^2(t) + \theta u_2(t), \\ \dot{u}_2(t) = b u_1(t) - (\theta + r_2) u_2(t). \end{gathered} \tag{4.5}$$

Since (H2) holds, by Lemma 2.5 it follows from (4.5) that

$$\begin{aligned} \lim_{t\to+\infty} u_1(t) &= \frac{\left(-r_1 + a_{21}\left(M_1^x + \varepsilon\right)\right)(\theta + r_2) + b\theta}{a_{22}(\theta + r_2)}, \\ \lim_{t\to+\infty} u_2(t) &= \frac{b\left(-r_1 + a_{21}\left(M_1^x + \varepsilon\right)\right)(\theta + r_2) + b^2\theta}{a_{22}(\theta + r_2)^2}. \end{aligned} \tag{4.6}$$

By comparison, we obtain that

$$\begin{aligned} U_1 = \limsup_{t\to+\infty} y_1(t) &\le \frac{\left(-r_1 + a_{21}\left(M_1^x + \varepsilon\right)\right)(\theta + r_2) + b\theta}{a_{22}(\theta + r_2)}, \\ U_2 = \limsup_{t\to+\infty} y_2(t) &\le \frac{b\left(-r_1 + a_{21}\left(M_1^x + \varepsilon\right)\right)(\theta + r_2) + b^2\theta}{a_{22}(\theta + r_2)^2}. \end{aligned} \tag{4.7}$$

Since these inequalities are true for arbitrary $\varepsilon > 0$ sufficiently small, it follows that $U_1 \le M_1^{y_1}$, $U_2 \le M_1^{y_2}$, where

$$\begin{aligned} M_1^{y_1} &= \frac{\left(-r_1 + a_{21} M_1^x\right)(\theta + r_2) + b\theta}{a_{22}(\theta + r_2)}, \\ M_1^{y_2} &= \frac{b\left(-r_1 + a_{21} M_1^x\right)(\theta + r_2) + b^2\theta}{a_{22}(\theta + r_2)^2}. \end{aligned} \tag{4.8}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_2 \ge T_1 + \tau$ such that if $t > T_2$, $y_1(t) \le M_1^{y_1} + \varepsilon$, $y_2(t) \le M_1^{y_2} + \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we derive from the first equation of system (1.6) that, for $t > T_2$,

$$\dot{x}(t) \geq x(t)\left(r - a_{11}x(t) - a_{12}\left(M_1^{y_1} + \varepsilon\right)\right). \tag{4.9}$$

By comparison it follows that

$$V = \liminf_{t\to+\infty} x(t) \geq \frac{r - a_{12}\left(M_1^{y_1} + \varepsilon\right)}{a_{11}}. \tag{4.10}$$

Since this is true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V \geq N_1^x$, where

$$N_1^x = \frac{r - a_{12}M_1^{y_1}}{a_{11}}. \tag{4.11}$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is a $T_3 \geq T_2$ such that if $t > T_3$, $x(t) \geq N_1^x - \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we derive from the second and the third equations of system (1.6) that, for $t > T_3 + \tau$,

$$\begin{aligned} \dot{y}_1(t) &\geq -r_1y_1(t) + a_{21}\left(N_1^x - \varepsilon\right)y_1(t-\tau) - a_{22}y_1^2(t) + \theta y_2(t), \\ \dot{y}_2(t) &= by_1(t) - (\theta + r_2)y_2(t). \end{aligned} \tag{4.12}$$

Consider the following auxiliary equations:

$$\begin{aligned} \dot{u}_1(t) &= -r_1u_1(t) + a_{21}\left(N_1^x - \varepsilon\right)u_1(t-\tau) - a_{22}u_1^2(t) + \theta u_2(t), \\ \dot{u}_2(t) &= bu_1(t) - (\theta + r_2)u_2(t). \end{aligned} \tag{4.13}$$

Since (H2) and (H3) hold, by Lemma 2.5 it follows from (4.13) that

$$\begin{aligned} \lim_{t\to+\infty} u_1(t) &= \frac{\left(-r_1 + a_{21}\left(N_1^x - \varepsilon\right)\right)(\theta + r_2) + b\theta}{a_{22}(\theta + r_2)}, \\ \lim_{t\to+\infty} u_2(t) &= \frac{b\left(-r_1 + a_{21}\left(N_1^x - \varepsilon\right)\right)(\theta + r_2) + b^2\theta}{a_{22}(\theta + r_2)^2}. \end{aligned} \tag{4.14}$$

