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Research Article

Nonlocal Boundary Value Problems for Elliptic-Parabolic Differential and Difference Equations

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The abstract nonlocal boundary value problem $-d^2u(t)/dt^2 + Au(t) = g(t), 0 < t < 1, du(t)/dt - Au(t) = f(t), 1 < t < 0, u(1) = u(-1) + \mu$ for differential equations in a Hilbert space H with the self-adjoint positive definite operator A is considered. The well-posedness of this problem in Hölder spaces with a weight is established. The coercivity inequalities for the solution of boundary value problems for elliptic-parabolic equations are obtained. The first order of accuracy difference scheme for the approximate solution of this nonlocal boundary value problem is presented. The well-posedness of this difference scheme in Hölder spaces is established. In applications, coercivity inequalities for the solution of a difference scheme for elliptic-parabolic equations are obtained.

1. Introduction

It is known that various problems in fluid mechanics and other areas of engineering, physics, and biological systems lead to partial differential equations of variable types. Methods of solutions of nonlocal boundary value problems for partial differential equations of variable type have been studied extensively by many researchers (see, e.g., [1–4] and the references given therein).

The nonlocal boundary value problem

$$\begin{aligned} -\frac{d^2u(t)}{dt^2} + Au(t) &= g(t), \quad 0 < t < 1, \\ \frac{du(t)}{dt} - Au(t) &= f(t), \quad -1 < t < 0, \\ u(1) &= u(-1) + \mu \end{aligned} \tag{1.1}$$

for differential equations in a Hilbert space H with the self-adjoint positive definite operator A is considered.

Let us denote by $C^{\alpha}_{0,1}([-1,1],H),\ 0<\alpha<1$ the Banach space obtained by completion of the set of all smooth H-valued function $\varphi(t)$ on $[-1,1]$ in the norm

$$\begin{aligned}\|\varphi\|_{C^{\alpha}_{0,1}([-1,1],H)} = \|\varphi\|_{C([-1,1],H)} &+ \sup_{-1<t<t+\tau<0}\frac{(-t)^{\alpha}\|\varphi(t+\tau)-\varphi(t)\|_H}{\tau^{\alpha}}\\ &+ \sup_{0<t<t+\tau<1}\frac{(1-t)^{\alpha}(t+\tau)^{\alpha}\|\varphi(t+\tau)-\varphi(t)\|_H}{\tau^{\alpha}},\end{aligned}\tag{1.2}$$

and denote by $C^{\alpha}_{0,1}([0,1],H),\ 0<\alpha<1$ the Banach space obtained by completion of the set of all smooth H-valued function $\varphi(t)$ on $[0,1]$ in the norm

$$\|\varphi\|_{C^{\alpha}_{0,1}([0,1],H)} = \|\varphi\|_{C([0,1],H)} + \sup_{0<t<t+\tau<1}\frac{(1-t)^{\alpha}(t+\tau)^{\alpha}\|\varphi(t+\tau)-\varphi(t)\|_H}{\tau^{\alpha}},\tag{1.3}$$

finally denote by $C^{\alpha}_{0}([-1,0],H),\ 0<\alpha<1$ the Banach space obtained by completion of the set of all smooth H-valued function $\varphi(t)$ on $[-1,0]$ in the norm

$$\|\varphi\|_{C^{\alpha}_{0}([-1,0],H)} = \|\varphi\|_{C([-1,0],H)} + \sup_{-1<t<t+\tau<0}\frac{(-t)^{\alpha}\|\varphi(t+\tau)-\varphi(t)\|_H}{\tau^{\alpha}}.\tag{1.4}$$

Here $C([a,b],H)$ stands for the Banach space of all continuous functions $\varphi(t)$ defined on $[a,b]$ with values in H equipped with the norm

$$\|\varphi\|_{C([a,b],H)} = \max_{a\le t\le b}\|\varphi(t)\|_H.\tag{1.5}$$

A function $u(t)$ is called a solution of problem (1.1) if the following conditions are satisfied.

(i) $u(t)$ is twice continuously differentiable on the segment $(0,1]$ and continuously differentiable on the segment $[-1,1]$; the derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives.

(ii) The element $u(t)$ belongs to the domain $D(A)$ of A for all $t\in[-1,1]$, and the function $Au(t)$ is continuous on the segment $[-1,1]$.

(iii) $u(t)$ satisfies the equations and the nonlocal boundary condition (1.1).

A solution of problem (1.1) defined in this manner will henceforth be referred to as a solution of problem (1.1) in the space $C(H)=C([-1,1],H)$.

We say that problem (1.1) is well-posed in $C(H)$, if there exists a unique solution $u(t)$ in $C(H)$ of problem (1.1) for any $g(t)\in C([0,1],H),\ f(t)\in C([-1,0],H)$, and $\mu\in D(A)$, and the following coercivity inequality is satisfied:

$$\|u''\|_{C([0,1],H)} + \|u'\|_{C([-1,0],H)} + \|Au\|_{C(H)} \le M\left[\|g\|_{C([0,1],H)} + \|f\|_{C([-1,0],H)} + \|A\mu\|_H\right],\tag{1.6}$$

where M is independent of μ, $f(t)$, and $g(t)$.

Problem (1.1) is not well-posed in $C(H)$ [5]. The well-posedness of the boundary value problem (1.1) can be established if one considers this problem in certain spaces $F(H)$ of smooth H-valued functions on $[-1,1]$.

