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Research Article

Triumph over Your Rivals in Dynamic Oligopoly

Weihong Huang

Division of Economics, School of Humanities and Social Science, Nanyang Technological University, Nanyang Avenue, Singapore 639798

Correspondence should be addressed to Weihong Huang, awhhuang@ntu.edu.sg

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Recommended by Akio Matsumoto

Challenging the conventional belief that sophistication in strategy is always better, it was found in W. Huang (2002a) that a price-taker who adopts the Cobweb strategy yields higher profits than those who adopt more sophisticated strategies. This study explores the possibility of improving further the relative profit advantage that the price-taker has over its counterparts through incorporating the growth-rate adjustment strategy. A linear heterogenous oligopoly model is used to illustrate the merits of such strategy in the case of disequilibrium. It is shown in theory and supported with numerical simulations that the adoption of growth-rate adjustment strategy together with price-taking strategy confers on the price-taker the stabilization power in a dynamically unstable market in addition to better relative performance in terms of major performance measures.

1. Introduction

One of the lessons learned from basic microeconomics is that for a profit maximizing, firm will always leverage on the market information as well as the behavioral rule of its rivals when making its output decision. In other words, a firm who ignores its market power and simply behaves a price-taker appears to be economically irrational. A typical example of this belief is demonstrated by the classic Cournot model in which each output decision of profit-maximizing firm is based on a best response function that is derived using the firm's forecast about its rivals' output level. However, such beliefs were first challenged in [1, 2] where an oligopoly that consists of a price-taker and many sophisticated firms, with identical technology, was studied. A counter-intuitive phenomenon is revealed—no matter what strategies the sophisticated firms may adopt, the price-taker always triumphs over them in terms of relative profitability at any intertemporal equilibrium. This result is found to hold regardless of the strategy the sophisticated firms adopt. It was further demonstrated in [3, 4] that either in dynamical transitionary periods or when the economy turns cyclic or chaotic, a combination of the price-taking strategy with a simple cautious adjustment strategy can also lead to relatively higher average profits for a firm than its rival, if the latter adopts a myopic Cournot best-response strategy.

The current research follows the same trend of research and explores the relative performance of price-taking strategy in terms of profit and sales revenue when a novel adjustment strategy—the growth-rate adjustment strategy—is implemented to prevent the economy collapse, or to force the market price to stay in an economically meaningful region. Using a traditional linear economy with linear demand and marginal cost, we are able to show in theory and demonstrate by numerical simulations that by combining the price-taking strategy with growth-rate adjustment strategy, a firm commands both an unbeatable advantage in terms of the relative performance (such as profit and sales revenue) as well as the stabilization power in a dynamically unstable market.

The article is organized as follows. A heterogeneous oligopoly model is set up in Section 2 and the relevant dynamical characteristics are summarized in Section 3. Section 4 examines the relative performance measures in equilibrium and their implications in general. The analysis for the relative performance in disequilibrium is presented in Section 5. Section 6 concludes the research.

2. A heterogeneous oligopoly model

Consider an oligopoly market, in which $N = 1 + m$ firms produce a homogeneous product with quantity q_t^i, $i = 1, 2, \ldots, 1 + m$, at period t. The inverse market demand for the product is given by $p_t = D(Q_t)$, where $D'(\cdot) \leq 0$, with equality holding only at finite number of points. The conventional assumption that $Q_t = \sum_{i=1}^{N} q_t^i$, that is, *the actual market price adjusts to the demand so as to clear the market at every period,* applies.

While all firms are assumed to have an identical technology and hence an identical convex cost function C, with $C' > 0$ and $C'' > 0$, they adopt different production strategies. We assume that all firms can be classified into two categories: *the price-takers* and *the Counort optimizers.*

The first firm is the price-taker, who is either deficient in market information or less strategic in market competition. Its production target, denoted by $\widehat{x}_t$ is derived by equating the marginal cost (i.e., the first-order derivative of cost) with market price if last period, that is, $C'(\widehat{x}_t) = p_{t-1}$. Notice that $p_{t-1} = D(Q_{t-1})$, the production target $\widehat{x}_t$ can be expressed as

$$\widehat{x}_t = MC^{-1}(D(Q_{t-1})), \tag{2.1}$$

where MC^{-1} denotes the inverse function of C'.

Instead of producing at $\widehat{x}_t$, the price-taker is assumed to implement the *growth-rate adjustment strategy* so as to limit the production growth rate to a positive constant γ in the sense that its output, denoted by x_t, is determined from

$$\frac{x_t - x_{t-1}}{x_{t-1}} = \gamma(\widehat{x}_t - x_{t-1}), \tag{2.2}$$

or, equivalently, from

$$x_t = x_{t-1}(1 + \gamma(MC^{-1}(D(Q_{t-1})) - x_{t-1})). \tag{2.3}$$

Remark 2.1. This mechanism was proposed in [2] as one of the adaptive adjustment mechanisms purely for the purpose of stabilizing unstable dynamics or controlling chaos.

Compared to other approaches, the mechanism inherits two unique advantages of feedback-adjustment type of methodologies: (i) requiring neither prior information about the system itself nor any externally generated control signal(s); (ii) forcing the adjusted system to converge to generic periodic points. Moreover, it is very effective and efficient in implementing and achieving the goal of stabilization.

In contrast, the rest m firms are *Counort optimizers*. They know exactly the price-taker's output. To take this information advantage and the market power to maximum extent, they form a collusion and produce an identical quantity y_t to maximize the individual profit given by

$$\pi^y(x_t, y_t) = s^y(x_t, y_t) - C(y_t) = D(Q_t)y_t - C(y_t), \tag{2.4}$$

where $s^y(x_t, y_t) = p_t y_t = D(Q_t)y_t$ is the sales revenue.

Then the first-order profit maximization condition $\partial\pi^y/\partial y_t = 0$ yields

$$D(Q_t) + \frac{\partial D(Q_t)}{\partial y_t} y_t - C'(y_t) = 0. \tag{2.5}$$

Since $Q_t = x_t + my_t$, we have $\partial D(Q_t)/\partial y_t = mD'(Q_t)$ and *so that* (2.5) is simplified to

$$D(Q_t) + my_t D'(Q_t) = C'(y_t). \tag{2.6}$$

Assume that appropriate second-order condition is satisfied so that $y_t = R_y(x_t)$ implicitly solved from (2.6) leads to a profit maximum.

For the convenience of reference, we will call the dynamical process given by (2.3) and (2.6) together as a *general heterogeneous oligopoly model* (GHO model). To avoid running into conclusions that are sensible in mathematics but meaningless in economics, we will focus on the scenarios that only bring economical meaningful solutions. To this end, an economically meaningful concept needs to be defined formally.

Definition 2.2. An output bundle (x_t, y_t) for the *general heterogeneous oligopoly model* is said to be *economically meaningful*, if the following inequalities are met:

(i) $x_t \geq 0$ and $y_t \geq 0$, but $q_t > 0$, that is, nonnegative market supply;

(ii) $0 < p_t = D(q_t) < \infty$, that is, positive and limited price.