By comparison, we obtain that

$$V_1 = \liminf_{t\to+\infty} y_1(t) \geq \frac{\left(-r_1 + a_{21}\left(N_1^x - \varepsilon\right)\right)(\theta + r_2) + b\theta}{a_{22}(\theta + r_2)},$$
$$V_2 = \liminf_{t\to+\infty} y_2(t) \geq \frac{b\left(-r_1 + a_{21}\left(N_1^x - \varepsilon\right)\right)(\theta + r_2) + b^2\theta}{a_{22}(\theta + r_2)^2}. \tag{4.15}$$

Since these two inequalities hold for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V_1 \geq N_1^{y_1}$, $V_2 \geq N_1^{y_2}$, where

$$N_1^{y_1} = \frac{\left(-r_1 + a_{21}N_1^x\right)(\theta + r_2) + b\theta}{a_{22}(\theta + r_2)},$$
$$N_1^{y_2} = \frac{b\left(-r_1 + a_{21}N_1^x\right)(\theta + r_2) + b^2\theta}{a_{22}(\theta + r_2)^2}. \tag{4.16}$$

Therefore, for $\varepsilon > 0$ sufficiently small, there exists a $T_4 \geq T_3 + \tau$ such that if $t > T_4$, $y_1(t) \geq N_1^{y_1} - \varepsilon$, $y_2(t) \geq N_1^{y_2} - \varepsilon$.

For $\varepsilon > 0$ sufficiently small, it follows from the first equation of system (1.6) that, for $t > T_4$,

$$\dot{x}(t) \leq x(t)\left(r - a_{11}x(t) - a_{12}\left(N_1^{y_1} - \varepsilon\right)\right). \tag{4.17}$$

A comparison argument yields

$$U = \limsup_{t\to+\infty} x(t) \leq \frac{r - a_{12}\left(N_1^{y_1} - \varepsilon\right)}{a_{11}}. \tag{4.18}$$

Since this is true for arbitrary $\varepsilon > 0$, we conclude that $U \leq M_2^x$, where

$$M_2^x = \frac{r - a_{12}N_1^{y_1}}{a_{11}}. \tag{4.19}$$

Hence, for $\varepsilon > 0$ sufficiently small, there exists a $T_5 \geq T_4$ such that if $t > T_5$, $x(t) \leq M_2^x + \varepsilon$.

Again, we derive from the second and the third equations of system (1.6) that for $t > T_5 + \tau$,

$$\dot{y}_1(t) \leq -r_1 y_1(t) + a_{21}\left(M_2^x + \varepsilon\right)y_1(t - \tau) - a_{22}y_1^2(t) + \theta y_2(t),$$
$$\dot{y}_2(t) = by_1(t) - (\theta + r_2)y_2(t). \tag{4.20}$$

Since (H2) and (H3) hold, by Lemma 2.5, a comparison argument shows that

$$U_1 = \limsup_{t\to+\infty} y_1(t) \le \frac{\left(-r_1 + a_{21}\left(M_2^x + \varepsilon\right)\right)(\theta + r_2) + b\theta}{a_{22}(\theta + r_2)},$$
$$U_2 = \limsup_{t\to+\infty} y_2(t) \le \frac{b\left(-r_1 + a_{21}\left(M_2^x + \varepsilon\right)\right)(\theta + r_2) + b^2\theta}{a_{22}(\theta + r_2)^2}. \tag{4.21}$$

Since these inequalities are true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $U_1 \le M_2^{y_1}$, $U_2 \le M_2^{y_2}$, where

$$M_2^{y_1} = \frac{\left(-r_1 + a_{21}M_2^x\right)(\theta + r_2) + b\theta}{a_{22}(\theta + r_2)},$$
$$M_2^{y_2} = \frac{b\left(-r_1 + a_{21}M_2^x\right)(\theta + r_2) + b^2\theta}{a_{22}(\theta + r_2)^2}. \tag{4.22}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_6 \ge T_5 + \tau$ such that if $t > T_6$, $y_1(t) \le M_2^{y_1} + \varepsilon$, $y_2(t) \le M_2^{y_2} + \varepsilon$.