A function $u(t)$ is said to be a solution of problem (1.1) in $F(H)$ if it is a solution of this problem in $C(H)$ and the functions $u''(t)$ $(t \in 0,1])$, $u'(t)$ $(t \in -1,1])$ and $Au(t)$ $(t \in -1,1])$ belong to $F(H)$.

As in the case of the space $C(H)$, we say that problem (1.1) is well-posed in $F(H)$, if the following coercivity inequality is satisfied:

$$\|u''\|_{F([0,1],H)} + \|u'\|_{F([-1,0],H)} + \|Au\|_{F(H)} \leq M\Big[\|g\|_{F([0,1],H)} + \|f\|_{F([-1,0],H)} + \|A\mu\|_H\Big], \tag{1.7}$$

where M is independent of μ, $f(t)$, and $g(t)$.

If we set $F(H)$ equal to $C^{\alpha}_{0,1}(H) = C^{\alpha}_{0,1}([-1,1],H)$ $(0 < \alpha < 1)$, then we can establish the following coercivity inequality.

Theorem 1.1. *Suppose $\mu \in D(A)$. Then the boundary value problem* (1.1) *is well-posed in a Hölder space $C^{\alpha}_{0,1}(H)$ and the following coercivity inequality holds:*

$$\begin{aligned} &\|u''\|_{C^{\alpha}_{0,1}([0,1],H)} + \|u'\|_{C^{\alpha}_{0}([-1,0],H)} + \|Au\|_{C^{\alpha}_{0,1}(H)} \\ &\qquad \leq M\left[\frac{1}{\alpha(1-\alpha)}\left[\|f\|_{C^{\alpha}_{0}([-1,0],H)} + \|g\|_{C^{\alpha}_{0,1}([0,1],H)}\right] + \|A\mu\|_H\right]. \end{aligned} \tag{1.8}$$

Here M is independent of $f(t)$, $g(t)$, and μ.

The proof of this assertion follows from the scheme of the proof of the theorem on well-posedness of paper [5] and is based on the following formulas:

$$\begin{aligned} u(t) &= \left(I - e^{-2A^{1/2}}\right)^{-1}\Big[\left(e^{-tA^{1/2}} - e^{-(-t+2)A^{1/2}}\right)u_0 \\ &\qquad + \left(e^{-(1-t)A^{1/2}} - e^{-(t+1)A^{1/2}}\right)u_1\Big] + \left(I - e^{-2A^{1/2}}\right)^{-1} \\ &\quad \times \left(e^{-(1-t)A^{1/2}} - e^{-(t+1)A^{1/2}}\right)\int_0^1 A^{-1/2}2^{-1}\left(e^{-(1-s)A^{1/2}} - e^{-(s+1)A^{1/2}}\right)g(s)ds \\ &\quad - \int_0^1 A^{-1/2}2^{-1}\left(e^{-(t+s)A^{1/2}} - e^{-|t-s|A^{1/2}}\right)g(s)ds, \quad 0 \leq t \leq 1, \\ u(t) &= e^{tA}u_0 + \int_0^t e^{(t-s)A}f(s)ds, \quad -1 \leq t \leq 0, \\ u_0 &= \left(I + e^{-2A^{1/2}} + A^{1/2}\left(I - e^{-2A^{1/2}}\right) - 2e^{-(A^{1/2}+A)}\right)^{-1} \\ &\quad \times \left[e^{-A^{1/2}}\left[2\int_0^{-1} e^{-(1+s)A}f(s)ds + \int_0^1 A^{-1/2}\left(e^{-(1-s)A^{1/2}} - e^{-(s+1)A^{1/2}}\right)g(s)ds\right] + 2e^{-A^{1/2}}\mu\right] \\ &\quad + \left(I - e^{-2A^{1/2}}\right)\left(I + e^{-2A^{1/2}} + A^{1/2}\left(I - e^{-2A^{1/2}}\right) - 2e^{-(A^{1/2}+A)}\right)^{-1} \\ &\quad \times \left[-A^{-1/2}f(0) + \int_0^1 A^{-1/2}e^{-sA^{1/2}}g(s)ds\right] \end{aligned} \tag{1.9}$$

for the solution of problem (1.1) and on the estimates

$$\begin{aligned}
&\left\| \left(I - e^{-2A^{1/2}}\right)^{-1} \right\|_{H\to H} \le M, \\
&\left\| \left(I + e^{-2A^{1/2}} + A^{1/2}\left(I - e^{-2A^{1/2}}\right) - 2e^{-(A^{1/2}+A)}\right)^{-1} \right\|_{H\to H} \le M, \\
&\left\| A^{1/2}\left(I + e^{-2A^{1/2}} + A^{1/2}\left(I - e^{-2A^{1/2}}\right) - 2e^{-(A^{1/2}+A)}\right)^{-1} \right\|_{H\to H} \le M, \\
&\left\| (A^{1/2})^{\alpha} e^{-tA^{1/2}} \right\|_{H\to H} \le t^{-\alpha}, \qquad t > 0,\ 0 \le \alpha \le 1, \\
&\| A^{\alpha} e^{-tA} \|_{H\to H} \le t^{-\alpha}, \qquad t > 0,\ 0 \le \alpha \le 1.
\end{aligned} \tag{1.10}$$

Remark 1.2. The nonlocal boundary value problem for the elliptic-parabolic equation

$$\begin{aligned}
&\frac{du(t)}{dt} + Au(t) = f(t), \quad 0 < t < 1, \\
&-\frac{d^2u(t)}{dt^2} + Au(t) = g(t), \quad -1 < t < 0, \\
&u(1) = u(-1) + \mu
\end{aligned} \tag{1.11}$$

in a Hilbert space H with a self-adjoint positive definite operator A is considered in paper [6]. The well-posedness of this problem in Hölder spaces $C^{\alpha}(H)$ without a weight was established under the strong condition on μ.

