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Research Article

On Dichotomous Behavior of Variational Difference Equations and Applications

Bogdan Sasu

Department of Mathematics, Faculty of Mathematics and Computer Science, West University of Timişoara, V. Pârvan Blvd. No. 4, 300223 Timişoara, Romania

Correspondence should be addressed to Bogdan Sasu, bsasu@math.uvt.ro

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We give new and very general characterizations for uniform exponential dichotomy of variational difference equations in terms of the admissibility of pairs of sequence spaces over $\mathbb{N}$ with respect to an associated control system. We establish in the variational case the connections between the admissibility of certain pairs of sequence spaces over $\mathbb{N}$ and the admissibility of the corresponding pairs of sequence spaces over $\mathbb{Z}$. We apply our results to the study of the existence of exponential dichotomy of linear skew-product flows.

1. Introduction

In recent years, a number of papers have added important contributions to the existing literature on the relation between exponential dichotomy of systems and solvability properties of associated difference or differential equations, or the so-called admissibility properties with sequence or function spaces (see [1–10]). In the case of classical differential equations, the literature on this subject is rich and the main techniques are presented in the valuable contributions of Coffman and Schäffer [11], Coppel [12], Daleckii and Kreĭn [13], and Massera and Schäffer [14]. In the last few years, the methods have been improved and extended for general cases like those of evolution families or variational equations. The recent development in the theory of linear skew-product flows led to important generalizations of the classical results (see [4, 6, 7, 15–17]). A significant achievement was obtained in [15], where Chow and Leiva deduced the structure of the stable and unstable subspaces for an exponentially dichotomic linear skew-product flow. Various discrete-time characterizations for uniform exponential dichotomy of linear skew-product flows were obtained in [4, 6, 7]. In [4] Chow and Leiva introduced and characterized the concept of pointwise discrete dichotomy for a skew-product sequence over $X \times \Theta$, with X a Banach space and Θ a compact Haussdorff space.

Let X be a Banach space, let Θ be a metric space, and let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E} = X \times \Theta$, that is, $\sigma : \Theta \times \mathbb{R} \to \Theta$, $\sigma(\theta, 0) = \theta$ and $\sigma(\theta, t+s) = \sigma(\sigma(\theta, t), s)$, for all $(\theta, t, s) \in \Theta \times \mathbb{R}^2$, and the mapping $\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{L}(X)$ satisfies the following conditions: $\Phi(\theta, 0) = I$, (the identity operator on X), $\Phi(\theta, t+s) = \Phi(\sigma(\theta, t), s)\Phi(\theta, t)$, for all $(\theta, t, s) \in \Theta \times \mathbb{R}_+^2$ and there are $M \geq 1, \omega > 0$ such that $\|\Phi(\theta, t)\| \leq Me^{\omega t}$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$. We associate with π the variational discrete-time system

$$x(\theta)(n+1) = \Phi(\sigma(\theta, n), 1)x(\theta)(n), \quad \forall(\theta, n) \in \Theta \times \mathbb{N}. \tag{A_π}$$

Using the consequences given by the pointwise behavior, Chow and Leiva established in [4] an important result concerning the exponential dichotomy of linear skew-product flows.

Theorem 1.1. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E} = X \times \Theta$. Then π is uniformly exponentially dichotomic if and only if the system (A_π) is uniformly exponentially dichotomic.*

A direct proof of the above theorem was presented in [6], without requiring continuity properties.

The impressive development of difference equations in the past few years (see, e.g., [1–4, 6–10, 15–25] and the references therein) led to important contributions at the study of the qualitative behavior of solutions of variational equations via discrete-time techniques. If $\{A(\theta)\}_{\theta\in\Theta} \subset \mathcal{L}(X)$, then one considers the linear system of variational difference equations

$$x(\theta)(n+1) = A(\sigma(\theta, n))x(\theta)(n), \quad (\theta, n) \in \Theta \times \mathbb{N}, \tag{A}$$

and associates to (A) the control system:

$$\gamma(\theta)(m+1) = A(\sigma(\theta, m))\gamma(\theta)(m) + s(m+1), \quad (\theta, m) \in \Theta \times \mathbb{Z}. \tag{S_A}$$

We denote by (Q_A) the restriction of (S_A) to $\mathbb{N}$. One of the main results in [6] states that a linear skew-product flow is uniformly exponentially dichotomic if and only if the pair $(c_0(\mathbb{N}, X), c_{00}(\mathbb{N}, X))$ is admissible with respect to the system (Q_A) and the space X may be decomposed at every moment into a sum of stable fiber and unstable fiber, that is, $X = \mathcal{S}_0(\theta) + \mathcal{U}_0(\theta)$, for all $\theta \in \Theta$. Here, $c_0(\mathbb{N}, X)$ denotes the space of all sequences $s : \mathbb{N} \to X$ with $\lim_{n\to\infty} s(n) = 0$ and $c_{00}(\mathbb{N}, X) = \{s \in c_0(\mathbb{N}, X) : s(0) = 0\}$.

In [7], we introduced distinct concepts of admissibility considering as the input space $\Delta(\mathbb{Z}, X)$-the space of all sequences of finite support and we gave a unified treatment considering both cases when the output space is $\ell^p(\mathbb{Z}, X)$ or $c_0(\mathbb{Z}, X) := \{s : \mathbb{Z} \to X : \lim_{k\to\pm\infty} s(k) = 0\}$. The approach given in [7] relies on the unique solvability of the discrete-time system (S_A) between certain sequence spaces. The main results in [7] expressed the uniform exponential dichotomy of a discrete linear skew-product flow π in terms of the uniform q-admissibility ($q \in (1, \infty]$) of one of the pairs $(\ell^p(\mathbb{Z}, X), \Delta(\mathbb{Z}, X))$ and $(c_0(\mathbb{Z}, X), \Delta(\mathbb{Z}, X))$, with respect to the system (S_A).

The natural question arises whether in the characterization of the exponential dichotomy the *unique* solvability of the associated discrete control system can be dropped. Another question is which are the connections between the solvability of the system (S_A) and the solvability of the system (Q_A). Using constructive techniques, we will provide complete answers for both questions and we will obtain new and optimal characterizations for uniform exponential dichotomy of variational difference equations. We denote by $\Delta_0(\mathbb{N}, X)$ the space of sequences $s : \mathbb{N} \to X$ with finite support and $s(0) = 0$. The admissibility concepts introduced in the present paper are new and simplify the solvability conditions in [6]. Specifically, the input space considered is the smallest possible input space, the solvability study is reduced to the behavior on the half-line and the boundedness condition is imposed only on the unstable fiber. Moreover we extend the applicability area to the general context of discrete variational systems.