3. Dynamical analysis of a linear model

To have a deeper understanding of the role of the price-taker in our model, a concrete example that can be manipulated analytically and verified is indispensable. For this purpose, we will illustrate our main points with a *linear heterogeneous oligopoly model* (LHO model), in which the market demand is assumed to be linear so that its inverse demand function is given by

$$p_t = D(x_t + my_t) = 1 - x_t - my_t, \tag{3.1}$$

whereas the marginal cost is linear so that the cost function adopts the form of

$$C(q) = C_0 + \frac{c}{2}q, \tag{3.2}$$

where both C_0 and c are positive constants. Without loss of generality, we assume that $C_0 = 0$.

Substituting (3.1) and (3.2) into (2.3) yields

$$x_t = x_{t-1}\left(1+\gamma\left(\frac{(1-x_{t-1}-my_{t-1})}{c}-x_{t-1}\right)\right). \tag{3.3}$$

On the other hand, (2.6) defines an optimal reaction function for the Cournot optimizers:

$$y_t = R_y(x_t) = \frac{1-x_t}{2m+c}. \tag{3.4}$$

Substituting (3.4) into (3.3) and manipulating them yield a quadratic recurrence relation

$$x_t = f(x_{t-1}) \doteq a(\gamma)x_{t-1}(1-b(\gamma)x_{t-1}), \tag{3.5}$$

where

$$\begin{aligned} a(\gamma) &\doteq 1+\frac{\gamma(c+m)}{c(c+2m)},\\ b(\gamma) &\doteq \gamma\frac{c+m+c(c+2m)}{(c+m)\gamma+c(c+2m)}. \end{aligned} \tag{3.6}$$

For the discrete process (3.5), the trajectory $\{x_t\}_{t=0}$ is bounded by $x_0^*(\gamma) = 1/b(\gamma)$ if and only if $0 \le x_0 \le x_0^*(\gamma)$ and $a(\gamma) \le 4$. However, to generate an economically meaningful trajectory $\{x_t\}_{t=0}$, we need also ensure that $x_0 < 1$, which is guaranteed if we have $x_0^*(\gamma) < 1$ and $a(\gamma) \le 4$, or equivalently, if

$$1 < \gamma \le \gamma_{\max}, \tag{3.7}$$

where

$$\gamma_{\max} = \frac{3c(c+2m)}{c+m}. \tag{3.8}$$

Figure 1 illustrates the impact of γ on the functional shape of f for the case of $c = 1$. We see that

$$\begin{aligned} \frac{\partial x_0^*(\gamma)}{\partial\gamma} &= -\frac{c}{\gamma^2}\frac{c+2m}{c+m+c(c+2m)} < 0,\\ \frac{\partial x_0^*(\gamma)}{\partial m} &= \frac{c^2}{\gamma}\frac{1-\gamma}{\left(c+m+2cm+c^2\right)^2} < 0 \quad \text{if } \gamma > 1. \end{aligned} \tag{3.9}$$

The dynamic characteristics of (3.5) have been studied substantially in literature and presented in many textbooks. So we will summarize some well-known results directly without repeating the details.

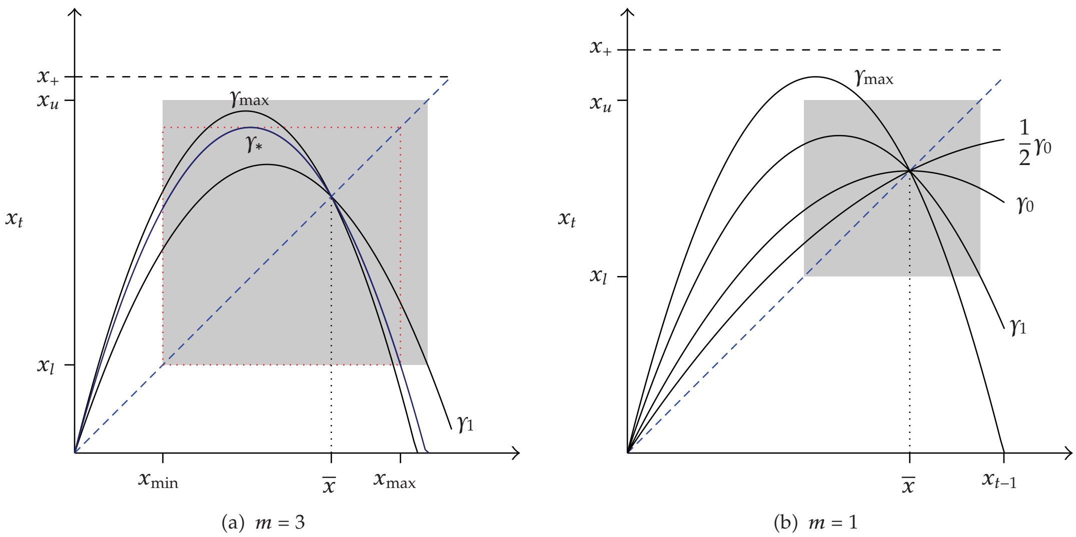


Figure 1: Illustration of $x_t = f(x_{t-1})$, relative profitability regime Ω^p and the trapping set $\Omega(\gamma)$.

Theorem 3.1. *For the discrete dynamic process* (3.5)*, when inequality* (3.7) *is satisfied, one has the following:*

(i) *there exists a unique nontrivial equilibrium* $\overline{x}$*:*

$$\overline{x} = \frac{m+c}{m+c+2mc+c^2}, \tag{3.10}$$

at which the derivative of f *is given by*

$$\sigma \doteq \left.\frac{df(x)}{dx}\right|_{\overline{x}} = 1 - \frac{\gamma(c+m)}{c(c+2m)}; \tag{3.11}$$

(ii) *the equilibrium* $\overline{x}$ *is stable for* $\gamma < \gamma_1$*, and the converge to the equilibrium is monotonic if* $\gamma < \gamma_0$ *and cyclical if* $\gamma > \gamma_0$*, where*

$$\gamma_0 = \frac{c(c+2m)}{(c+m)}, \tag{3.12}$$

$$\gamma_1 = 2\gamma_0; \tag{3.13}$$

(iii) *increasing* γ *from* γ_1 *to* γ_∞*, where*

$$\gamma_\infty \approx 2.5699\,\gamma_0 \tag{3.14}$$

will lead to a sequence of period-doubling bifurcations, that is, stable cycles with periods 2^n *appear right after the cycles with periods* 2^{n-1} *become unstable, for* $n \geq 1$*;*

(iv) *increasing* γ *from* γ_∞ *to* γ_{max}*, stable cycles for all periods can appear at some specific intervals for* γ *while aperiodic oscillations occur for the rest value of* γ*;*

(v) *for* $\gamma = \gamma_{\max}$, *no stable periodic-cycles exist for any order and the system is pure chaotic in the topological sense, that is, sensitive dependence on the initial condition, and the probabilistic sense, that is, the frequency from the random-like trajectories is however independent of the initial conditions and obeys an invariant density* φ *that is absolutely continuous with respect to the Lebesgue measure, where*

$$\varphi_f(x) = \frac{1}{\pi\sqrt{x(\overline{x}_{\max} - x)}} \quad \text{for } x \in J \doteq [0, \overline{x}_{\max}], \tag{3.15}$$

where

$$\overline{x}_{\max} = x_0^*(\gamma_{\max}) = \frac{4(c+m)}{3(c+m+2cm+c^2)}; \tag{3.16}$$

(vi) *for* $\gamma > \gamma_{\max}$, *the trajectory starting from almost all* $x_0 > 0$ *becomes unbounded and hence is not economically meaningful.*

Proof. At first, we point out that the map f is always topologically conjugated with the logistic map, via a linear homeomorphism. Condition (i) can be directly verified. Conditions (ii)–(iv) and (vi) follow from the standard textbook analysis.