Now, the applications of this abstract results are presented.

First, the mixed boundary value problem for the elliptic-parabolic equations

$$\begin{aligned}
&ga - u_{tt} - (a(x)u_x)_x + \delta u = g(t,x), \quad 0 < t < 1,\ 0 < x < 1, \\
&u_t + (a(x)u_x)_x - \delta u = f(t,x), \quad -1 < t < 0,\ 0 < x < 1, \\
&u(t,0) = u(t,1), \quad u_x(t,0) = u_x(t,1), \quad -1 \le t \le 1, \\
&u(1,x) = u(-1,x) + \mu(x), \quad 0 \le x \le 1, \\
&u(0+,x) = u(0-,x), u_t(0+,x) = u_t(0-,x), \quad 0 \le x \le 1
\end{aligned} \tag{1.12}$$

is considered. Problem (1.12) has a unique smooth solution $u(t,x)$ for $a(x) \ge a > 0$ $(x \in (0,1))$, and $g(t,x)$ $(t \in 0,1], x \in [0,1])$, $f(t,x)$ $(t \in [-1,0], x \in 0,1])$ the smooth functions and $\delta = \text{const} > 0$. This allows us to reduce the mixed problem (1.12) to the nonlocal boundary value problem (1.1) in the Hilbert space $H = L_2[0,1]$ with a self-adjoint positive definite operator A defined by (1.12).

Theorem 1.3. *The solutions of the nonlocal boundary value problem* (1.12) *satisfy the coercivity inequality*

$$\begin{aligned}
&\|u_{tt}\|_{C^{\alpha}_{0,1}([0,1],L_2[0,1])} + \|u_t\|_{C^{\alpha}_{0}([-1,0],L_2[0,1])} + \|u\|_{C^{\alpha}_{0,1}([-1,1],W^2_2[0,1])} \\
&\quad \le M\left[\frac{1}{\alpha(1-\alpha)}\left[\|g\|_{C^{\alpha}_{0,1}([0,1],L_2[0,1])} + \|f\|_{C^{\alpha}_{0}([-1,0],L_2[0,1])}\right] + \|\mu\|_{W^2_2[0,1]}\right],
\end{aligned} \tag{1.13}$$

where M is independent of $f(t,x)$, $g(t,x)$, and $\mu(x)$.

The proof of Theorem 1.3 is based on the abstract Theorem 1.1 and the symmetry properties of the space operator are generated by problem (1.12).

Second, let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \le k \le n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[-1,1] \times \Omega$, the boundary value problem for the multidimensional elliptic-parabolic equation

$$\begin{gathered}
-u_{tt} - \sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} = g(t,x), \quad 0 < t < 1, \; x \in \Omega, \\
u_t + \sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} = f(t,x), \quad -1 < t < 0, \; x \in \Omega, \\
u(t,x) = 0, \quad x \in S, \quad -1 \le t \le 1; \qquad u(1,x) = u(-1,x) + \mu(x), \quad x \in \overline{\Omega}, \\
u(0+,x) = u(0-,x), \quad u_t(0+,x) = u_t(0-,x), \quad x \in \overline{\Omega}
\end{gathered} \tag{1.14}$$

is considered. Problem (1.14) has a unique smooth solution $u(t,x)$ for $a_r(x) \ge a > 0$ $(x \in \Omega)$ and $g(t,x)$ $(t \in (0,1),\, x \in \overline{\Omega})$, $f(t,x)$ $(t \in (-1,0),\, x \in \overline{\Omega})$, the smooth functions. This allows us to reduce the mixed problem (1.14) to the nonlocal boundary value problem (1.1) in the Hilbert space $H = L_2(\overline{\Omega})$ of all the integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{L_2(\overline{\Omega})} = \left\{ \int \cdots \int_{x \in \overline{\Omega}} |f(x)|^2 dx_1 \cdots dx_n \right\}^{1/2} \tag{1.15}$$

with a self-adjoint positive definite operator A defined by (1.14).

Theorem 1.4. *The solution of the nonlocal boundary value problem* (1.14) *satisfies the coercivity inequality*

$$\begin{aligned}
&\|u_{tt}\|_{C^{\alpha}_{0,1}([0,1],L_2(\overline{\Omega}))} + \|u_t\|_{C^{\alpha}_{0}([-1,0],L_2(\overline{\Omega})])} + \|u\|_{C^{\alpha}_{0,1}([-1,1],W^2_2(\overline{\Omega}))} \\
&\quad \le M\left[\frac{1}{\alpha(1-\alpha)}\left[\|g\|_{C^{\alpha}_{0,1}([0,1],L_2(\overline{\Omega}))} + \|f\|_{C^{\alpha}_{0}([-1,0],L_2(\overline{\Omega}))}\right] + \|\mu\|_{W^2_2(\overline{\Omega})}\right],
\end{aligned} \tag{1.16}$$

where M is independent of $f(t,x)$, $g(t,x)$, and $\mu(x)$.