Another purpose of this paper is to provide a complete study concerning the relation between the discrete admissibility on the whole line and the corresponding discrete admissibility on the half-line. Our main strategy will be to provide an almost exhaustive analysis, providing the connections between admissibility of concrete pairs of sequence spaces defined on $\mathbb{N}$ and the uniform exponential dichotomy of a general system of discrete variational equations, emphasizing the appropriate techniques for each case considered therein. Finally, applying the main results, we will deduce new characterizations for uniform exponential dichotomy of linear skew-product flows, in terms of the uniform q-admissibility of the pairs $(\ell^p(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ and $(c_0(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$.

2. Uniform Exponential Dichotomy of Variational Difference Equations

Let X be a real or complex Banach space, let (Θ, d) be a metric space and let $\mathcal{E} = X \times \Theta$. The norm on X and on $\mathcal{L}(X)$-the Banach algebra of all bounded linear operators on X will be denoted by $\|\cdot\|$.

Notations

Let $\mathbb{Z}$ denote the set of the integers, let $\mathbb{N}$ be the set of the nonnegative integers ($m \in \mathbb{Z}, m \geq 0$) and let $\mathbb{Z}_-$ be the set of the nonpositive integers ($m \in \mathbb{Z}, m \leq 0$). For every $A \subset \mathbb{Z}$, let χ_A denote the characteristic function of A. If $J \in \{\mathbb{Z}, \mathbb{N}, \mathbb{Z}_-\}$ let $\Delta(J, X)$ be the linear space of all sequences $s : J \to X$ with the property that the set $\{k \in J : s(k) \neq 0\}$ is finite and let $\Delta_0(\mathbb{N}, X) := \{s \in \Delta(\mathbb{N}, X) : s(0) = 0\}$.

If $p \in [1, \infty)$ let $\ell^p(J, X)$ denote the set of all sequences $s : J \to X$ with the property that $\sum_{j\in J} \|s(j)\|^p < \infty$, which is a Banach space with respect to the norm

$$\|s\|_p := \left(\sum_{j\in J} \|s(j)\|^p \right)^{1/p}. \tag{2.1}$$

Let $\ell^\infty(J, X)$ be the set of all bounded sequences $s : J \to X$, which is a Banach space with respect to the norm $\|s\|_\infty := \sup_{j\in J} \|s(j)\|$. If $\omega \in \{-\infty, \infty\}$, then $c_0(J, X)$-the set of all sequences $s : J \to X$ with $\lim_{j\to\omega} s(j) = 0$, is a closed linear subspace of $\ell^\infty(J, X)$. For $p \in [1, \infty]$, we set $\ell^p_0(\mathbb{N}, X) := \{s \in \ell^p(\mathbb{N}, X) : s(0) = 0\}$. Similarly, $c_{00}(\mathbb{N}, X) := \{s \in c_0(\mathbb{N}, X) : s(0) = 0\}$.

Definition 2.1. A mapping $\sigma : \Theta \times \mathbb{Z} \to \Theta$ is called *discrete flow* on Θ if $\sigma(\theta, 0) = \theta$ and $\sigma(\theta, m + n) = \sigma(\sigma(\theta, m), n)$, for all $(\theta, m, n) \in \Theta \times \mathbb{Z}^2$.

Let $\{A(\theta)\}_{\theta\in\Theta} \subset \mathcal{L}(X)$. We consider the linear system of variational difference equations (A). The discrete cocycle associated with the system (A) is

$$\Phi : \Theta \times \mathbb{N} \longrightarrow \mathcal{L}(X), \qquad \Phi(\theta, n) = \begin{cases} A(\sigma(\theta, n-1)) \cdots A(\theta), & n \in \mathbb{N}^*, \\ I, & n = 0, \end{cases} \tag{2.2}$$

where I is the identity operator on X. It is easy to see that $\Phi(\theta, m+n) = \Phi(\sigma(\theta, n), m)\Phi(\theta, n)$, for all $(\theta, m, n) \in \Theta \times \mathbb{N}^2$.

Definition 2.2. The system (A) is said to be *uniformly exponentially dichotomic* if there exist a family of projections $\{P(\theta)\}_{\theta\in\Theta} \subset \mathcal{L}(X)$ and two constants $K \geq 1$ and $\nu > 0$ such that the following properties hold:

(i) $\Phi(\theta, n)P(\theta) = P(\sigma(\theta, n))\Phi(\theta, n)$, for all $(\theta, n) \in \Theta \times \mathbb{N}$;

(ii) $\|\Phi(\theta, n)x\| \leq Ke^{-\nu n}\|x\|$, for all $x \in \operatorname{Im} P(\theta)$ and all $(\theta, n) \in \Theta \times \mathbb{N}$;

(iii) $\|\Phi(\theta, n)y\| \geq (1/K)e^{\nu n}\|y\|$, for all $y \in \operatorname{Ker} P(\theta)$ and all $(\theta, n) \in \Theta \times \mathbb{N}$;

(iv) the restriction $\Phi(\theta, n)_| : \operatorname{Ker} P(\theta) \rightarrow \operatorname{Ker} P(\sigma(\theta, n))$ is an isomorphism, for all $(\theta, n) \in \Theta \times \mathbb{N}$.

For every $\theta \in \Theta$ we denote by $\mathcal{F}(\theta)$ the linear space of all sequences $\varphi : \mathbb{Z}_- \rightarrow X$ with the property $\varphi(m) = A(\sigma(\theta, m-1))\varphi(m-1)$, for all $m \in \mathbb{Z}_-$.

Let $p \in [1, \infty]$. For every $\theta \in \Theta$ we consider the stable space

$$\mathcal{S}_p(\theta) = \{x \in X : \Phi(\theta, \cdot)x \in \ell^p(\mathbb{N}, X)\}, \tag{2.3}$$

and the unstable space

$$\mathcal{U}_p(\theta) = \{x \in X : \text{ there exists } \varphi \in \mathcal{F}(\theta) \cap \ell^p(\mathbb{Z}_-, X) \text{with } \varphi(0) = x\}. \tag{2.4}$$

We note that $\mathcal{S}_p(\theta)$ and $\mathcal{U}_p(\theta)$ are linear subspaces, for every $\theta \in \Theta$.