(v) When $\gamma = \gamma_{\max}$, f is topologically conjugated with

$$X_t = F(x_{t-1}) \doteq 4X_{t-1}(1 - X_{t-1}) \tag{3.17}$$

in the sense that $f = h^{-1} \circ F \circ h$, where $h : J \to [0,1]$ is a homeomorphism defined by

$$h(x) = \frac{x}{\overline{x}_{\max}}, \quad \text{for } x \in J. \tag{3.18}$$

By the law of conjugation, if $\varphi_F(X)$ is the invariant density for F, then the invariant density for f is given by

$$\varphi_f(x) = \varphi_F(h(x))\left|\frac{dh}{dx}\right|. \tag{3.19}$$

It is well known that the invariant density preserved by (3.17) is given by

$$\varphi_F(X) = \frac{1}{\pi\sqrt{X(1-X)}} \quad \text{for } X \in [0,1]. \tag{3.20}$$

Substituting with h defined in (3.18) into (3.19) results in (3.15). □

Remark 3.2. Details about the chaos in probabilistic sense and relevant issues of statistic dynamics can be found in [5, 6]. Chaos in probabilistic sense occurs for infinitely many other values in the compact interval $[\gamma_\infty, \gamma_{\max}]$ as well. However, the invariant density in general cannot be expressed as the composition of finite terms of basic functions.

Remark 3.3. While the value of the nontrivial steady-state $\overline{x}$ is independent of the adjustment rate γ, the stability does depend on γ. For a fixed $\gamma > \gamma_0$, the stability of $\overline{x}$ is characterized by the derivative given by (3.11), from which we have

$$\begin{aligned} \frac{\partial\sigma}{\partial c} &= \gamma\frac{(c+m)^2 + m^2}{c^2(c+2m)^2} > 0, \\ \frac{\partial\sigma}{\partial m} &= \frac{\gamma}{(c+2m)^2} > 0. \end{aligned} \tag{3.21}$$

Thus, increasing number of Cournot optimizers stabilizes the economy, but increasing the technological level (less c) destabilizes the economy.

Figures 2(a) and 2(b) depict the bifurcation diagram with respect to γ and c, respectively. A structural similarity is apparently observed between them.

4. Relative performance in equilibrium

To see the merits of (3.3), we will proceed to evaluate the relative performance between the price-taker and the Cournot optimizer. The *profits* and *sales revenue* as two of most important performance measures used in the business and accounting literature will be considered. However, sales revenue is taken as auxiliary index evaluated from the price-taker's point of view instead of from the collusion's point of view since the latter is assumed to be a profit-maximizer as implied by the optimal reaction given by (2.6). Unlike being treated in [7], we do not evaluate these indices in the aggregated sense because the long-run averages of the relevant performance measures provide equivalent information both in cyclical and chaotic market environment.

4.1. Profits

At each period, the price is given by

$$p_t = 1 - x_t - my_t = \frac{m+c}{2m+c}(1-x_t), \tag{4.1}$$

that is, the realized price is in fact negatively related to the price-taker's output. However, compared to the case in which there is no counter-reaction from the Cournot optimizers, the sensitivity is reduced from unity to a fraction (since $(m+c)/(2m+c) < 1$).

The profits made by the price-taker and by any of the Cournot optimizers are

$$\begin{aligned} \pi_t^x &= \left(p_t - \frac{cx_t}{2}\right)x_t = \frac{m+c}{2m+c}\left(1 - \frac{2(m+c)+c(c+2m)}{2(m+c)}x_t\right)x_t, \\ \pi_t^y &= \left(p_t - \frac{cy_t}{2}\right)y_t = \left(m + \frac{c}{2}\right)y_t^2 = \frac{(1-x_t)^2}{2(2m+c)}, \end{aligned} \tag{4.2}$$

respectively.

We see that, $\pi_t^y \geq 0$ for all t, that is, the Cournot optimizers make no profit loss at each and every move due to its optimization strategy. (This result is actually independent of the specific demand function and the cost function. It can be shown that it is a typical characteristic of a firm that adopts Cournot-type strategy with full-information about its rival's outputs when the fixed cost is negligible.) On the contrary, the price-taker may suffer from profit loss due to its lagged reaction to the market changes. In fact, $\pi_t^x > 0$ if and only if $x_t < x_+$, where x_+ is defined by

$$x_+ = \frac{2(m+c)}{2(m+c)+c(2m+c)} < 1. \tag{4.3}$$

Simple mathematical manipulations indicate that

$$\Delta\pi_t^{xy} \doteq \pi_t^x - \pi_t^y = \frac{(c+2m+1)(1+c)}{2(2m+c)}(x^* - x_t)(x_t - x_*), \tag{4.4}$$

where $x_* \doteq 1/(1+c+2m)$ and $x^* \doteq 1/(1+c)$.

Therefore, $\pi_t^x \geq \pi_t^y$ if and only if $x_t \in \Omega^p \doteq [x_*, x^*]$. Ω^p so defined will be referred to as the *relative profitability regime* for the price-taker. It is worthwhile to note that the *relative profitability regime* is independent of the adjustment rate, γ, which enables a firm to adjust the trapping set (to be discussed in Section 5) of the trajectory so as to maintain the relative profitability advantage over the Cournot optimizers.

4.2. Sales revenue

In many business competition, sales revenue is a benchmark for market share. In our settings, they are given by

$$s_t^x = p_t x_t = \frac{m+c}{2m+c} x_t (1 - x_t),$$
$$s_t^y = p_t y_t = \frac{m+c}{(2m+c)^2} (1 - x_t)^2. \tag{4.5}$$

We see that s_t^x and s_t^y are always positive (due to the fact that $x_t < 1$ is implicitly guaranteed). Moreover, since

$$\Delta s_t^{xy} \doteq s_t^x - s_t^y = p_t (x_t - y_t), \tag{4.6}$$

we have $\Delta s_t^{xy} > 0$ so long as $x_t > y_t$, which occurs when

$$x_t > \frac{x_* = 1}{(1 + c + 2m)}, \tag{4.7}$$

that is, when the price-taker's output exceeds the lower bound of the relative profitability regime Ω^p.

4.3. Relative performance

At the unique nontrivial equilibrium $\overline{x}$ given by (3.10), the other relevant equilibrium values are summarized in the second column of Table 1. It can be verified that $\overline{x} \in \Omega^x$, regardless of the values of c, m, and γ. Therefore, *at the equilibrium, the price-taker enjoys a higher sale revenue as well as a higher profit than the Cournot optimizers*.

For benchmarks, two extreme situations are also included in Table 1, one with all $m+1$ firms adopting the price-taking strategy and the other with all $m+1$ firms forming a collusion.

In the former case where all firms behave as the price-taker, the equilibrium is known as a *Walrasian equilibrium* (competitive equilibrium). For the latter case, all firms behave as if they were a monopolist and maximize the total profit with a constant output

$$q_u = \frac{1}{2m + 2 + c}, \tag{4.8}$$

from which the other performance measure can be evaluated. (Again we do not consider the case in which the monopoly maximizes the sales revenue.)

While Figure 3 provides graphical illustrations of the outputs, the sales revenue, and the profits at the equilibrium with $m = 1$ and $m = 3$.