For $\varepsilon > 0$ sufficiently small, it follows from the first equation of system (1.6) that, for $t > T_6$,

$$\dot{x}(t) \ge x(t)\left(r - a_{11}x(t) - a_{12}\left(M_2^{y_1} + \varepsilon\right)\right). \tag{4.23}$$

By comparison we obtain that

$$V = \liminf_{t\to+\infty} x(t) \ge \frac{r - a_{12}\left(M_2^{y_1} + \varepsilon\right)}{a_{11}}. \tag{4.24}$$

Since this is true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V \ge N_2^x$, where

$$N_2^x = \frac{r - a_{12}M_2^{y_1}}{a_{11}}. \tag{4.25}$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is a $T_7 \ge T_6$ such that if $t > T_7$, $x(t) \ge N_2^x - \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we derive from the second and the third equations of system (1.6) that for $t > T_7 + \tau$,

$$\dot{y}_1(t) \ge -r_1y_1(t) + a_{21}\left(N_2^x - \varepsilon\right)y_1(t-\tau) - a_{22}y_1^2(t) + \theta y_2(t),$$
$$\dot{y}_2(t) = by_1(t) - (\theta + r_2)y_2(t). \tag{4.26}$$

Since (H2) and (H3) hold, by Lemma 2.5 and by comparison, it follows from (4.26) that

$$V_1 = \liminf_{t\to+\infty} y_1(t) \geq \frac{\left(-r_1 + a_{21}\left(N_2^x - \varepsilon\right)\right)(\theta + r_2) + b\theta}{a_{22}(\theta + r_2)},$$
$$V_2 = \liminf_{t\to+\infty} y_2(t) \geq \frac{b\left(-r_1 + a_{21}\left(N_2^x - \varepsilon\right)\right)(\theta + r_2) + b^2\theta}{a_{22}(\theta + r_2)^2}. \tag{4.27}$$

Since these two inequalities hold for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V_1 \geq N_2^{y_1}, V_2 \geq N_2^{y_2}$, where

$$N_2^{y_1} = \frac{\left(-r_1 + a_{21}N_2^x\right)(\theta + r_2) + b\theta}{a_{22}(\theta + r_2)},$$
$$N_2^{y_2} = \frac{b\left(-r_1 + a_{21}N_2^x\right)(\theta + r_2) + b^2\theta}{a_{22}(\theta + r_2)^2}. \tag{4.28}$$

Continuing this process, we derive six sequences M_n^x, $M_n^{y_1}$, $M_n^{y_2}$, N_n^x, $N_n^{y_1}$, $N_n^{y_2}$ $(n = 1, 2, \ldots)$ such that, for $n \geq 2$,

$$M_n^x = \frac{r - a_{12}N_{n-1}^{y_1}}{a_{11}},$$
$$M_n^{y_1} = \frac{(-r_1 + a_{21}M_n^x)(\theta + r_2) + b\theta}{a_{22}(\theta + r_2)},$$
$$M_n^{y_2} = \frac{b(-r_1 + a_{21}M_n^x)(\theta + r_2) + b^2\theta}{a_{22}(\theta + r_2)^2},$$
$$N_n^x = \frac{r - a_{12}M_n^{y_1}}{a_{11}},$$
$$N_n^{y_1} = \frac{(-r_1 + a_{21}N_n^x)(\theta + r_2) + b\theta}{a_{22}(\theta + r_2)},$$
$$N_n^{y_2} = \frac{b(-r_1 + a_{21}N_n^x)(\theta + r_2) + b^2\theta}{a_{22}(\theta + r_2)^2}. \tag{4.29}$$

It is readily seen that

$$N_n^x \leq V \leq U \leq M_n^x, \qquad N_n^{y_i} \leq V_i \leq U_i \leq M_n^{y_i} \quad (i = 1, 2). \tag{4.30}$$

We derive (4.29) that

$$M_{n+1}^{y_1} = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}^2 a_{22}^2}\left(\frac{a_{11}b\theta}{\theta + r_2} + a_{21}r - a_{11}r_1\right) + \frac{a_{12}^2 a_{21}^2}{a_{11}^2 a_{22}^2}M_n^{y_1}. \tag{4.31}$$

Noting that $M_n^{y_1} \geq y_1^*$ and $a_{11}a_{22} > a_{12}a_{21}$, it follows from (4.31) that

$$\begin{aligned} M_{n+1}^{y_1} - M_n^{y_1} &= \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}^2 a_{22}^2}\left(\frac{a_{11}b\theta}{\theta + r_2} + a_{21}r - a_{11}r_1\right) + \left(\frac{a_{12}^2 a_{21}^2}{a_{11}^2 a_{22}^2} - 1\right)M_n^{y_1} \\ &\leq \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}^2 a_{22}^2}\left(\frac{a_{11}b\theta}{\theta + r_2} + a_{21}r - a_{11}r_1\right) + \left(\frac{a_{12}^2 a_{21}^2}{a_{11}^2 a_{22}^2} - 1\right)y_1^* \\ &= 0. \end{aligned} \tag{4.32}$$