Remark 2.3. If the system (A) is uniformly exponentially dichotomic and

$$\sup_{\theta\in\Theta} \|A(\theta)\| < \infty, \tag{2.5}$$

then the family of projections given by Definition 2.2 is uniquely determined and

$$\sup_{\theta\in\Theta} \|P(\theta)\| < \infty. \tag{2.6}$$

Moreover, for every $p \in [1, \infty]$, we have that $\operatorname{Im} P(\theta) = \mathcal{S}_p(\theta)$ and $\operatorname{Ker} P(\theta) = \mathcal{U}_p(\theta)$, for all $\theta \in \Theta$ (see [7, Proposition 2.1]).

We associate to the system (A) the input-output control system (S_A).

Definition 2.4. Let $p, q \in [1, \infty]$. The pair $(\ell^p(\mathbb{Z}, X), \Delta(\mathbb{Z}, X))$ is said to be *uniformly q -admissible* for the system (S_A) if the following assertions hold:

(i) for every $s \in \Delta(\mathbb{Z}, X)$ there is a unique $\gamma_s : \Theta \to \ell^p(\mathbb{Z}, X)$ solution of the system (S_A) corresponding to the input sequence s;

(ii) there is $\lambda > 0$ such that $\|\gamma_s(\theta)\|_p \le \lambda \|s\|_q$, for all $(\theta, s) \in \Theta \times \Delta(\mathbb{Z}, X)$.

For the proof of the next result we refer to [7, Theorem 3.6].

Theorem 2.5. *Let $p, q \in [1, \infty]$ be such that $(p, q) \neq (\infty, 1)$. The following assertions hold:*

(i) *if the pair $(\ell^p(\mathbb{Z}, X), \Delta(\mathbb{Z}, X))$ is uniformly q-admissible for the system (S_A), then the system (A) is uniformly exponentially dichotomic;*

(ii) *if $p \ge q$ and $\sup_{\theta\in\Theta}\|A(\theta)\| < \infty$, then the system (A) is uniformly exponentially dichotomic if and only if the pair $(\ell^p(\mathbb{Z}, X), \Delta(\mathbb{Z}, X))$ is uniformly q-admissible for the system (S_A).*

We consider the input-output system

$$\alpha(\theta)(n+1) = A(\sigma(\theta, n))\alpha(\theta)(n) + u(n+1), \quad (\theta, n) \in \Theta \times \mathbb{N}. \qquad (Q_A)$$

The main question arises whether in the characterization of the exponential dichotomy the *unique* solvability of the associated discrete equation can be dropped. Another question is which are the connections between the solvability of the system (Q_A) and the solvability of the system (S_A). In what follows we will give complete answers for both questions, our study focusing on these central purposes.

Definition 2.6. Let $p, q \in [1, \infty]$. The pair $(\ell^p(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ is said to be *uniformly q-*admissible for the system (Q_A) if there is $\lambda > 0$ such that the following assertions hold:

(i) for every $u \in \Delta_0(\mathbb{N}, X)$ there is $\alpha : \Theta \to \ell^p(\mathbb{N}, X)$ solution of the system (Q_A) corresponding to u;

(ii) if $u \in \Delta_0(\mathbb{N}, X)$ and $\alpha : \Theta \to \ell^p(\mathbb{N}, X)$ is a solution of (Q_A) corresponding to u with the property that $\alpha(\theta)(0) \in \mathcal{U}_p(\theta)$, for every $\theta \in \Theta$, then

$$\|\alpha(\theta)\|_p \le \lambda \|u\|_q, \quad \forall\theta \in \Theta. \qquad (2.7)$$

Lemma 2.7. *Let $p, q \in [1, \infty]$. If the pair $(\ell^p(\mathbb{Z}, X), \Delta(\mathbb{Z}, X))$ is uniformly q-admissible for the system (S_A) then $\mathcal{S}_p(\theta) + \mathcal{U}_p(\theta) = X$, for all $\theta \in \Theta$.*

Proof. Let $x \in X$. We consider the sequence $s : \mathbb{Z} \to X$, $s(k) = \chi_{\{0\}}(k)x$. From hypothesis, there is γ_s solution of the system (S_A) corresponding to s.

Let $\theta \in \Theta$. From $\gamma_s(\theta)(n) = \Phi(\theta, n)\gamma_s(\theta)(0)$ for all $n \in \mathbb{N}$ and since $\gamma_s(\theta) \in \ell^p(\mathbb{Z}, X)$ we deduce that $\gamma_s(\theta)(0) \in \mathcal{S}_p(\theta)$. In addition, considering the sequence $\varphi_\theta : \mathbb{Z}_- \to X, \varphi_\theta(k) = \chi_{\{0\}}(k)x - \gamma_s(\theta)(k)$ we have that $\varphi_\theta \in \mathcal{F}(\theta)$ and $\varphi_\theta \in \ell^p(\mathbb{Z}_-, X)$. This shows that $x - \gamma_s(\theta)(0) = \varphi_\theta(0) \in \mathcal{U}_p(\theta)$. It follows that $x = \gamma_s(\theta)(0) + (x - \gamma_s(\theta)(0)) \in \mathcal{S}_p(\theta) + \mathcal{U}_p(\theta)$. Since $\theta \in \Theta$ was arbitrary, the proof is complete. □

The first main result of this section is as follows.

Theorem 2.8. *Let $p, q \in [1, \infty]$. The following assertions are equivalent:*

(i) *the pair $(\ell^p(\mathbb{Z}, X), \Delta(\mathbb{Z}, X))$ is uniformly q-admissible for the system (S_A);*

(ii) *the pair $(\ell^p(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ is uniformly q-admissible for the system (Q_A) and $\mathcal{S}_p(\theta) + \mathcal{U}_p(\theta) = X$, for all $\theta \in \Theta$.*

Proof. (i)⇒(ii) Let $\lambda > 0$ be given by Definition 2.4. Let $u \in \Delta_0(\mathbb{N}, X)$. Then the sequence $s_u : \mathbb{Z} \to X, s_u(k) = \chi_{\mathbb{N}}(k)u(k)$ belongs to $\Delta(\mathbb{Z}, X)$. From the uniform q-admissibility of the pair $(\ell^p(\mathbb{Z}, X), \Delta(\mathbb{Z}, X))$ it follows that there is a unique $\gamma_u : \Theta \to \ell^p(\mathbb{Z}, X)$ solution of (S_A) corresponding to s_u. Moreover

$$\left\|\gamma_u(\theta)\right\|_p \leq \lambda \, \|s_u\|_q, \quad \forall \theta \in \Theta. \tag{2.8}$$

Taking

$$\alpha_u : \Theta \longrightarrow \ell^p(\mathbb{N}, X), \qquad \alpha_u(\theta)(n) = \gamma_u(\theta)(n), \tag{2.9}$$

we have that α_u is a solution of (Q_A) corresponding to u.