In fact, the conclusions for the equilibrium analysis are generic in the sense that they are independent of the concrete functional forms of the linear demand and the quadratic cost functions illustrated in our analysis.

Theorem 4.1. *Consider the GHO model given by*

$$x_t = x_{t-1}\left(1 + \gamma\left(MC^{-1}\left(D(Q_t)\right) - x_{t-1}\right)\right), \tag{4.9}$$

$$C'(y_t) = D(Q_t) + m y_t D'(Q_t), \tag{4.10}$$

where $Q_t \doteq x_t + m y_t$, *the cost function is strictly convex* ($C' > 0$ *and* $C'' > 0$) *and the demand function* ($D' < 0$) *satisfies the additional assumptions that both* $D(Q_t)$ *and* $D'(Q_t)$ *are bounded for* $Q_t \neq 0$, *the following facts hold.*

Table 1: Equilibrium outcomes.

	All price-taker	1 price-taker m Colluded	All Colluded
Price	$\overline{p}_w = \dfrac{c}{1+m+c}$	$\overline{p} = \dfrac{c(m+c)}{2cm+c^2+m+c}$	$\overline{p}_u = \dfrac{m+c+1}{2m+2+c}$
Outputs	$\overline{q}_w = \dfrac{1}{1+m+c}$	$\overline{x} = \dfrac{m+c}{2cm+c^2+m+c}$ $\overline{y} = \dfrac{c}{2cm+c^2+m+c}$	$\overline{q}_u = \dfrac{1}{2m+2+c}$
Profits	$\overline{\pi}_w = \dfrac{c}{2(1+m+c)^2}$	$\overline{\pi}^x = \dfrac{1}{2}\dfrac{c(m+c)^2}{(2cm+c^2+m+c)^2}$ $\overline{\pi}^y = \dfrac{1}{2}\dfrac{c^2(2m+c)}{(2cm+c^2+m+c)^2}$	$\overline{\pi}_u = \dfrac{1}{2(2+2m+c)}$
Sales	$\overline{s}_w = \dfrac{c}{(1+m+c)^2}$	$\overline{s}^x = \dfrac{c(m+c)^2}{(2cm+c^2+m+c)^2}$ $\overline{s}^y = \dfrac{c^2(m+c)}{(2cm+c^2+m+c)^2}$	$\overline{s}_u = \dfrac{m+c+1}{(2m+2+c)^2}$

(i) *All nontrivial intertemporal equilibria are economically meaningful.*

(ii) $\overline{x} > \overline{y} > 0$ *so that* $\overline{s}^x > \overline{s}^y > 0$.

(iii) $\overline{\pi}^x > \overline{\pi}^y$. *Moreover,* $\overline{\pi}^x > \overline{\pi}^y > 0$ *if* $C(0) = 0$.

Proof. (i)-(ii) At a nontrivial intertemporal equilibrium (steady-states $(\overline{x}, \overline{y})$), we mean $\overline{x} > 0$. It follows from (4.9) that we must have

$$\overline{p} = C'(\overline{x}). \tag{4.11}$$

The facts that $C'' > 0$ thus implies $\overline{p} > 0$. What remains to be verified is the positiveness of $\overline{y}$. Equation (4.10) yields

$$C'(\overline{x}) + mD'(\overline{Q})\overline{y} = C'(\overline{y}), \tag{4.12}$$

where $\overline{Q} = \overline{x} + m\overline{y}$. Since $D'(\overline{Q})$ is bounded even when $\overline{y} = 0$, a hypothesis that $\overline{y} = 0$ would lead to

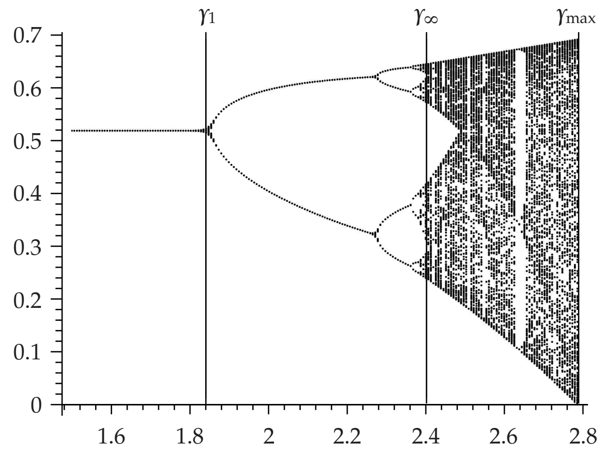
$$C'(\overline{x}) = C'(\overline{y}) = C'(0). \tag{4.13}$$

However, (4.13) in turn suggests that $\overline{x} = 0$ by the monotonicity characteristics implied from the convexity assumption of C, which is a contradiction to the fact that $\overline{x} > 0$.

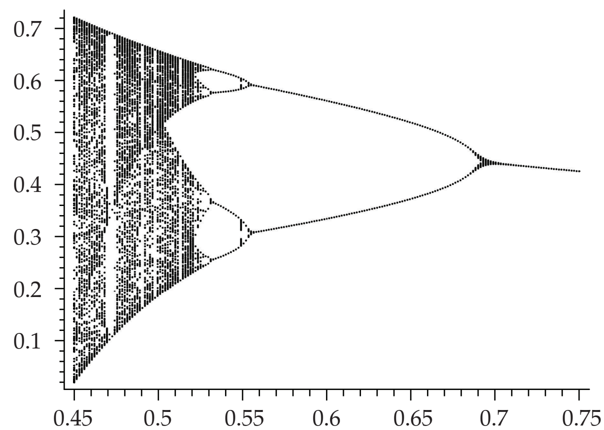
Again from (4.12), a positive $\overline{y}$ and $D' < 0$ together thus suggest $\overline{x} > \overline{y}$. Under the same equilibrium price $\overline{p}$, we thus have $\overline{s}^x > \overline{s}^y > 0$.

(iii) For the profit difference at the nontrivial equilibrium, (4.11) suggests that

$$\Delta\overline{\pi}^{xy} = \overline{\pi}^x - \overline{\pi}^y = C'(\overline{x})(\overline{x} - \overline{y}) - \left(C(\overline{x}) - C(\overline{y})\right). \tag{4.14}$$



(a) Bifurcation with respect to $\gamma(c = 1/2)$



(b) Bifurcation with respect to $c(\gamma = 2.5)$

Figure 2: Bifurcations of $x_t = f(x_{t-1})$, $m = 3$.