Thus, the sequence $M_n^{y_1}$ is nonincreasing. Hence, $\lim_{n \to +\infty} M_n^{y_1}$ exists. Taking $n \to +\infty$, it follows from (4.31) that

$$\lim_{n \to +\infty} M_n^{y_1} = \frac{(\theta b a_{11}/(\theta + r_2)) + a_{21}r - a_{11}r_1}{a_{12}a_{21} + a_{11}a_{22}} = y_1^*. \tag{4.33}$$

We therefore obtain from (4.30) and (4.33) that

$$\begin{gathered} \lim_{n \to +\infty} M_n^{y_2} = y_2^*, \qquad \lim_{n \to +\infty} N_n^{y_1} = y_1^*, \qquad \lim_{n \to +\infty} N_n^{y_2} = y_2^*, \\ \lim_{n \to +\infty} M_n^{x} = x^*, \qquad \lim_{n \to +\infty} N_n^{x} = x^*. \end{gathered} \tag{4.34}$$

It follows from (4.30), (4.33) and (4.34) that

$$U = V = x^*, \qquad U_1 = V_1 = y_1^*, \qquad U_2 = V_2 = y_2^*. \tag{4.35}$$

We therefore have

$$\lim_{n \to +\infty} x(t) = x^*, \qquad \lim_{n \to +\infty} y_1(t) = y_1^*, \qquad \lim_{n \to +\infty} y_2(t) = y_2^*. \tag{4.36}$$

Noting that if $a_{11}a_{22} > a_{12}a_{21}$, then $p_0 - q_0 > 2(\theta + r_2)a_{11}a_{21}(x^*)^2 > 0$. By Theorem 3.1, the positive equilibrium E^* is locally stable. We therefore conclude that E^* is globally stable. The proof is complete. □

Theorem 4.2. *If $r_1(\theta + r_2) > b\theta + a_{21}r/a_{11}(\theta + r_2)$, the equilibrium E_1 of system* (1.6) *is globally asymptotically stable.*

Proof. Let $(x(t), y_1(t), y_2(t))$ be any positive solution of system (1.6) with initial conditions (1.7). It follows from the first equation of system (1.6) that

$$\dot{x}(t) \leq x(t)(r - a_{11}x(t)). \tag{4.37}$$

A standard comparison argument shows that

$$\limsup_{t \to +\infty} x(t) \leq \frac{r}{a_{11}}. \tag{4.38}$$

Choose $\varepsilon > 0$ sufficiently small such that

$$r_1(\theta + r_2) > b\theta + a_{21}\left(\frac{r}{a_{11}} + \varepsilon\right)(\theta + r_2). \tag{4.39}$$

Hence, for $\varepsilon > 0$ sufficiently small satisfying (4.39), there is a $T_1 > 0$ such that if $t > T_1$, then $x(t) \le r/a_{11} + \varepsilon$. We derive from the second and the third equations of system (1.6) that for $t > T_1 + \tau$,

$$\begin{gathered} \dot{y}_1(t) \le -r_1 y_1(t) + a_{21}\left(\frac{r}{a_{11}} + \varepsilon\right) y_1(t-\tau) - a_{22} y_1^2(t) + \theta y_2(t), \\ \dot{y}_2(t) = b y_1(t) - (\theta + r_2) y_2(t). \end{gathered} \tag{4.40}$$

Consider the following auxiliary equations:

$$\begin{gathered} \dot{u}_1(t) = -r_1 u_1(t) + a_{21}\left(\frac{r}{a_{11}} + \varepsilon\right) u_1(t-\tau) - a_{22} u_1^2(t) + \theta u_2(t), \\ \dot{u}_2(t) = b u_1(t) - (\theta + r_2) u_2(t). \end{gathered} \tag{4.41}$$

Since (4.39) holds, by Lemma 2.5 it follows from (4.41) that

$$\lim_{t \to +\infty} u_1(t) = 0, \qquad \lim_{t \to +\infty} u_2(t) = 0. \tag{4.42}$$

By comparison, we obtain that

$$\lim_{t \to +\infty} y_1(t) = 0, \qquad \lim_{t \to +\infty} y_2(t) = 0. \tag{4.43}$$

Hence, for $\varepsilon > 0$ sufficiently small satisfying (4.39), there is a $T_2 > T_1$ such that if $t > T_2$, $y_1(t) < a_{11}\varepsilon$.