The proof of Theorem 1.4 is based on the abstract Theorem 1.1 and the symmetry properties of the space operator generated by problem (1.14) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 1.5. *For the solution of the elliptic differential problem*

$$\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r} = \omega(x), \quad x \in \Omega, \tag{1.17}$$

$$u(x) = 0, \quad x \in S, \tag{1.18}$$

the following coercivity inequality holds [7]:

$$\sum_{r=1}^{n}\|u_{x_r}x_r\|_{L_2(\overline{\Omega})} \le M\|\omega\|_{L_2(\overline{\Omega})}. \tag{1.19}$$

2. The first order of accuracy difference scheme

Let us associate the boundary-value problem (1.1) with the corresponding first order of accuracy difference scheme

$$\begin{gathered}
-\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k = g_k, \\
g_k = g(t_k), \quad t_k = k\tau, \ 1 \le k \le N-1, \\
\tau^{-1}(u_k - u_{k-1}) - Au_{k-1} = f_k, \quad f_k = f(t_{k-1}), \\
t_{k-1} = (k-1)\tau, \quad -N+1 \le k \le 0, \\
u_N = u_{-N} + \mu, \qquad u_1 - u_0 = u_0 - u_{-1}.
\end{gathered} \tag{2.1}$$

A study of discretization, over time only, of the nonlocal boundary value problem also permits one to include general difference schemes in applications if the differential operator in space variables, A, is replaced by the difference operators A_h that act in the Hilbert spaces H_h and are uniformly self-adjoint positive definite in h for $0 < h \le h_0$.

Let $P = P(\tau A) = (I + \tau A)^{-1}$. Then the following estimates are satisfied [8]:

$$\|P^k\|_{H\to H} \le M(1+\delta\tau)^{-k}, \quad k\tau\|AP^k\|_{H\to H} \le M, \quad k \ge 1, \ \delta > 0, \tag{2.2}$$

$$\|A^{\beta}(P^{k+r} - P^k)\|_{H\to H} \le M\frac{(r\tau)^{\alpha}}{(k\tau)^{\alpha+\beta}}, \quad 1 \le k < k+r \le N, \ 0 \le \alpha, \ \beta \le 1. \tag{2.3}$$

Furthermore, for a self-adjoint positive definite operator A it follows that the operator $R = (I + \tau B)^{-1}$ is defined on the whole space H, it is a bounded operator, and the following estimates hold:

$$\|R^k\|_{H\to H} \le M(1+\delta\tau)^{-k}, \quad k\tau\|BR^k\|_{H\to H} \le M, \quad k \ge 1,\ \delta > 0, \tag{2.4}$$

$$\|B^{\beta}(R^{k+r}-R^k)\|_{H\to H} \le M\frac{(r\tau)^{\alpha}}{(k\tau)^{\alpha+\beta}}, \quad 1 \le k < k+r \le N,\ 0 \le \alpha,\ \beta \le 1. \tag{2.5}$$

Here $B = (1/2)(\tau A + \sqrt{A(4+\tau^2 A)})$. From (2.2) and (2.4), it follows that

$$||(I-R^{2N})^{-1}||_{H\to H} \le M, \tag{2.6}$$

$$\begin{aligned}\Big\|&(I+(I+\tau A)(I+2\tau A)^{-1}R^{2N-1}+B^{-1}A(I+2\tau A)^{-1}(I-R^{2N-1})\\ &-(2I+\tau B)(I+2\tau A)^{-1}R^N P^{N-1})^{-1}\Big\|_{H\to H} \le M.\end{aligned} \tag{2.7}$$

Theorem 2.1. *For any g_k, $1 \le k \le N-1$ and f_k, $-N+1 \le k \le 0$, the solution of problem* (2.1) *exists and the following formulas hold:*

$$\begin{aligned}u_k = (I-R^{2N})^{-1}\Bigg\{&[R^k - R^{2N-k}]u_0\\ &+[R^{N-k}-R^{N+k}]\left[P^N u_0 - \tau\sum_{s=-N+1}^{0}P^{s+N}f_s+\mu\right]\\ &-[R^{N-k}-R^{N+k}](I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}[R^{N-s}-R^{N+s}]g_s\tau\Bigg\}\\ &+(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}[R^{|k-s|}-R^{k+s}]g_s\tau, \quad 1 \le k \le N,\end{aligned} \tag{2.8}$$

$$u_k = P^{-k}u_0 - \tau\sum_{s=k+1}^{0}P^{s-k}f_s, \quad -N \le k \le 0, \tag{2.9}$$

$$\begin{aligned}u_0 = T_\tau(I+2\tau A)^{-1}(I+\tau A)\Bigg\{\Bigg\{&(2+\tau B)R^N\left[-\tau\sum_{s=-N+1}^{0}P^{s+N}f_s+\mu\right]\\ &-R^{N-1}B^{-1}\sum_{s=1}^{N-1}[R^{N-s}-R^{N+s}]g_s\tau\Bigg\}\\ &+(I-R^{2N})B^{-1}\sum_{s=1}^{N-1}R^{s-1}g_s\tau-(I-R^{2N})(I+\tau B)B^{-1}Pf_0\Bigg\},\end{aligned} \tag{2.10}$$

where

$$T_\tau = \left(I+(I+\tau A)(I+2\tau A)^{-1}R^{2N-1}+B^{-1}A(I+2\tau A)^{-1}(I-R^{2N-1})-(2I+\tau B)(I+2\tau A)^{-1}R^N P^{N-1}\right)^{-1}. \tag{2.11}$$