It follows from the assumption of $C''(\cdot) > 0$ and $\overline{x} \neq \overline{y}$ that $C'(\overline{x})(\overline{x}-\overline{y})-(C(\overline{x})-C(\overline{y})) > 0$, or equivalently,

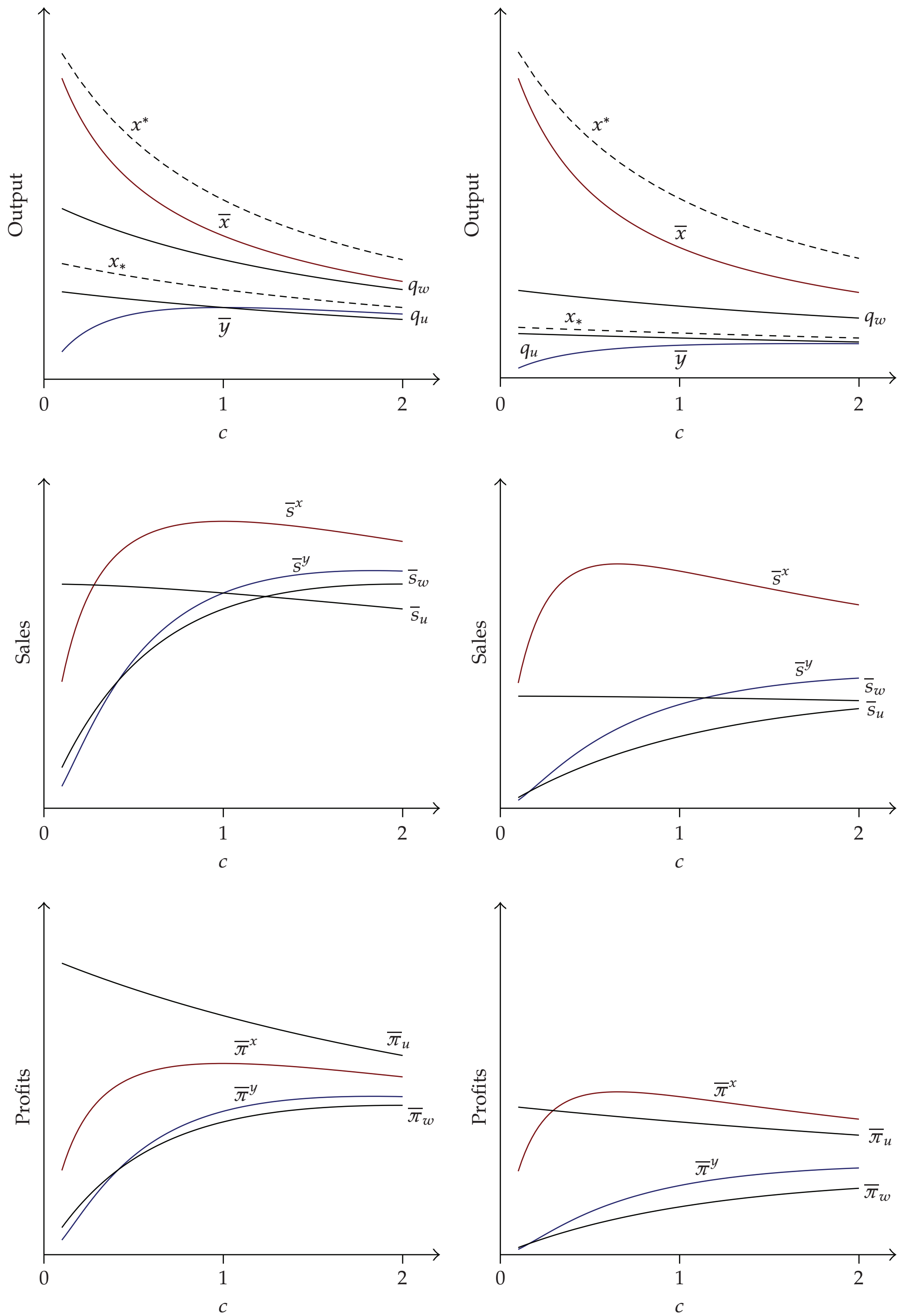
$$\overline{\pi}^x > \overline{\pi}^y. \tag{4.15}$$

Next we need to show that $\overline{\pi}^y > 0$ if the fixed cost is negligible. It follows from (4.12) that $\overline{p} = C'(\overline{y}) - mD'(\overline{Q})\overline{y}$ so that

$$\begin{aligned}\overline{\pi}^y &= \overline{p}\,\overline{y} - C(\overline{y}) = \left(C'(\overline{y}) - mD'(\overline{Q})\overline{y}\right)\overline{y} - C(\overline{y}) \\ &= C'(\overline{y})\overline{y} - C(\overline{y}) + \left(- mD'(\overline{Q})\overline{y}^2\right),\end{aligned} \tag{4.16}$$

which is positive since $C'(\overline{y})\overline{y} > C(\overline{y})$ if $C(0) = 0$. □

Remark 4.2. Due to the limit scope, theoretical issues such as the existence of nontrivial equilibrium and the selection mechanism when multiple nontrivial equilibria are presented



(a) $m = 1$ (b) $m = 3$

Figure 3: Critical outputs and performance measures.

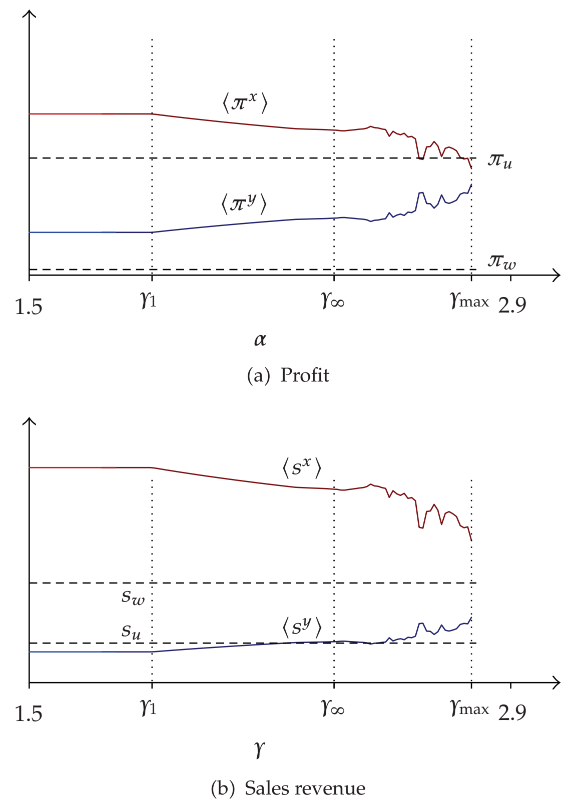


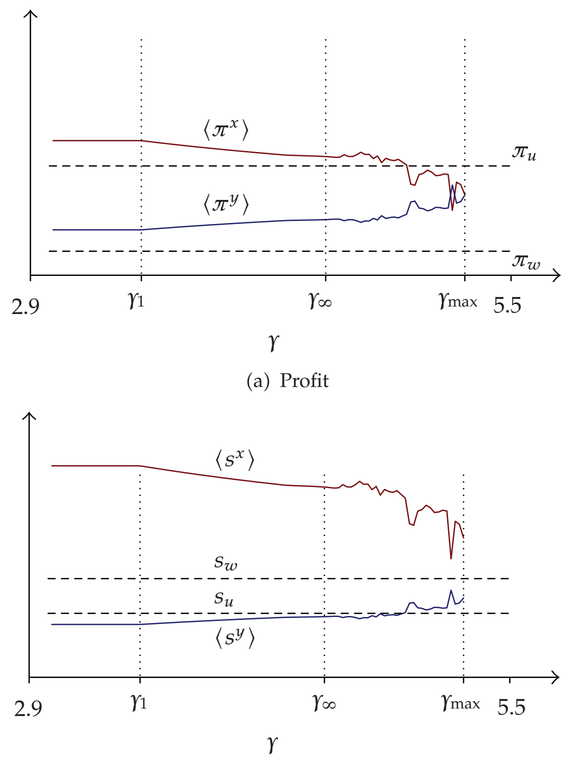
Figure 4: Long-run averages with respect to γ ($m = 3, c = 1/2$).

are not discussed since we are only concerned with with the situations that can lead to economically meaningful outcomes, based on which the relative performance of two types of firms can be evaluated. Nevertheless, it can be proved that a unique nontrivial equilibrium exists when a classical assumption about the marginal revenue as justified in [8, 9] is assumed.