For $\varepsilon > 0$ sufficiently small satisfying (4.39) and $a_{11}\varepsilon < r$, we derive from the first equation of system (1.6) that for $t > T_2$

$$\dot{x}(t) \ge x(t)(r - a_{11}x(t) - a_{11}\varepsilon). \tag{4.44}$$

By comparison, we obtain that

$$\liminf_{t \to +\infty} x(t) \ge \frac{r}{a_{11}} - \varepsilon, \tag{4.45}$$

which, together with (4.38), yields

$$\lim_{t \to +\infty} x(t) = \frac{r}{a_{11}}. \tag{4.46}$$

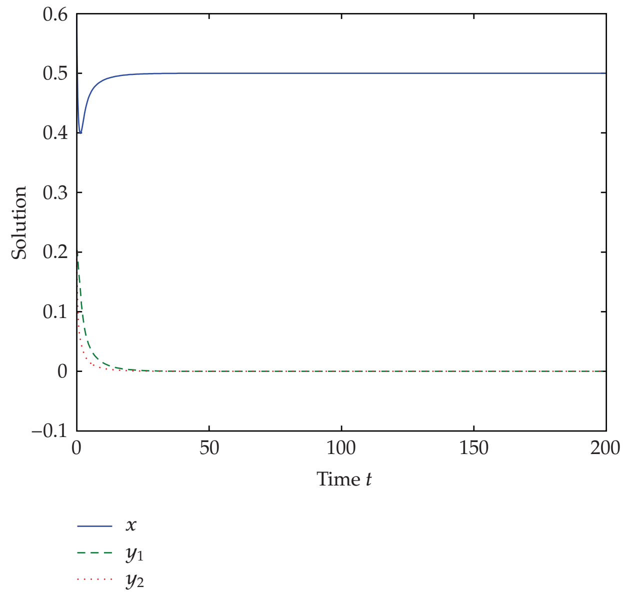

Figure 1: The temporal solution found by numerical integration of system (1.6) with $r = 2,\ r_1 = 2,\ r_2 = 1,\ a_{11} = 4,\ a_{12} = 3,\ a_{21} = 2,\ a_{22} = 4,\ b = 1,\ \theta = 2,\ \tau = 1$ and initial value is $(0.6, 0.2, 0.2)$.

By Theorem 3.1, if $r_1(\theta + r_2) > b\theta + a_{21}r/a_{11}(\theta + r_2)$, the boundary equilibrium E_1 is locally stable. We therefore conclude that E_1 is globally stable in this case. This completes the proof. □

Theorem 4.3. *Let* (H1) *hold. If* $(\theta + r_2)(a_{22}r + a_{12}r_1) < a_{12}\theta b$, *then the equilibrium* E_2 *of system* (1.6) *is globally asymptotically stable.*

Proof. Let $(x(t), y_1(t), y_2(t))$ be any positive solution of system (1.6) with initial conditions (1.7). We derive from the second and the third equations of system (1.6) that, for $t > T_1 + \tau$,

$$\begin{aligned} \dot{y}_1(t) &\geq -r_1 y_1(t) - a_{22}y_1^2(t) + \theta y_2(t), \\ \dot{y}_2(t) &= b y_1(t) - (\theta + r_2)y_2(t). \end{aligned} \tag{4.47}$$

Consider the following auxiliary equations:

$$\begin{aligned} \dot{u}_1(t) &= -r_1 u_1(t) - a_{22}u_1^2(t) + \theta u_2(t), \\ \dot{u}_2(t) &= b u_1(t) - (\theta + r_2)u_2(t). \end{aligned} \tag{4.48}$$

Since (H1) holds, by Lemma 2.5 it follows from (4.48) that

$$\begin{aligned} \lim_{t\to+\infty} u_1(t) &= \frac{1}{a_{22}}\left(\frac{\theta b}{\theta + r_2} - r_1\right) = \tilde{y}_1, \\ \lim_{t\to+\infty} u_2(t) &= \frac{b}{a_{22}(\theta + r_2)}\left(\frac{\theta b}{\theta + r_2} - r_1\right) = \tilde{y}_2. \end{aligned} \tag{4.49}$$