Let $\alpha : \Theta \to \ell^p(\mathbb{N}, X)$ be a solution of (Q_A) corresponding to u with the property that $\alpha(\theta)(0) \in \mathcal{U}_p(\theta)$, for all $\theta \in \Theta$. Then for every $\theta \in \Theta$, there is $\varphi_\theta \in \ell^p(\mathbb{Z}_-, X)$ with $\varphi_\theta(0) = \alpha(\theta)(0)$ and

$$\varphi_\theta(m) = A(\sigma(\theta, m-1))\varphi_\theta(m-1), \quad \forall m \in \mathbb{Z}_-. \tag{2.10}$$

Considering the sequence

$$\gamma_\theta : \mathbb{Z} \longrightarrow X, \quad \gamma_\theta(m) = \begin{cases} \alpha(\theta)(m), & m \in \mathbb{N}, \\ \varphi_\theta(m), & m \in \mathbb{Z} \setminus \mathrm{N}, \end{cases} \tag{2.11}$$

we obtain that $\gamma_\theta \in \ell^p(\mathbb{Z}, X)$ and

$$\gamma_\theta(m+1) = A(\sigma(\theta, m))\gamma_\theta(m) + s_u(m+1), \quad \forall m \in \mathbb{Z}. \tag{2.12}$$

This implies that

$$\tilde{\gamma} : \Theta \longrightarrow \ell^p(\mathbb{Z}, X), \qquad \tilde{\gamma}(\theta) = \gamma_\theta \tag{2.13}$$

is solution of the system (S_A) corresponding to s_u. From the uniqueness, we deduce that $\tilde{\gamma} = \gamma_u$. Then, using (2.8) we have that

$$\|\alpha(\theta)\|_p \leq \left\|\gamma_\theta\right\|_p = \left\|\gamma_u(\theta)\right\|_p \leq \lambda \, \|s_u\|_q = \lambda \, \|u\|_q, \quad \forall \theta \in \Theta. \tag{2.14}$$

Taking into account that λ does not depend on u or θ we obtain that the pair $(\ell^p(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ is uniformly q-admissible for the system (Q_A).

In addition, from Lemma 2.7 we have that $\mathcal{S}_p(\theta) + \mathcal{U}_p(\theta) = X$, for all $\theta \in \Theta$.

(ii)⇒(i) Let $\lambda > 0$ be given by Definition 2.6.

Let $s \in \Delta(\mathbb{Z}, X)$. Then, there is $h \in \mathbb{Z}_-$ such that $s(j) = 0$, for all $j \le h$.

Consider the sequence $u : \mathbb{N} \to X$, $u(n) = s(n+h)$. Then $u \in \Delta_0(\mathbb{N}, X)$. From the uniform q-admissibility of the pair $(\ell^p(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ there is $\alpha : \Theta \to \ell^p(\mathbb{N}, X)$ solution of the system (Q_A) corresponding to u.

Let $\theta \in \Theta$. Since $\mathcal{S}_p(\theta) + \mathcal{U}_p(\theta) = X$ there is $x_s^\theta \in \mathcal{S}_p(\theta)$ and $x_u^\theta \in \mathcal{U}_p(\theta)$ such that $\alpha(\theta)(0) = x_s^\theta + x_u^\theta$. Let

$$\delta_\theta : \mathbb{N} \longrightarrow X, \qquad \delta_\theta(n) = \alpha(\theta)(n) - \Phi(\theta, n)x_s^\theta. \tag{2.15}$$

Then $\delta_\theta \in \ell^p(\mathbb{N}, X), \delta_\theta(0) = x_u^\theta$ and

$$\delta_\theta(n+1) = A(\sigma(\theta, n))\delta_\theta(n) + u(n+1), \quad \forall n \in \mathbb{N}. \tag{2.16}$$

From $x_u^\theta \in \mathcal{U}_p(\theta)$ it follows that there is $\varphi_\theta \in \ell^p(\mathbb{Z}_-, X)$ with $\varphi_\theta(0) = x_u^\theta$ and

$$\varphi_\theta(m) = A(\sigma(\theta, m-1))\varphi_\theta(m-1), \quad \forall m \in \mathbb{Z}_-. \tag{2.17}$$

Let

$$v_\theta : \mathbb{Z} \longrightarrow X, \qquad v_\theta(k) = \begin{cases} \delta_\theta(k), & k \in \mathbb{N}, \\ \varphi_\theta(k), & k \in \mathbb{Z} \setminus \mathbb{N}. \end{cases} \tag{2.18}$$

Then $v_\theta \in \ell^p(\mathbb{Z}, X)$ and from (2.16) and (2.17) we obtain that

$$v_\theta(m+1) = A(\sigma(\theta, m))v_\theta(m) + s(m+h+1), \quad \forall m \in \mathbb{Z}. \tag{2.19}$$

We define

$$\gamma : \Theta \longrightarrow \ell^p(\mathbb{Z}, X), \qquad \gamma(\theta)(m) = v_{\sigma(\theta,h)}(m-h). \tag{2.20}$$

Then, using (2.19) we have that

$$v_{\sigma(\theta,h)}(m+1) = A(\sigma(\theta, m+h))v_{\sigma(\theta,h)}(m) + s(m+h+1), \quad \forall m \in \mathbb{Z}, \ \forall \theta \in \Theta, \tag{2.21}$$

which implies that

$$\gamma(\theta)(k+1) = A(\sigma(\theta, k))\gamma(\theta)(k) + s(k+1), \quad \forall k \in \mathbb{Z}, \ \forall \theta \in \Theta. \tag{2.22}$$

This shows that γ is a solution of the system (S_A) corresponding to s.