Remark 4.3. In the proof of part (iii), we do not use the fact that the outputs of the Cournot optimizers follow (4.10), thus the conclusion for the relative profitability advantage of the price-taker holds for more general situations where each Cournot optimizer adopts different strategy (including partial and full collusion), provided that its equilibrium output, $\overline{y}^k$, $k = 1, 2, \ldots, m$, is different from $\overline{x}$. Moreover, the assumption for the strict convexity can be relaxed to convexity so long as the marginal cost is not a constant.

Remark 4.4. The relative profitability in an equilibrium for the price-taker, though seemingly contradictory to economic intuition, is still justified by economic theory. In fact, at an equilibrium, the market price approaches to a constant and does not respond to the variation of individual output. As a result, equating marginal cost to the market price is the optimal decision.

An equilibrium may exist in theory but may never be reached (converged to) in reality. When γ is sufficiently large ($\gamma > \gamma_\infty$ in LHO model), the asymptotic behavior becomes very



(a) Profit

(b) Sales revenue

Figure 5: Long-run averages with respect to γ ($m = 3, c = 1$).

complex, ranging from stable periodic cycles to aperiodic cyclic fluctuations, as well as the situations in which a stable cycle coexists with infinitively many unstable cycles. Therefore, to complete the analysis of relative performance, we need to investigate what happens when the trivial equilibrium becomes unstable, which leads us to the next section.

5. Relative performance in disequilibrium

Due to complex nature of the nonlinear dynamics for an unstable GHO model, we need to resort to a concrete example like LHO to provide a clearer picture.

As indicated in Section 3, when $\gamma > \gamma_1$, stable and unstable periodic cycles as well as aperiodic cycles may appear. In these cases, we are interested in comparing the long-run average performance of the price-taker and the Cournot optimizers. A formal definition for the average is needed.

Definition 5.1. Let $G \in C^1$ be a performance measure function of x_t. Given an adjustment speed γ and an associated periodic-k cycle $\{\overline{x}_i(\gamma)\}_{i=1}^{k}$, its *k-period average* is defined as

$$\langle G_k \rangle_\gamma \doteq \frac{1}{k \sum_{i=1}^{k} G(\overline{x}_i(\gamma))}, \tag{5.1}$$

and its *long-run average* for a trajectory $\{x_t\}_{t=0}^{\infty}$ is defined as

$$\langle G\rangle_\gamma \doteq \frac{\lim_{T\to\infty}1}{T\sum_{t=0}^{T}G(x_t)}. \tag{5.2}$$

For the discrete dynamic process (3.5), if $\{\overline{x}_i(\gamma)\}_{i=1}^{k}$ is a stable periodic cycle, then we have

$$\langle G\rangle_\gamma = \langle G_k\rangle_\gamma. \tag{5.3}$$

Apparently, both the performance measure functions: profits and sales revenue, are continuous at least up to the first-order differential, so are the respective differences between the price-taker and the Cournot optimizer.

Before proceeding to evaluate the periodic-k average and the long-run average of these measures in the relative terms, we have to make sure that they are economically meaningful.

Theorem 5.2. *For the LHO model given by* (3.5) *and* (3.4), *for* $\gamma \in (0, \gamma_{\max})$, *where* $\gamma_{\max}$ *is defined in* (3.8), *one always has*

$$\langle G_k^x\rangle_\gamma > 0, \quad \langle G_k^y\rangle_\gamma > 0 \quad \text{for any } k \geq 1, \tag{5.4}$$

and more generally,

$$\langle G^x\rangle_\gamma > 0, \qquad \langle G^y\rangle_\gamma > 0, \tag{5.5}$$

where G denotes π *and s.*

Proof. The conclusion for the sales revenue s follows straightforwardly from the fact that $x_t > 0$ for all t if $\gamma \in (0, \gamma_{\max})$ and $x_0 < x_0^*(\gamma)$, as discussed in Section 3.

For the profit π, we note first that (4.9) implies that

$$\frac{1}{\gamma}\left(\frac{x_{t+1}}{x_t} - 1\right) + x_t = MC^{-1}(p_t), \tag{5.6}$$

that is,

$$p_t = \frac{c}{\gamma}\left(\frac{x_{t+1}}{x_t - 1}\right) + cx_t, \tag{5.7}$$

which leads to

$$\pi^x(x_t) = p_t x_t - C(x_t) = \frac{c}{\gamma}(x_{t+1} - x_t) + \frac{c}{2}x_t^2. \tag{5.8}$$

For a given periodic-k cycle $\{\overline{x}_i\}_{i=1}^{k}$, we have

$$\begin{aligned}\langle \pi_k^x\rangle_\gamma &\doteq \frac{1}{k\sum_{i=1}^{k}\pi^x(\overline{x}_i)} = \frac{1}{k\sum_{i=1}^{k}\left((c/\gamma)(\overline{x}_{i+1} - \overline{x}_i) + (c/2)\overline{x}_i^2\right)} \\ &= \frac{c}{k\gamma}(\overline{x}_{k+1} - \overline{x}_1) + \frac{c}{2k}\sum_{i=1}^{k}\overline{x}_i^2 = \frac{c}{2k}\sum_{i=1}^{k}\overline{x}_i^2 > 0,\end{aligned} \tag{5.9}$$

where we utilize the fact that $\overline{x}_{k+1} = \overline{x}_1$.

Similarly, for any chaotical trajectory $\{x_t\}_{t=1}^{\infty}$, the long-run average of profit for the price-taker can be recast as

$$\langle \pi^x \rangle_\gamma \doteq \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \pi^x(x_t) = \frac{c}{\gamma} \lim_{T\to\infty} \frac{1}{T}(x_T - x_1) + \frac{c}{2} \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{T} x_i^2 = \frac{c}{2} \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{T} x_i^2 > 0, \quad (5.10)$$

which holds because the limit term $\lim_{T\to\infty}(1/T)(x_T - x_1)$ approaches zero since both x_T and x_1 are less than unity by assumption. □

Next, we turn to explore the relative performance between two types of firms in disequilibrium. Since the requirements for the relative performance in terms of sales-revenue are weaker than those in terms of profits, we will focus our discussion more on the latter.

From the analysis in Section 4, we see that there exists a relative profitability regime Ω^p that is independent of the adjustment rate γ such that the price-taker will make more profit than the Cournot optimizer whenever the former's output falls in that regime.

On the other hand, as illustrated in Figure 1, with fixed m and c, for $\gamma_0 < \gamma \le \gamma_{\max}$, where γ_0 and $\gamma_{\max}$ are, respectively, defined in (3.12) and (3.8), there exists a trapping set $\Omega(\gamma) \doteq [x_{\min}(\gamma), x_{\max}(\gamma)]$ such that starting from any initial output $x_0 \in (0, \overline{x}_0(\gamma))$, the output trajectories $\{x_t\}_{t=1}^{\infty}$ will be eventually trapped in $\Omega(\gamma)$ and remain there forever. $x_{\min}(\gamma)$ and $x_{\max}(\gamma)$ are determined from

$$\begin{aligned} x_{\max}(\gamma) &= f\left(\frac{x_0^*(\gamma)}{2}\right) = \frac{(\gamma(c+m) + c(c+2m))^2}{4c\gamma(c+2m)(c+m+c(c+2m))}, \\ x_{\min}(\gamma) &= f(x_{\max}(\gamma)) = \frac{(\gamma(c+m) + c(c+2m))^3(3c(c+2m) - \gamma(c+m))}{16c^3\gamma(c+2m)^3(c+m+c(c+2m))}. \end{aligned} \quad (5.11)$$

For $\gamma < \gamma_{\max}$, $x_{\min}(\gamma) > 0$. Moreover, $\partial x_{\max}(\gamma)/\partial\gamma > 0$ and $\partial x_{\min}(\gamma)/\partial\gamma < 0$ for $\gamma_0 \le \gamma \le \gamma_{\max}$.