Proof. By [8, 9],

$$u_k = (I - R^{2N})^{-1}\Big\{[R^k - R^{2N-k}]\xi + [R^{N-k} - R^{N+k}]\varphi - [R^{N-k} - R^{N+k}](I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}[R^{N-s} - R^{N+s}]g_s\tau\Big\} + (I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}[R^{|k-s|} - R^{k+s}]g_s\tau, \quad 1 \le k \le N, \tag{2.12}$$

is the solution of the boundary value difference problem

$$\begin{gathered} -\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k = g_k, \\ g_k = g(t_k), \quad t_k = k\tau, \quad 1 \le k \le N-1, \\ u_0 = \xi, \qquad u_N = \varphi, \end{gathered} \tag{2.13}$$

$$u_k = P^{-k}\xi - \tau\sum_{s=k+1}^{0}P^{s-k}f_s, \quad -N \le k \le 0 \tag{2.14}$$

is the solution of the inverse Cauchy problem

$$\begin{gathered} \tau^{-1}(u_k - u_{k-1}) - Au_{k-1} = f_k, \quad f_k = f(t_{k-1}), \\ t_{k-1} = (k-1)\tau, \quad -N+1 \le k \le 0, \ u_0 = \xi. \end{gathered} \tag{2.15}$$

Exploiting (2.12), (2.14), and the formulas

$$\varphi = u_{-N} + \mu, \quad \xi = u_0, \tag{2.16}$$

we obtain formulas (2.8) and (2.9). For u_0, using (2.8), (2.9), and the formula

$$u_1 - u_0 = u_0 - u_{-1}, \tag{2.17}$$

we obtain the operator equation

$$(I - R^{2N})^{-1}\Big\{[R - R^{2N-1}]u_0 + [R^{N-1} - R^{N+1}] \times \Big[P^N u_0 - \tau\sum_{s=-N+1}^{0}P^{s+N}f_s + \mu\Big] - [R^{N-1} - R^{N+1}](I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}[R^{N-s} - R^{N+s}]g_s\tau\Big\} + (I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}[R^{s-1} - R^{1+s}]g_s\tau = 2u_0 - Pu_0 + \tau Pf_0. \tag{2.18}$$

The operator

$$I+(I+\tau A)(I+2\tau A)^{-1}R^{2N-1}+B^{-1}A(I+2\tau A)^{-1}(I-R^{2N-1})-(2I+\tau B)(I+2\tau A)^{-1}R^{N}P^{N-1} \tag{2.19}$$

has an inverse

$$T_\tau=\left(I+(I+\tau A)(I+2\tau A)^{-1}R^{2N-1}+B^{-1}A(I+2\tau A)^{-1}(I-1R^{2N-1})-(2I+\tau B)(I+2\tau A)^{-1}R^{N}P^{N-1}\right)^{-1}, \tag{2.20}$$

and the following formula

$$\begin{aligned} u_0 = T_\tau(I+\tau A)(I+2\tau A)^{-1}\Bigg\{\Bigg\{&(2+\tau B)R^N\Bigg[-\tau\sum_{s=-N+1}^{0}P^{s+N}f_s+\mu\Bigg] \\ &-R^{N-1}B^{-1}\sum_{s=1}^{N-1}[R^{N-s}-R^{N+s}]g_s\tau\Bigg\} \\ &+(I-R^{2N})B^{-1}\sum_{s=1}^{N-1}R^{s-1}g_s\tau-(I-R^{2N})(I+\tau B)B^{-1}Pf_0\Bigg\} \end{aligned} \tag{2.21}$$

is satisfied. This concludes the proof of Theorem 2.1. □

Let $F_\tau(H)=F([a,b]_\tau,H)$ be the linear space of mesh functions $\varphi^\tau=\{\varphi_k\}_{N_a}^{N_b}$ defined on $[a,b]_\tau=\{t_k=kh,\ N_a\le k\le N_b,\ N_a\tau=a,\ N_b\tau=b\}$ with values in the Hilbert space H. Next on $F_\tau(H)$ we denote by $C([a,b]_\tau,H)$ and $C_{0,1}^\alpha([-1,1]_\tau,H),\ C_{0,1}^\alpha([-1,0]_\tau,H),\ C_0^\alpha([0,1]_\tau,H)(0<\alpha<1)$ Banach spaces with the norms

$$\begin{aligned} \|\varphi^\tau\|_{C([a,b]_\tau,H)} &= \max_{N_a\le k\le N_b}\|\varphi_k\|_H, \\ \|\varphi^\tau\|_{C_{0,1}^\alpha([-1,1]_\tau,H)} &= \|\varphi^\tau\|_{C([-1,1]_\tau,H)}+\sup_{-N\le k<k+r\le 0}\|\varphi_{k+r}-\varphi_k\|_E\frac{(-k)^\alpha}{r^\alpha} \\ &\quad+\sup_{1\le k<k+r\le N-1}\|\varphi_{k+r}-\varphi_k\|_E\frac{((k+r)\tau)^\alpha(N-k)^\alpha}{r^\alpha}, \\ \|\varphi^\tau\|_{C_0^\alpha([-1,0]_\tau,H)} &= \|\varphi^\tau\|_{C([-1,0]_\tau,H)}+\sup_{-N\le k<k+r\le 0}\|\varphi_{k+r}-\varphi_k\|_E\frac{(-k)^\alpha}{r^\alpha}, \\ \|\varphi^\tau\|_{C_{0,1}^\alpha([0,1]_\tau,H)} &= \|\varphi^\tau\|_{C([0,1]_\tau,H)}+\sup_{1\le k<k+r\le N-1}\|\varphi_{k+r}-\varphi_k\|_E\frac{((k+r)\tau)^\alpha(N-k)^\alpha}{r^\alpha}. \end{aligned} \tag{2.22}$$

The nonlocal boundary value problem (2.1) is said to be stable in $F([-1,1]_\tau,H)$ if we have the inequality

$$\|u^\tau\|_{F([-1,1]_\tau,H)}\le M[\|f^\tau\|_{F([-1,0]_\tau,H)}+\|g^\tau\|_{F([0,1]_\tau,H)}+\|\mu\|_H], \tag{2.23}$$

where M is independent of not only $f^\tau,\ g^\tau,\ \mu$ but also τ.