Let $\tilde{\gamma} : \Theta \to \ell^p(\mathbb{Z}, X)$ be a solution of (S_A) corresponding to s and let $\beta = \tilde{\gamma} - \gamma$. Then

$$\beta(\theta)(m+1) = A(\sigma(\theta, m))\beta(\theta)(m), \quad \forall (m, \theta) \in \mathbb{Z} \times \Theta. \tag{2.23}$$

Let $m \in \mathbb{Z}$. Considering

$$\alpha : \Theta \to \ell^p(\mathbb{N}, X), \qquad \alpha(\theta)(n) = \beta(\sigma(\theta, -m))(n+m), \tag{2.24}$$

we have that $\alpha(\theta)(n+1) = A(\sigma(\theta, n))\alpha(\theta)(n)$, for all $(n, \theta) \in \mathbb{N} \times \Theta$, so α is a solution of the system (Q_A) corresponding to $w = 0$. From (2.23) it follows that $\alpha(\theta)(0) \in \mathcal{U}_p(\theta)$, for all $\theta \in \Theta$. Then, from the uniform q-admissibility of the pair $(\ell^p(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ we obtain that $\alpha = 0$. In particular $\beta(\theta)(m) = \alpha(\sigma(\theta, m))(0) = 0$, for all $\theta \in \Theta$. Since $m \in \mathbb{Z}$ was arbitrary it follows that $\beta(\theta) = 0$, for all $\theta \in \Theta$, so γ was uniquely determined.

It remains to verify that condition (ii) in Definition 2.4 is fulfilled. Let $s \in \Delta(\mathbb{Z}, X)$ and let $\gamma \in \ell^p(\mathbb{Z}, X)$ be the solution of (S_A) corresponding to the input s. Let $h \in \mathbb{Z}_-$ be such that $s(j) = 0$, for all $j \leq h$. We consider the sequence $u_h : \mathbb{N} \to X$, $u_h(n) = s(n+h)$ and let

$$\alpha_h : \Theta \longrightarrow \ell^p(\mathbb{N}, X), \qquad \alpha_h(\theta)(n) = \gamma(\sigma(\theta, -h))(n+h). \tag{2.25}$$

Then we have that α_h is the solution of the system (Q_A) corresponding to u_h. From

$$\gamma(\theta)(m) = A(\sigma(\theta, m-1))\gamma(\theta)(m-1), \quad \forall m \leq h, \ \forall \theta \in \Theta, \tag{2.26}$$

we deduce that

$$\gamma(\sigma(\theta, -h))(k+h) = A(\sigma(\theta, k-1))\gamma(\sigma(\theta, -h))(k+h-1), \quad \forall k \in \mathbb{Z}_-, \forall \theta \in \Theta. \tag{2.27}$$

Let $\theta \in \Theta$. Denoting by $\varphi_\theta : \mathbb{Z}_- \to X$, $\varphi_\theta(k) = \gamma(\sigma(\theta, -h))(k+h)$ we deduce that

$$\varphi_\theta(k) = A(\sigma(\theta, k-1))\varphi_\theta(k-1), \quad \forall k \in \mathbb{Z}_-. \tag{2.28}$$

Moreover, since $\gamma(\tilde{\theta}) \in \ell^p(\mathbb{Z}, X)$, for all $\tilde{\theta} \in \Theta$, it follows that $\varphi_\theta \in \ell^p(\mathbb{Z}_-, X)$.

This implies that $\alpha_h(\theta)(0) = \varphi_\theta(0) \in \mathcal{U}_p(\theta)$, for all $\theta \in \Theta$. From the uniform q-admissibility of the pair $(\ell^p(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ we obtain that

$$\|\alpha_h(\theta)\|_p \leq \lambda \, \|u_h\|_q, \quad \forall \theta \in \Theta. \tag{2.29}$$

Since $||u_h||_q = ||s||_q$, from (2.29) we deduce that

$$\left(\sum_{j=0}^{\infty} \|\alpha_h(\theta)(j)\|^p \right)^{1/p} \leq \lambda \, \|s\|_q, \quad \forall \theta \in \Theta, \tag{2.30}$$

which is equivalent with

$$\left(\sum_{j=0}^{\infty}\left\|\gamma(\sigma(\theta,-h))(j+h)\right\|^{p}\right)^{1/p} \leq \lambda\ \|s\|_{q}, \quad \forall\theta\in\Theta, \tag{2.31}$$

and, respectively, with

$$\left(\sum_{k=h}^{\infty}\left\|\gamma(\theta)(k)\right\|^{p}\right)^{1/p} \leq \lambda\ \|s\|_{q}, \quad \forall\theta\in\Theta. \tag{2.32}$$

Since relation (2.32) holds for all $h \in \mathbb{Z}_-$ with the property that $s(j) = 0$, for all $j \leq h$, for $h \to -\infty$ in (2.32) we obtain that $\|\gamma(\theta)\|_p \leq \lambda\ \|s\|_q$, for all $\theta \in \Theta$. Taking into account that λ does not depend on θ or s, we conclude that the pair $(\ell^p(\mathbb{Z}, X), \Delta(\mathbb{Z}, X))$ is uniformly q-admissible for the system (S_A) and the proof is complete. □

The second main result of this section is as follows.

Theorem 2.9. *Let $p, q \in [1,\infty]$ be such that $(p,q) \neq (\infty, 1)$. The following assertions hold:*

(i) *if the pair $(\ell^p(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ is uniformly q-admissible for the system (Q_A) and $\mathcal{S}_p(\theta) + \mathcal{U}_p(\theta) = X$, for all $\theta \in \Theta$, then the system (A) is uniformly exponentially dichotomic;*

(ii) *if $p \geq q$ and $\sup_{\theta\in\Theta}\|A(\theta)\| < \infty$, then the system (A) is uniformly exponentially dichotomic if and only if the pair $(\ell^p(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ is uniformly q-admissible for the system (Q_A) and $\mathcal{S}_p(\theta) + \mathcal{U}_p(\theta) = X$, for all $\theta \in \Theta$.*

Proof. This follows from Theorems 2.5 and 2.8. □

Remark 2.10. Naturally, the question arises whether, generally, the uniform 1-admissibility of the pair $(\ell^\infty(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ for the system (Q_A) and the property that $\mathcal{S}_\infty(\theta) + \mathcal{U}_\infty(\theta) = X$, for all $\theta \in \Theta$, implies the uniform exponential dichotomy of the system (A). The answer is negative, as the following example shows.