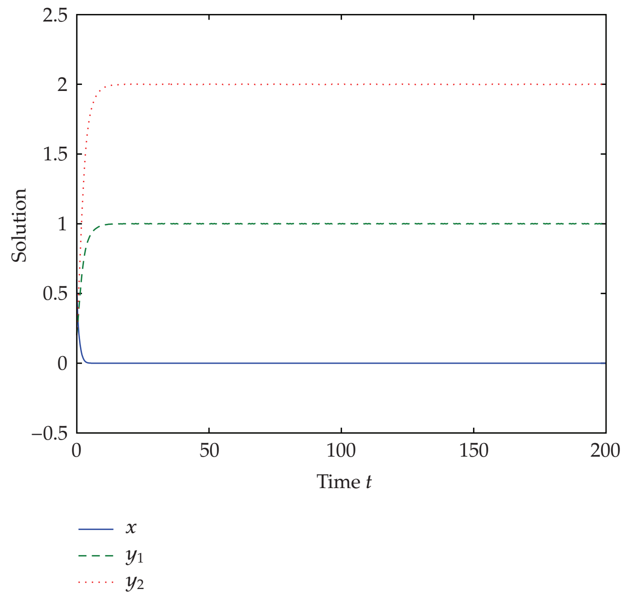

Figure 2: The temporal solution found by numerical integration of system (1.6) with $r = 1,\ r_1 = 1,\ r_2 = 1,\ a_{11} = 4,\ a_{12} = 3,\ a_{21} = 2,\ a_{22} = 1,\ b = 4,\ \theta = 1,\ \tau = 1$ and initial value is $(0.6, 0.2, 0.2)$.

By comparison we derive that

$$\lim_{t\to+\infty} y_1(t) \geq \frac{1}{a_{22}}\left(\frac{\theta b}{\theta + r_2} - r_1\right) = \tilde{y}_1,$$
$$\lim_{t\to+\infty} y_2(t) \geq \frac{b}{a_{22}(\theta + r_2)}\left(\frac{\theta b}{\theta + r_2} - r_1\right) = \tilde{y}_2. \tag{4.50}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a T_1 such that if $t > T_1$, $y_1(t) \geq \tilde{y}_1 - \varepsilon$, $y_2(t) \geq \tilde{y}_2 - \varepsilon$.

For $\varepsilon > 0$ sufficiently small, it follows from the first equation of system (1.6) that for $t > T_1$,

$$\dot{x}(t) \leq x(t)\left(r - a_{11}x(t) - a_{12}\left(\tilde{y}_1 - \varepsilon\right)\right). \tag{4.51}$$

Since $(\theta + r_2)(a_{22}r + a_{12}r_1) < a_{12}\theta b$ and $\varepsilon > 0$ is sufficiently small, by comparison we derive that

$$\limsup_{t\to+\infty} x(t) \leq 0. \tag{4.52}$$

We therefore have $\lim_{t\to+\infty} x(t) = 0$. Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_2 > T_1 + \tau$ such that if $t > T_2$, $x(t) \leq \varepsilon$.

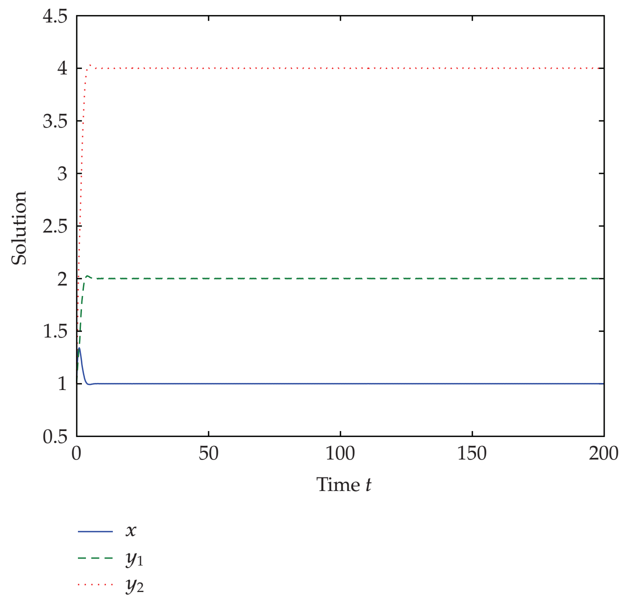

Figure 3: The temporal solution found by numerical integration of system (1.6) with $r = 4$, $r_1 = 1$, $r_2 = 1$, $a_{11} = 2$, $a_{12} = 1$, $a_{21} = 2$, $a_{22} = 1.5$, $b = 4$, $\theta = 1$, $\tau = 1$ and initial value is $(0.8, 1.5, 3.6)$.