Theorem 2.2. *The nonlocal boundary value problem* (2.1) *is stable in* $C([-1,1]_\tau, H)$ *norm.*

Proof. By [9],

$$\left\|\{u_k\}_{-N}^{0}\right\|_{C([-1,0]_\tau,H)} \leq M[\|f^\tau\|_{C([-1,0]_\tau,H)} + \|u_0\|_H] \tag{2.24}$$

for the solution of the inverse Cauchy difference problem (2.15) and

$$\left\|\{u_k\}_{1}^{N-1}\right\|_{C([0,1]_\tau,H)} \leq M[\|g^\tau\|_{C([0,1]_\tau,H)} + \|u_0\|_H + \|u_N\|_H] \tag{2.25}$$

for the solution of the boundary value problem (2.13). The proof of Theorem 2.2 is based on the stability inequalities (2.24), (2.25), and on the estimates

$$\begin{aligned} \|u_0\|_H &\leq M[\|f^\tau\|_{C([-1,0]_\tau,H)} + \|g^\tau\|_{C([0,1]_\tau,H)} + \|\mu\|_H], \\ \|u_N\|_H &\leq M[\|f^\tau\|_{C([-1,0]_\tau,H)} + \|g^\tau\|_{C([0,1]_\tau,H)} + \|\mu\|_H] \end{aligned} \tag{2.26}$$

for the solution of the boundary value problem (2.1). Estimates (2.26) are derived from formula (2.10) and estimates (2.2), (2.4), (2.7). This concludes the proof of Theorem 2.2. □

The nonlocal boundary value problem (2.1) is said to be coercively stable (well-posed) in $F([-1,1]_\tau, H)$ if we have the coercive inequality

$$\begin{aligned} &\|\{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1}\|_{F([0,1]_\tau,H)} \\ &\quad + \|\{\tau^{-1}(u_k - u_{k-1})\}_{-N+1}^{0}\|_{F([-1,0]_\tau,H)} + \left\|\{Au_k\}_{-N}^{N-1}\right\|_{F([-1,1]_\tau,H)} \\ &\leq M[\|f^\tau\|_{F([-1,0]_\tau,H)} + \|g^\tau\|_{F([0,1]_\tau,H)} + \|A\mu\|_H], \end{aligned} \tag{2.27}$$

where M is independent of not only f^τ, g^τ, μ but also τ.

Since the nonlocal boundary value problem (1.1) in the space $C([0,1], H)$ of continuous functions defined on $[-1,1]$ and with values in H is not well-posed for the general positive unbounded operator A and space H, then the well-posedness of the difference nonlocal boundary value problem (2.1) in $C([-1,1]_\tau, H)$ norm does not take place uniformly with respect to $\tau > 0$. This means that the coercive norm

$$\begin{aligned} \|u^\tau\|_{K_\tau(E)} = &\|\{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1}\|_{C([0,1]_\tau,H)} \\ &+ \|\{\tau^{-1}(u_k - u_{k-1})\}_{-N+1}^{0}\|_{C([-1,0]_\tau,H)} + \|\{Au_k\}_{-N}^{N-1}\|_{C([-1,1]_\tau,H)} \end{aligned} \tag{2.28}$$

tends to ∞ as $\tau \to 0^+$. The investigation of the difference problem (2.1) permits us to establish the order of growth of this norm to ∞.

Theorem 2.3. *Assume that $\mu \in D(A)$ and $f_0 \in D(I+\tau B)$. Then for the solution of the difference problem* (2.1) *we have the almost coercivity inequality*

$$\begin{aligned}\|u^\tau\|_{K_\tau(E)} \leq M[&\|A\mu\|_H + \|(I+\tau B)f_0\|_H \\ &+ \min\left\{\ln\frac{1}{\tau}, 1+|\ln\|A\|_{H\to H}|\right\}[\|f^\tau\|_{C([-1,0]_\tau,H)} + \|g^\tau\|_{C([0,1]_\tau,H)}]],\end{aligned} \tag{2.29}$$

where M is independent of not only f^τ, g^τ, μ but also τ.

Proof. By [9],

$$\begin{aligned}&\|\{\tau^{-1}(u_k - u_{k-1})\}_{-N+1}^{0}\|_{C([-1,0]_\tau,H)} + \|\{Au_k\}_{-N}^{0}\|_{C([-1,0]_\tau,H)} \\ &\quad\leq M\left[\min\left\{\ln\frac{1}{\tau}, 1+|\ln\|A\|_{H\to H}|\right\}\|f^\tau\|_{C([-1,0]_\tau,H)} + \|Au_0\|_H\right]\end{aligned} \tag{2.30}$$

for the solution of the inverse Cauchy difference problem (2.15) and

$$\begin{aligned}&\left\|\{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1}\right\|_{C([0,1]_\tau,H)} + \left\|\{Au_k\}_1^{N-1}\right\|_{C([0,1]_\tau,H)} \\ &\quad\leq M\left[\min\left\{\ln\frac{1}{\tau}, 1+|\ln\|A\|_{H\to H}|\right\}\|g^\tau\|_{C([0,1]_\tau,H)} + \|Au_0\|_H + \|Au_N\|_H\right]\end{aligned} \tag{2.31}$$