Example 2.11. Let W be a Banach space and let $X = W \times W$. On X we consider the norm $\|(w_1, w_2)\| = \|w_1\|_W + \|w_2\|_W$. Let

$$f : \mathbb{Z} \longrightarrow (0,\infty), \qquad f(k) = \begin{cases} \dfrac{1}{k+1}, & k \in \mathbb{N}, \\ 1-k, & k \in \mathbb{Z}\setminus\mathbb{N}. \end{cases} \tag{2.33}$$

Let $\Theta = \mathbb{Z}$ and let $\sigma : \Theta \times \mathbb{Z} \to \Theta, \sigma(\theta, m) = \theta + m$. For every $\theta \in \Theta$ we consider the operator

$$A(\theta) : X \longrightarrow X, \qquad A(\theta)(w_1, w_2) = \left(\frac{f(\theta+1)}{f(\theta)}\ w_1, \frac{f(\theta)}{f(\theta+1)} w_2\right). \tag{2.34}$$

The discrete cocycle associated with the system

$$x(\theta)(n+1) = A(\sigma(\theta,n))x(\theta)(n), \quad \forall(\theta,n) \in \Theta \times \mathbb{N}, \tag{A'}$$

is given by

$$\Phi(\theta,n) : X \longrightarrow X, \qquad \Phi(\theta,n)(w_1,w_2) = \left(\frac{f(\theta+n)}{f(\theta)}\ w_1, \frac{f(\theta)}{f(\theta+n)}w_2\right) \tag{2.35}$$

for all $(\theta,n) \in \Theta \times \mathbb{N}$.

Step 1. We prove that $\mathcal{S}_\infty(\theta) + \mathcal{U}_\infty(\theta) = X$, for all $\theta \in \Theta$.

Let $\theta \in \Theta$. Using (2.35) it is easy to see that $\mathcal{S}_\infty(\theta) = W \times \{0\}$. We prove that $\mathcal{U}_\infty(\theta) = \{0\} \times W$. Indeed, if $x = (0,w) \in \{0\} \times W$, by defining $\varphi : \mathbb{Z}_- \to X, \varphi(m) = (f(\theta)/f(\theta+m))\ \ x$, we observe that $\varphi \in \mathcal{F}(\theta)$ and $\varphi(0) = x$. This shows that $x \in \mathcal{U}_\infty(\theta)$, so $\{0\} \times W \subset \mathcal{U}_\infty(\theta)$.

Conversely, let $x \in \mathcal{U}_\infty(\theta)$. Then there is $\varphi \in \mathcal{F}(\theta) \cap \ell^\infty(\mathbb{Z}_-, X)$ with $\varphi(0) = x$. If $\varphi = (\varphi_1, \varphi_2)$, in particular, we have that

$$\varphi_1(0) = \frac{f(\theta)}{f(\theta+k)}\varphi_1(k), \quad \forall k \in \mathbb{Z}_-. \tag{2.36}$$

Since $\varphi \in \ell^\infty(\mathbb{Z}_-, X)$ we have that $\varphi_1 \in \ell^\infty(\mathbb{Z}_-, W)$. Then, from (2.36) we obtain that

$$\|\varphi_1(0)\| \le \frac{f(\theta)}{f(\theta+k)}\|\varphi_1\|_\infty, \quad \forall k \in \mathbb{Z}_-. \tag{2.37}$$

As $k \to -\infty$ in (2.37) it follows that $\varphi_1(0) = 0$. Then $x = (0, \varphi_2(0)) \in \{0\} \times W$. This shows that $\mathcal{U}_\infty(\theta) = \{0\} \times W$. So, $\mathcal{S}_\infty(\theta) + \mathcal{U}_\infty(\theta) = X$, for all $\theta \in \Theta$.

Step 2. We prove that the pair $(\ell^\infty(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ is uniformly 1-admissible for the system (Q_A). Let $u \in \Delta_0(\mathbb{N}, X)$ and let $\theta \in \Theta$. If $u = (u_1, u_2)$, then we define

$$\alpha_\theta : \mathbb{N} \longrightarrow X, \qquad \alpha_\theta(n) = \left(\sum_{k=0}^{n}\frac{f(\theta+n)}{f(\theta+k)}u_1(k), -\sum_{k=n+1}^{\infty}\frac{f(\theta+k)}{f(\theta+n)}u_2(k)\right). \tag{2.38}$$

Since $u \in \Delta_0(\mathbb{N}, X)$ there is $l \in \mathbb{N}^*$ such that $u(k) = 0$, for all $k \ge l$. Then

$$\alpha_\theta(n) = \left(f(\theta+n)\sum_{k=0}^{l}\frac{u_1(k)}{f(\theta+k)},\ 0\right), \quad \forall n \ge l. \tag{2.39}$$

From (2.39) it follows that $\alpha_\theta \in c_0(\mathbb{N}, X)$, so, in particular, $\alpha_\theta \in \ell^\infty(\mathbb{N}, X)$. We define $\alpha : \Theta \to \ell^\infty(\mathbb{N}, X), \alpha(\theta) = \alpha_\theta$ and an easy computation shows that α is a solution of the system (Q_A) corresponding to u.

Let $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2) : \Theta \to \ell^\infty(\mathbb{N}, X)$ be a solution of (Q_A) corresponding to u with the property that $\tilde{\alpha}(\theta)(0) \in \mathcal{U}_\infty(\theta)$, for all $\theta \in \Theta$.

Let $\theta \in \Theta$. From $\tilde{\alpha}(\theta)(0) \in \mathcal{U}_\infty(\theta)$ and Step 1 it follows that $\tilde{\alpha}_1(\theta)(0) = 0$. This implies that $\tilde{\alpha}_1(\theta)(1) = u_1(1)$, so

$$\|\tilde{\alpha}_1(\theta)(2)\| \leq \sum_{k=1}^{2} \|u_1(k)\|. \tag{2.40}$$

Inductively, we obtain that

$$\|\tilde{\alpha}_1(\theta)(n)\| \leq \sum_{k=1}^{n} \|u_1(k)\|, \quad \forall n \in \mathbb{N} \tag{2.41}$$

so

$$\|\tilde{\alpha}_1(\theta)\|_\infty \leq \|u_1\|_1. \tag{2.42}$$

Since $u(k) = 0$, for all $k \geq l$, we have that

$$\tilde{\alpha}_2(\theta)(k) = \frac{f(\theta + l - 1)}{f(\theta + k)} \tilde{\alpha}_2(\theta)(l-1), \quad \forall k \geq l - 1. \tag{2.43}$$

Taking into account that $\tilde{\alpha}(\theta) \in \ell^\infty(\mathbb{N}, X)$, using relation (2.43) we deduce that

$$\|\tilde{\alpha}_2(\theta)(l-1)\| \leq \frac{f(\theta + k)}{f(\theta + l - 1)} \|\tilde{\alpha}_2(\theta)\|_\infty, \quad \forall k \geq l - 1. \tag{2.44}$$