For $\varepsilon > 0$ sufficiently small, it follows from the first equation of system (1.6) that for $t > T_2 + \tau$

$$\begin{gathered} \dot{y}_1(t) \leq -r_1 y_1(t) + a_{21}\varepsilon y_1(t-\tau) - a_{22} y_1^2(t) + \theta y_2(t), \\ \dot{y}_2(t) = b y_1(t) - (\theta + r_2) y_2(t). \end{gathered} \tag{4.53}$$

Since (H1) holds, by Lemma 2.5, a comparison argument shows that

$$\begin{gathered} \lim_{t \to +\infty} y_1(t) \leq \frac{(-r_1 + a_{21}\varepsilon)(\theta + r_2) + b\theta}{a_{22}(\theta + r_2)}, \\ \lim_{t \to +\infty} y_2(t) \leq \frac{b(-r_1 + a_{21}\varepsilon)(\theta + r_2) + b^2\theta}{a_{22}(\theta + r_2)^2}, \end{gathered} \tag{4.54}$$

Together with (4.49), yields

$$\begin{gathered} \lim_{t \to +\infty} y_1(t) = \frac{1}{a_{22}} \left(\frac{\theta b}{\theta + r_2} - r_1 \right) = \tilde{y}_1, \\ \lim_{t \to +\infty} y_2(t) = \frac{b}{a_{22}(\theta + r_2)} \left(\frac{\theta b}{\theta + r_2} - r_1 \right) = \tilde{y}_2. \end{gathered} \tag{4.55}$$

Noting that if (H1) holds and $(\theta + r_2)(a_{22} r + a_{12} r_1) < a_{12}\theta b$, the equilibrium E_2 of system (1.6) is locally stable. We therefore conclude that E_2 is globally stable. The proof is complete. □

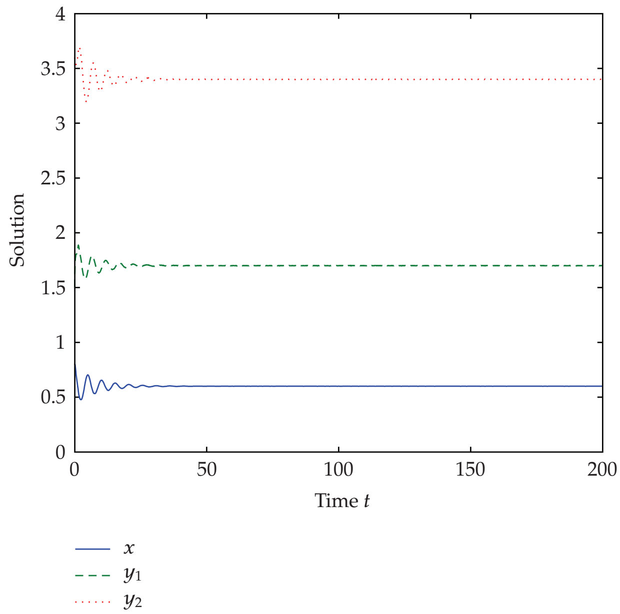

Figure 4: When $\tau = 1 < \tau_0$, the positive equilibrium E^* is asymptotically stable. Here $r = 4,\ r_1 = 1,\ r_2 = 1,\ a_{11} = 1,\ a_{12} = 2,\ a_{21} = 4,\ a_{22} = 2,\ b = 4,\ \theta = 1$ and initial value is $(0.8, 1.5, 3.6)$.

5. Numerical Examples

In this section, we give some examples to illustrate the main results.

Example 5.1. In system (1.6), we let $r = 2,\ r_1 = 2,\ r_2 = 1,\ a_{11} = 4,\ a_{12} = 3,\ a_{21} = 2,\ a_{22} = 4,\ b = 1,\ \theta = 2,$ and $\tau = 1$. It is easy to show that $r_1(\theta + r_2) - b\theta - a_{21}r(\theta + r_2)/a_{11} = 1 > 0$. By Theorem 4.2 we see that the equilibrium $E_1(0.5, 0, 0)$ of system (1.6) is globally stable. Numerical simulation illustrates our result (see Figure 1).