for the solution of the boundary value problem (2.13). Then the proof of Theorem 2.3 is based on the almost coercivity inequalities (2.30), (2.31), and on the estimates

$$\begin{aligned}\|Au_0\|_H \leq M[&\|A\mu\|_H + \|(I+\tau B)f_0\|_H \\ &+ \min\left\{\ln\frac{1}{\tau}, 1+|\ln\|A\|_{H\to H}|\right\}[\|f^\tau\|_{C([-1,0]_\tau,H)} + \|g^\tau\|_{C([0,1]_\tau,H)}]], \\ \|Au_N\|_H \leq M[&\|A\mu\|_H + \|(I+\tau B)f_0\|_H \\ &+ \min\left\{\ln\frac{1}{\tau}, 1+|\ln\|A\|_{H\to H}|\right\}[\|f^\tau\|_{C([-1,0]_\tau,H)} + \|g^\tau\|_{C([0,1]_\tau,H)}]]\end{aligned} \tag{2.32}$$

for the solution of the boundary value problem (2.1). The proof of these estimates follows the scheme of papers [8, 9] and relies on formula (2.10) and on estimates (2.2), (2.4), and (2.7). This concludes the proof of Theorem 2.3. □

Theorem 2.4. *Let the assumptions of Theorem 2.3 be satisfied. Then the boundary value problem* (2.1) *is well-posed in a Hölder space $C_{0,1}^{\alpha}([-1,1]_\tau, H)$ and the following coercivity inequality holds:*

$$\begin{aligned}&\|\{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1}\|_{C_{0,1}^{\alpha}([0,1]_\tau,H)} \\ &\quad+ \left\|\{Au_k\}_{-N}^{N-1}\right\|_{C_{0,1}^{\alpha}([-1,1]_\tau,H)} + \|\{\tau^{-1}(u_k - u_{k-1})\}_{-N+1}^{0}\|_{C_0^{\alpha}([-1,0]_\tau,H)} \\ &\quad\leq M\left[\|A\mu\|_H + \|(I+\tau B)f_0\|_H + \frac{1}{\alpha(1-\alpha)}[\|f^\tau\|_{C_0^{\alpha}([-1,0]_\tau,H)} + \|g^\tau\|_{C_{0,1}^{\alpha}([0,1]_\tau,H)}]\right],\end{aligned} \tag{2.33}$$

where M is independent of not only f^τ, g^τ, μ but also τ and α.

Proof. By [8, 9],

$$\begin{aligned} &\left\|\{\tau^{-1}(u_k - u_{k-1})\}_{-N+1}^{0}\right\|_{C_0^{\alpha}([-1,0]_\tau,H)} + \left\|\{Au_k\}_{-N}^{0}\right\|_{C_0^{\alpha}([-1,0]_\tau,H)} \\ &\quad\le M\left[\frac{1}{\alpha(1-\alpha)}\|f^\tau\|_{C_0^{\alpha}([-1,0]_\tau,H)} + \|Au_0\|_H\right] \end{aligned} \tag{2.34}$$

for the solution of the inverse Cauchy difference problem (2.15) and

$$\begin{aligned} &\left\|\{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_{1}^{N-1}\right\|_{C_{0,1}^{\alpha}([0,1]_\tau,H)} + \left\|\{Au_k\}_{1}^{N-1}\right\|_{C_{0,1}^{\alpha}([0,1]_\tau,H)} \\ &\quad\le M\left[\frac{1}{\alpha(1-\alpha)}\|g^\tau\|_{C_{0,1}^{\alpha}([0,1]_\tau,H)} + \|Au_0\|_H + \|Au_N\|_H\right] \end{aligned} \tag{2.35}$$

for the solution of the boundary value problem (2.13). Then the proof of Theorem 2.4 is based on the coercivity inequalities (2.34), (2.35), and on the estimates

$$\begin{aligned} \|Au_0\|_H &\le M\left[\|A\mu\|_H + \|(I+\tau B)f_0\|_H + \frac{1}{\alpha(1-\alpha)}\left[\|f^\tau\|_{C_0^{\alpha}([-1,0]_\tau,H)} + \|g^\tau\|_{C_{0,1}^{\alpha}([0,1]_\tau,H)}\right]\right], \\ \|Au_N\|_H &\le M\left[\|A\mu\|_H + \|(I+\tau B)f_0\|_H + \frac{1}{\alpha(1-\alpha)}\left[\|f^\tau\|_{C_0^{\alpha}([-1,0]_\tau,H)} + \|g^\tau\|_{C_{0,1}^{\alpha}([0,1]_\tau,H)}\right]\right] \end{aligned} \tag{2.36}$$

for the solution of the boundary value problem (2.1). Estimates (2.36) are derived from the formulas

$$\begin{aligned} Au_0 &= T_\tau(I+2\tau A)^{-1}(I+\tau A) \\ &\quad\times\Bigg\{\Bigg\{(2+\tau B)R^N\left[-\tau\sum_{s=-N+1}^{0}AP^{s+N}(f_s - f_{-N+1}) + A\mu\right] \\ &\qquad\quad - R^{N-1}AB^{-2}\left\{\sum_{s=1}^{N-1}BR^{N-s}(g_s - g_{N-1})\tau + \sum_{s=1}^{N-1}BR^{N+s}(g_1 - g_s)\tau\right\}\Bigg\} \\ &\qquad\quad + (I-R^{2N})AB^{-2}\sum_{s=1}^{N-1}BR^{s-1}(g_s - g_1)\tau\Bigg\} \\ &\quad + T_\tau(I+2\tau A)^{-1}(I+\tau A)\big\{\big\{(2+\tau B)R^N(P^N - I)f_{-N+1} \\ &\qquad\qquad - R^{N-1}AB^{-2}\big\{(I-R^{N-1})g_{N-1} - (R^{N-2} - R^{2N-1})g_1\big\}\big\} \\ &\qquad\qquad + (I-R^{2N})AB^{-2}(I-R^{N-1})g_1 - (I-R^{2N})(I+\tau B)B^{-1}APf_0\big\}, \end{aligned}$$