For $k \to \infty$ in (2.44) we obtain that $\tilde{\alpha}_2(\theta)(l-1) = 0$. Then, from (2.43) it follows that $\tilde{\alpha}_2(\theta)(k) = 0$, for all $k \geq l - 1$. Moreover, if $l \geq 2$, from

$$0 = \tilde{\alpha}_2(\theta)(l-1) = \frac{f(\theta + l - 2)}{f(\theta + l - 1)} \tilde{\alpha}_2(\theta)(l-2) + u_2(l-1), \tag{2.45}$$

we deduce that $\|\tilde{\alpha}_2(\theta)(l-2)\| \leq \|u_2(l-1)\|$. Inductively, it follows that

$$\left\|\tilde{\alpha}_2(\theta)(j)\right\| \leq \sum_{k=j+1}^{l-1} \|u_2(k)\|, \quad \forall j \in \{0, \ldots, l-2\}. \tag{2.46}$$

In particular, this implies that

$$\|\tilde{\alpha}_2(\theta)\|_\infty = \sup_{j \in \{0, \ldots, l-2\}} \left\|\tilde{\alpha}_2(\theta)(j)\right\| \leq \|u_2\|_1. \tag{2.47}$$

From relations (2.42) and (2.47) we obtain that $\|\tilde{\alpha}(\theta)\|_\infty = \|\tilde{\alpha}_1(\theta)\|_\infty + \|\tilde{\alpha}_2(\theta)\|_\infty \leq \|u_1\|_1 + \|u_2\|_1 = \|u\|_1$, for all $\theta \in \Theta$. This shows that the pair $(\ell^\infty(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ is uniformly 1-admissible for the system (Q_A).

Step 3. We prove that the system (A) is not uniformly exponentially dichotomic.

Supposing that the system (A) is uniformly exponentially dichotomic, there exists a family of projections $\{P(\theta)\}_{\theta\in\Theta}$ and two constants $K, \nu > 0$ given by Definition 2.2. Then

$$\|\Phi(\theta, n)x\| \leq Ke^{-\nu n}\|x\|, \quad \forall x \in \text{Im } P(\theta),\ \forall(\theta, n) \in \Theta \times \mathbb{N}. \tag{2.48}$$

According to Remark 2.3 and Step 1 we have that Im $P(\theta) = \mathcal{S}_\infty(\theta) = W \times \{0\}$, for all $\theta \in \Theta$. Then (2.48) yields $f(\theta + n)/f(\theta) \leq \text{K}e^{-\nu n}$, for all $(\theta, n) \in \Theta \times \mathbb{N}$. In particular, for $\theta = 0$, from the above inequality we obtain that $1/(n + 1) \leq \text{K}e^{-\nu n}$, for all $n \in \mathbb{N}$, which is absurd. This shows that the system (A) is not uniformly exponentially dichotomic.

Definition 2.12. Let $q \in [1, \infty)$. The pair $(c_0(\mathbb{Z}, X), \Delta(\mathbb{Z}, X))$ is said to be *uniformly q-admissible* for the system (S_A) if the following assertions hold:

(i) for every $s \in \Delta(\mathbb{Z}, X)$ there is a unique $\gamma_s : \Theta \to c_0(\mathbb{Z}, X)$ solution of the system (S_A) corresponding to s;

(ii) there is $\lambda > 0$ such that $\|\gamma_s(\theta)\|_\infty \leq \lambda\ \|s\|_q$, for all $(\theta, s) \in \Theta \times \Delta(\mathbb{Z}, X)$.

For the proof of next theorem we refer to [7, Theorem 3.7].

Theorem 2.13. *Let $q \in (1, \infty)$. The following assertions hold:*

(i) *if the pair $(c_0(\mathbb{Z}, X), \Delta(\mathbb{Z}, X))$ is uniformly q-admissible for the system (S_A), then the system (A) is uniformly exponentially dichotomic;*

(ii) *if $\sup_{\theta\in\Theta}\|A(\theta)\| < \infty$, then the system (A) is uniformly exponentially dichotomic if and only if the pair $(c_0(\mathbb{Z}, X), \Delta(\mathbb{Z}, X))$ is uniformly q-admissible for the system (S_A).*

For every $\theta \in \Theta$ we consider the subspaces

$$\begin{gathered} \mathcal{S}_0(\theta) = \{x \in X : \Phi(\theta, \cdot)x \in c_0(\mathbb{N}, X)\}, \\ \mathcal{U}_0(\theta) = \{x \in X : \text{ there is } \varphi \in \mathcal{F}(\theta) \cap c_0(\mathbb{Z}_-, X) \text{ with } \varphi(0) = x\}. \end{gathered} \tag{2.49}$$

Definition 2.14. Let $q \in [1, \infty)$. The pair $(c_0(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ is said to be *uniformly q-admissible* for the system (Q_A) if there is $\lambda > 0$ such that the following assertions hold:

(i) for every $u \in \Delta_0(\mathbb{N}, X)$ there is $\alpha : \Theta \to c_0(\mathbb{N}, X)$ solution of the system (Q_A) corresponding to u;

(ii) if $u \in \Delta_0(\mathbb{N}, X)$ and $\alpha : \Theta \to c_0(\mathbb{N}, X)$ is a solution of (Q_A) corresponding to u with the property that $\alpha(\theta)(0) \in \mathcal{U}_0(\theta)$ for every $\theta \in \Theta$, then $\|\alpha(\theta)\|_\infty \leq \lambda\ \|u\|_q$ for all $\theta \in \Theta$.

Working with the subspaces $\mathcal{S}_0(\theta), \mathcal{U}_0(\theta)$ and using similar arguments with those in Lemma 2.7 and Theorem 2.8 we obtain the following.

Theorem 2.15. *Let $q \in [1, \infty)$. The following assertions are equivalent:*

(i) *the pair $(c_0(\mathbb{Z}, X), \Delta(\mathbb{Z}, X))$ is uniformly q-admissible for the system (S_A);*

(ii) *the pair $(c_0(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ is uniformly q-admissible for the system (Q_A) and $\mathcal{S}_0(\theta) + \mathcal{U}_0(\theta) = X$, for all $\theta \in \Theta$.*

As a consequence of Theorems 2.13 and 2.15 we deduce the following result.