From the fact that $x_{\min}(\gamma_0) = x_{\max}(\gamma_0) = \overline{x}$, we can infer that the trapping set $\Omega(\gamma)$ grows around $\overline{x}$ along with increasing value of γ. This in turn suggests that for a sufficiently small positive ϵ and $\gamma = \gamma_0 + \epsilon$, we are able to enforce $\Omega(\gamma) \subset \Omega^p$. That is to say, by limiting the magnitude of γ, the price-taker is able to secure that the trapping set falls within the relative profitability regime so as to achieve a relative advantage in terms of profit. Similar reasoning can be applied to the analysis of the sales revenue.

Since the condition for $s_t^x > s_t^y$ is just $x_t > x_*$, which can be guaranteed as long as the lower-bound of $\Omega(\gamma)$ coincides with the lower-bound of Ω^p, solving the identity

$$x_{\min}(\gamma_*^s) = x_* \quad (5.12)$$

yields an upper bound γ_*^s for the adjustment rate to ensure $s_t^x > s_t^y$ for all $x_t \in \Omega(\gamma)$.

Denote by γ^* the maximum possible γ such that the upper bound of the trapping set $\Omega(\gamma)$ coincides with the upper bound of the relative profitability regime Ω^p, then γ^* is a solution to the identity $x_{\max}(\gamma^*) = x^*$.

Finally, let $\gamma_*^p \doteq \min\{\gamma_*^s, \gamma^*\}$, then we have $\gamma_*^s \ge \gamma_*^p$ by definition and the following beautiful results.

Table 2: Some typical values.

	γ_*^p	γ_*^s
$c = 1/2$, $m = 1$	2.0723	2.0955
$c = 1/2$, $m = 3$	2.5848	2.5898
$c = 1$, $m = 1$	3.5777	3.5777
$c = 1$, $m = 3$	4.7387	4.7387

Theorem 5.3. *With the LHO model defined in* (3.4) *and* (3.5)*, for any given* m *and* c*,*

(i) *when* $\gamma \in [\gamma_0, \gamma_*^p]$*, one has* $\pi_t^x > \pi_t^y > 0$ *for all* $x_t \in \Omega(\gamma)$*, that is, the price-taker makes more profit than the individual Cournot optimizer at each move after the trajectory runs into the trapping set;*

(ii) *when* $\gamma \in [\gamma_0, \gamma_*^s]$*, one has* $s_t^x > s_t^y > 0$ *for all* $x_t \in \Omega(\gamma)$*, that is, the price-taker has higher sales revenue than the individual Cournot optimizer at each move after the trajectory runs into the trapping set.*

Table 2 provides a list of γ_*^s and γ_*^p values for several typical sets of parameters we have applied in our illustrations.

When γ exceeds the upper bounds γ_*^p and γ_*^s, respectively, $\Delta\pi_t^{xy}$ and ΔS_t^{xy} become negative from time to time. Then we are again forced to consider periodic-k average and the long-run averages of the performance measures.

In terms of profit, it follows from Theorem 5.3 that $\langle \Delta\pi^{xy} \rangle_\gamma > 0$ is guaranteed for all $x_t \in \Omega(\gamma)$, if $\gamma \in [\gamma_0, \gamma_*^p]$. By the continuity of π_t^x and π_t^y as functions of γ and the ergodicity of the dynamical process (3.5), we can hypothesize that there exists a $\overline{\gamma}_k^p$ such that so long as $\gamma_0 \le \gamma < \overline{\gamma}_k^p$, despite that $\Delta\pi_t^{xy} < 0$ for some t, we still have $\langle \Delta\pi_k^{xy} \rangle_\gamma > 0$ for any cyclic orbit. Apparently, the value of $\overline{\gamma}_k^p$ depends on the order k. For instance, for a period-2 cycle $(\overline{x}_1(\gamma), \overline{x}_2(\gamma))$, with $\overline{x}_1(\gamma) \le \overline{x}_2(\gamma)$, (4.4) suggests that

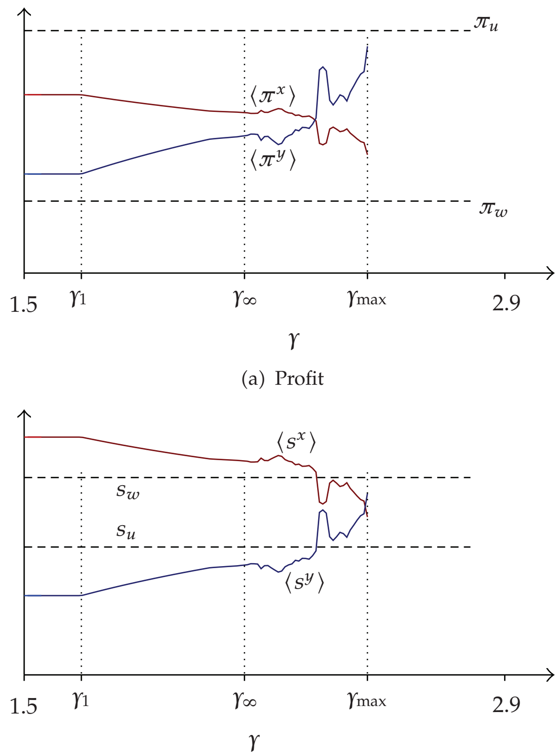
$$\begin{aligned} \left\langle \Delta\pi_2^{xy} \right\rangle_\gamma &= \frac{1}{2}\left(\Delta\pi^{xy}\left(\overline{x}_1(\gamma)\right) + \Delta\pi^{xy}\left(\overline{x}_2(\gamma)\right)\right) \\ &\bowtie \frac{1}{2}\sum_{i=1}^{2}\left(x^* - \overline{x}_i(\gamma)\right)\left(\overline{x}_i(\gamma) - x_*\right). \end{aligned} \tag{5.13}$$

The RHS of (5.13) is a continuous function of γ. In considering the fact that $\overline{x}_1(\gamma_0) = \overline{x}_2(\gamma_0) = \overline{x}$, where γ_0 is defined in (3.12), and the differential properties $\partial\overline{x}_1(\gamma)/\partial\gamma > 0$ and $\partial\overline{x}_2(\gamma)/\partial\gamma < 0$ for $\gamma_1 \le \gamma \le \gamma_{\max}$, we are able to conclude the existence of a $\overline{\gamma}_2^p, \overline{\gamma}_2^p > \gamma_0$, such that the equality $\langle \Delta\pi^{xy}(\gamma) \rangle_2 = 0$ holds.