Example 5.2. In system (1.6), we let $r = 1,\ r_1 = 1,\ r_2 = 1,\ a_{11} = 4,\ a_{12} = 3,\ a_{21} = 2,\ a_{22} = 1,\ b = 4,\ \theta = 1,$ and $\tau = 1$. It is easy to show that $b\theta - r_1(\theta + r_2) = 2 > 0,$ and $r - a12\tilde{y}_1 = -2 < 0$. By Theorem 4.3 we see that the equilibrium $E_2(0, 1, 2)$ of system (1.6) is globally stable, as depicted in Figure 2.

Example 5.3. In system (1.6), we let $r = 4,\ r_1 = 1,\ r_2 = 1,\ a_{11} = 2,\ a_{12} = 1,\ a_{21} = 2,\ a_{22} = 1.5,\ b = 4,\ \theta = 1,$ and $\tau = 1$. System (1.6) with above coefficients has a unique positive equilibrium $E^*(1.5, 2, 4)$. It is easy to show that $a_{11}a_{22} - a_{12}a_{21} = 1 > 0$. By Theorem 4.1 we see that the positive equilibrium $E^*(1.5, 2, 4)$ of system (1.6) is globally stable. Numerical simulation illustrates our result (see Figure 3).

Example 5.4. In system (1.6), we let $r = 4,\ r_1 = 1,\ r_2 = 1,\ a_{11} = 1,\ a_{12} = 2,\ a_{21} = 4,\ a_{22} = 2,\ b = 4,$ and $\theta = 1$. System (1.6) with above coefficients has a unique positive equilibrium $E^*(0.6, 1.7, 3.4)$. By computing, we can get $p_0 - q_0 = -6.48 < 0$ and $\omega_0 = 0.9326, \tau_0 = 1.664$. By Theorem 3.1 we see that the equilibrium E^* is locally stable for $\tau < \tau_0$; and E^* is unstable for $\tau > \tau_0$. And system (1.6) undergoes a Hopf Bifurcation at the positive equilibrium E^* when $\tau = \tau_0$. See Figures 4 and 5.

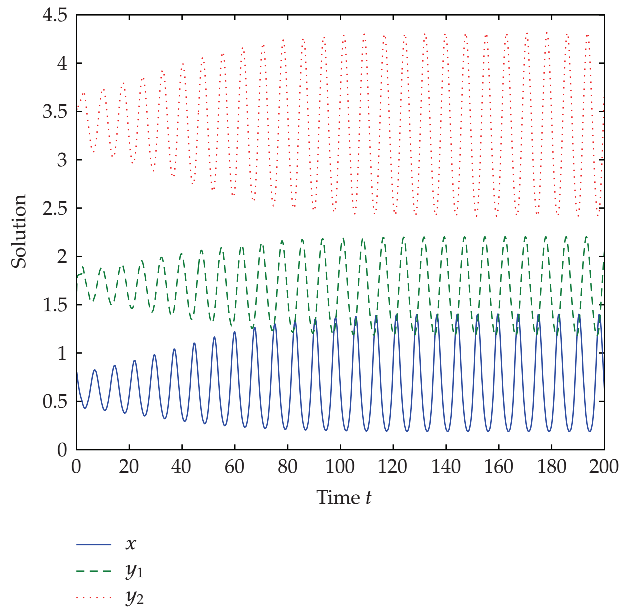

Figure 5: When $\tau = 2 > \tau_0$, the positive equilibrium E^* is asymptotically stable. Here $r = 4,\ r_1 = 1,\ r_2 = 1,\ a_{11} = 1,\ a_{12} = 2,\ a_{21} = 4,\ a_{22} = 2,\ b = 4,\ \theta = 1$ and the initial value is (0.8,1.5,3.6).

6. Discussion

In this paper, we considered a delayed predator-prey model with stage structure for the predator. By using the iteration technique and comparison argument, respectively, sufficient conditions were established for the global stability of the positive equilibrium and two boundary equilibria of system (1.6). By Theorems 4.1, 4.2 and 4.3, we see that: (i) If (H2) holds and $a_{11}a_{22} > a_{12}a_{21}$, system (1.6) is permanent. (ii) If $r_1(\theta + r_2) > b\theta + a_{21}r/a_{11}(\theta + r_2)$, the prey species is persistent but the predator becomes extinct. (iii) If (H1) holds and $(\theta + r_2)(a_{22}r + a_{12}r_1) < a_{12}\theta b$, the predator species is persistent but the prey species becomes extinct.

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