Theorem 2.16. *Let $q \in (1, \infty)$. The following assertions hold:*

(i) *if the pair $(c_0(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ is uniformly q-admissible for the system (Q_A) and $\mathcal{S}_0(\theta) + \mathcal{U}_0(\theta) = X$, for all $\theta \in \Theta$, then the system (A) is uniformly exponentially dichotomic;*

(ii) *if $\sup_{\theta \in \Theta} \|A(\theta)\| < \infty$, then the system (A) is uniformly exponentially dichotomic if and only if the pair $(c_0(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ is uniformly q-admissible for the system (Q_A) and $\mathcal{S}_0(\theta) + \mathcal{U}_0(\theta) = X$, for all $\theta \in \Theta$.*

Remark 2.17. It is easy to see that the system presented in Example 2.11 has the property that $\mathcal{S}_0(\theta) + \mathcal{U}_0(\theta) = X$, for all $\theta \in \Theta$. Using similar arguments with those from Example 2.11, we deduce that the pair $(c_0(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ is uniformly 1-admissible for the system (Q_A). But, for all that, the system (A) is not uniformly exponentially dichotomic.

Remark 2.18. As an immediate consequence of Theorem 2.16 we obtain the main result of the paper [6]. The techniques involved in the proofs from [6] are different from those presented above.

3. Applications for Uniform Exponential Dichotomy of Linear Skew-Product Flows

In this section we apply the results obtained in the previous section in order to deduce characterizations for uniform exponential dichotomy of linear skew-product flows.

Let X be a real or complex Banach space, let (Θ, d) be a metric space and let $\mathcal{E} = X \times \Theta$. A mapping $\sigma : \Theta \times \mathbb{R} \to \Theta$ is called *a flow* on Θ if $\sigma(\theta, 0) = \theta$ and $\sigma(\theta, t+s) = \sigma(\sigma(\theta, t), s)$, for all $(\theta, t, s) \in \Theta \times \mathbb{R}^2$.

Definition 3.1. A pair $\pi = (\Phi, \sigma)$ is called a *linear skew-product flow* if σ is a flow on Θ and the mapping $\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{L}(X)$ satisfies the following conditions:

(i) $\Phi(\theta, 0) = I$, (the identity operator), for all $\theta \in \Theta$;

(ii) $\Phi(\theta, t+s) = \Phi(\sigma(\theta, t), s)\Phi(\theta, t)$, for all $(\theta, t, s) \in \Theta \times \mathbb{R}_+^2$;

(iii) there are $M \geq 1$ and $\omega > 0$ such that $||\Phi(\theta, t)|| \leq Me^{\omega t}$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$.

Definition 3.2. A linear skew-product flow $\pi = (\Phi, \sigma)$ is said to be *uniformly exponentially dichotomic* if there are a family of projections $\{P(\theta)\}_{\theta \in \Theta}$ and two constants $K, \nu > 0$ such that the following properties hold:

(i) $\Phi(\theta, t)P(\theta) = P(\sigma(\theta, t))\Phi(\theta, t)$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$;

(ii) $||\Phi(\theta, t)x|| \leq \mathrm{K}e^{-\nu t}||x||$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$ and $x \in \operatorname{Im} P(\theta)$;

(iii) $||\Phi(\theta, t)x|| \geq (1/K)e^{\nu t}||x||$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$ and $x \in \text{Ker } P(\theta)$;

(iv) for every $(\theta, t) \in \Theta \times \mathbb{R}_+$, the operator $\Phi(\theta, t)_{|} : \text{Ker } P(\theta) \to \text{Ker } P(\sigma(\theta, t))$ is invertible.

Remark 3.3. Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E} = X \times \Theta$. We associate with π the variational discrete-time system (A_π). The discrete cocycle associated with the system (A_π) is $\{\Phi(\theta, n)\}_{(\theta,n)\in\Theta\times\mathbb{N}}$.

If $\pi = (\Phi, \sigma)$ is a linear skew-product flow on $\mathcal{E} = X \times \Theta$, we consider the system

$$\alpha(\theta)(n+1) = \Phi(\sigma(\theta, n), 1)\alpha(\theta)(n) + u(n+1), \quad \forall(\theta, n) \in \Theta \times \mathbb{N}. \tag{Q_{A_π}}$$

As consequences of the main results from the previous section we deduce the following.

Theorem 3.4. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E} = X \times \Theta$ and let $p, q \in [1, \infty]$ be such that $(p, q) \neq (\infty, 1)$. The following assertions hold:*

(i) *if the pair $(\ell^p(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ is uniformly q-admissible for the system (Q_{A_π}) and $\mathcal{S}_p(\theta) + \mathcal{U}_p(\theta) = X$, for all $\theta \in \Theta$, then π is uniformly exponentially dichotomic;*

(ii) *if $p \geq q$, then π is uniformly exponentially dichotomic if and only if the pair $(\ell^p(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ is uniformly q-admissible for the system (Q_{A_π}) and $\mathcal{S}_p(\theta) + \mathcal{U}_p(\theta) = X$, for all $\theta \in \Theta$.*

Proof. This follows from Theorems 2.9 and 1.1. □

Theorem 3.5. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E} = X \times \Theta$ and let $q \in (1, \infty)$. Then π is uniformly exponentially dichotomic if and only if the pair $(c_0(\mathbb{N}, X), \Delta_0(\mathbb{N}, X))$ is uniformly q-admissible for the system (Q_{A_π}) and $\mathcal{S}_0(\theta) + \mathcal{U}_0(\theta) = X$, for all $\theta \in \Theta$.*

Proof. This follows from Theorems 2.16 and 1.1. □

4. Conclusions

In order to study the existence of exponential dichotomy of a variational difference equation (A), we associate to (A) the linear control system:

$$\gamma(\theta)(m+1) = A(\sigma(\theta, m))\gamma(\theta)(m) + s(m+1), \quad (\theta, m) \in \Theta \times J \tag{4.1}$$

which is denoted by (S_A) if $J = \mathbb{Z}$ and by (Q_A) if $J = \mathbb{N}$. The uniform exponential dichotomy of (A) may be expressed either in terms of the *unique* solvability of the system (S_A) or using the solvability of the system (Q_A) in certain hypotheses, provided (in both cases) that the norm of the solution of the control system satisfies a boundedness condition with respect to the norm of the input sequence. In the first case the smaller input space is $\Delta(\mathbb{Z}, X)$, while in the second case the smaller input space may be considered $\Delta_0(\mathbb{N}, X)$. When the associated control system is on the half-line, then the uniqueness of solution may be dropped. In this case, the space X should be the sum between the stable and the unstable space at every point $\theta \in \Theta$, but the boundedness condition may hold only for output sequences starting in the unstable space. Our study is explicitly done for the case when the output space is an $\ell^p(J,X)$-space as well as when this a $c_0(J,X)$-space, emphasizing the particular properties of each case. In base of Theorem 1.1 the main results are applicable not only to the general case of variational difference equations but also to the class of linear skew-product flows in infinite-dimensional spaces, without requiring measurability or continuity properties.

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