Similar conclusion can be expected for an aperiodic orbit which occurs when $\gamma > \gamma_\infty$. There exists a $\overline{\gamma}^p$ value such that for all $\gamma < \overline{\gamma}^p$, we have $\langle \Delta\pi^{xy} \rangle_\gamma > 0$. There is no analytical formula to obtain the value of $\overline{\gamma}^p$.

Figures 4 to 6 depict the numerical simulations of relevant performance measures for several combinations of m and c values.

Comparing Figure 4(a) with Figure 5(a), we see that $\overline{\gamma}^p$ decreases with the increase in c. For a relatively small c (a more divergent economy caused by higher marginal cost), $\overline{\gamma}^p$ can even exceed its maximum possible growth-rate limit $\gamma_{\max}$, as illustrated in Figure 4. On the other hand, by comparing Figure 4(a) with Figure 6(a), we see that increasing m



(b) Sales revenue

Figure 6: Long-run averages with respect to γ ($m = 1, c = 1/2$).

(more Cournot optimizers) stabilizes the economy so that $\overline{\gamma}^p$ increases along with the increase of the maximum possible growth-rate limit $\gamma_{\max}$.

Similarly, there exist cases in which $\langle s^x \rangle_\gamma > \langle s^y \rangle_\gamma$ for all $\gamma < \gamma_{\max}$, as illustrated in Figures 4(b) and 5(b).

The above analysis and observations can be summarized into the following theorem.

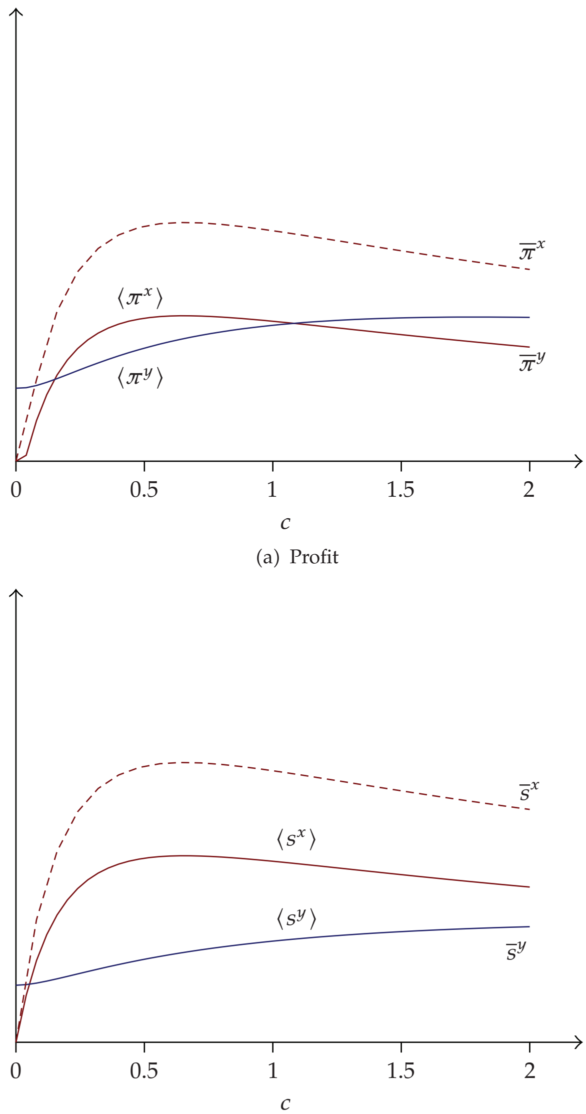
Theorem 5.4. *With the LHO model defined in* (3.4) *and* (3.5), *for any given m and c,*

(i) *there exists $\overline{\gamma}^p$, $\overline{\gamma}^p > \gamma_*^p$, such that for all $\gamma \in [\gamma_0, \overline{\gamma}^p]$, one has $\langle \Delta\pi^{xy} \rangle_\gamma > 0$, that is, the long-run average profits made by the price-taker is higher than that earned by each individual Cournot optimizer;*

(ii) *there exists $\overline{\gamma}^s$, $\overline{\gamma}^s \geq \gamma_*^s$, such that for all $\gamma \in [\gamma_0, \overline{\gamma}^s]$, one has $\langle \Delta s^{xy} \rangle_\gamma > 0$, that is, the long-run average sales-revenue earned by the price-taker is higher than that earned by each individual Cournot optimizer.*

When $\gamma = \gamma_{\max}$, process (3.5) becomes full-chaotic and preserves an ergodic invariant density φ_f given by (3.15) so that for any function $G \in C^1$, its space mean equals to the time mean in the sense of

$$\langle G \rangle_\gamma = \int_0^{\overline{x}_{\max}} G(x)\varphi_f(x)dx. \tag{5.14}$$



(a) Profit

(b) Sales revenue

Figure 7: Performance measures under full-chaos ($\gamma = \gamma_{\max}, m = 3$).

For instance, substituting $\Delta\pi_t^{xy}$ defined in (4.4) into (5.14) leads to

$$\begin{aligned} \langle \Delta\pi^{xy} \rangle_{\gamma_{\max}} &= \int_0^{\overline{x}_{\max}} \Delta\pi^{xy}(x)\varphi_f(x)dx \\ &\bowtie \int_0^{\overline{x}_{\max}} (x^* - x)(x - x_*)\varphi_f(x)dx \\ &\bowtie \int_0^{\overline{x}_{\max}} \frac{(x^* - x)(x - x_*)}{\sqrt{x(\overline{x}_{\max} - x)}}dx. \end{aligned} \tag{5.15}$$

As x^*, x_*, and $\overline{x}_{\max}$ all depend on m and c, so does the integral in (5.15). Analogously, all long-run averages of performance measures under full chaos can be numerically evaluated. Figure 7 provides such results with respect to c for $m = 3$, from which the impact of c is illustrated. Again, as expected, a range for γ can be identified, in which we have $\langle \Delta\pi^{xy} \rangle_{\gamma_{\max}} > 0$.

Although our analysis is conducted using the LHO model, the methodology applied and conclusions drawn in Theorem 5.4 are expected to hold for the GHO model in general. Conclusive results cannot be obtained without adequate analytical assumptions on the derivative properties of cost function and demand in general, and hence will be presented in the future report.

6. Conclusions

A heterogeneous oligopoly model consisting of a price-taking firm and a group of Cournot optimizers have been studied from both equilibrium and disequilibrium points of view. If the Cournot optimizers form a collusion and maximize their profit with the full information about the price-taker's output, a price-taker can triumph over her rivals in terms of major performance measures utilized in economics and business by adopting a combination of price-taking strategy and growth-rate adjustment strategy. Moreover, it is also the price-taker who is in control of the complexity of the oligopolistic dynamics. Numerical simulations successfully verified all theoretical conclusions.

In our analysis, the Cournot optimizers are assumed to command the full knowledge of the price-taker's current production. Such assumption reduces the oligopolistic dynamical system into a simple one-dimensional discrete process and hence greatly simplifies the analysis. If the Cournot optimizers know only the price-taker's outputs in previous periods, then a multidimensional discrete process is formed. Global analysis as implemented in [10] should be carried out.

Apparently, the current studies can be generalized in many ways. For instance, it is interesting to see the relative profitability of the different agents in an oligopolist economy with product differentiation. Analogous analysis can also be conducted for heterogeneous oligopsonist model. However, the analysis turns to be much complicated when multiple nontrivial equilibria may appear, under which a suitable "selection mechanism" as discussed in [11] needs to be considered.

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