$$
\begin{aligned}
Au_N = P^N \Bigg\{ & T_\tau (I+2\tau A)^{-1}(I+\tau A) \\
& \times \Bigg\{ \Bigg\{ (2+\tau B)R^N \Bigg[-\tau \sum_{s=-N+1}^{0} AP^{s+N}(f_s - f_{-N+1}) + A\mu \Bigg] \\
& \qquad - R^{N-1}AB^{-2} \Bigg\{ \sum_{s=1}^{N-1} BR^{N-s}(g_s - g_{N-1})\tau + \sum_{s=1}^{N-1} BR^{N+s}(g_1 - g_s)\tau \Bigg\} \Bigg\} \\
& \qquad + (I-R^{2N})AB^{-2} \sum_{s=1}^{N-1} BR^{s-1}(g_s - g_1)\tau \Bigg\} \Bigg\} \\
& - \tau \sum_{s=-N+1}^{0} AP^{s+N}(f_s - f_{-N+1}) + A\mu + (P^N - I) f_{-N+1} \\
& + P^N \{ T_\tau (I+2\tau A)^{-1}(I+\tau A)\{\{(2+\tau B)R^N(P^N - I)f_{-N+1} \\
& \qquad - R^{N-1}AB^{-2}\{(I-R^{N-1})g_{N-1} - (R^{N-2} - R^{2N-1})g_1\}\} \\
& \quad + (I-R^{2N})AB^{-2}(I-R^{N-1})g_1 - (I-R^{2N})(I+\tau B)B^{-1}APf_0\}\}
\end{aligned}
\tag{2.37}
$$

for the solution of problem (2.1) and estimates (2.2), (2.4), and (2.7). This concludes the proof of Theorem 2.4. □

Now, the applications of this abstract result to the approximate solution of the mixed boundary value problem for the elliptic-parabolic equation (1.14) are considered. The discretization of problem (1.14) is carried out in two steps. In the first step, the grid sets

$$
\begin{gathered}
\widetilde{\Omega}_h = \{x = x_m = (h_1 m_1, \dots, h_n m_n),\ m = (m_1, \dots, m_n), 0 \le m_r \le N_r,\ h_r N_r = 1,\ r = 1, \dots, n\}, \\
\Omega_h = \widetilde{\Omega}_h \cap \Omega, \qquad S_h = \widetilde{\Omega}_h \cap S
\end{gathered}
\tag{2.38}
$$

are defined. To the differential operator A generated by problem (1.14) we assign the difference operator A_h^x by the formula

$$
A_h^x u_x^h = -\sum_{r=1}^{n} \left(a_r(x) u_{\bar{x}_r}^h \right)_{x_r, j_r} \tag{2.39}
$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. With the help of A_h^x we arrive at the nonlocal boundary-value problem

$$
\begin{aligned}
& -\frac{d^2 u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = g^h(t,x), \quad 0 < t < 1,\ x \in \Omega_h, \\
& \frac{du^h(t,x)}{dt} - A_h^x u^h(t,x) = f^h(t,x), \quad -1 < t < 0,\ x \in \Omega_h,
\end{aligned}
$$

$$u^h(-1,x) = u^h(1,x) + \mu^h(x), \quad x \in \widetilde{\Omega}_h,$$

$$u^h(0+,x) = u^h(0-,x), \quad \frac{du^h(0+,x)}{dt} = \frac{du^h(0-,x)}{dt}, \quad x \in \widetilde{\Omega}_h \tag{2.40}$$

for an infinite system of ordinary differential equations.

In the second step problem (2) is replaced by the difference scheme (2.1):

$$\begin{gathered}
-\frac{u^h_{k+1}(x) - 2u^h_k(x) + u^h_{k-1}(x)}{\tau^2} + A^x_h u^h_k(x) = g^h_k(x), \\
g^h_k(x) = g^h(t_k,x), \quad t_k = k\tau,\ 1 \le k \le N-1, \quad N\tau = 1,\ x \in \Omega_h, \\
\frac{u^h_k(x) - u^h_{k-1}(x)}{\tau} - A^x_h u^h_{k-1}(x) = f^h_k(x), \\
f^h_k(x) = f^h(t_k,x), \quad t_{k-1} = (k-1)\tau, \quad -N+1 \le k \le -1,\ x \in \Omega_h, \\
u^h_{-N}(x) = u^h_N(x) + \mu^h(x), \quad x \in \widetilde{\Omega}_h, \\
u^h_1(x) - u^h_0(x) = u^h_0(x) - u^h_{-1}(x), \quad x \in \widetilde{\Omega}_h.
\end{gathered} \tag{2.41}$$

Based on the number of corollaries of the abstract theorems given above, to formulate the result, one needs to introduce the space $L_{2h} = L_2(\widetilde{\Omega}_h)$ of all the grid functions $\varphi^h(x) = \{\varphi(h_1 m_1, \ldots, h_n m_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norm

$$\|\varphi^h\|_{L_2(\widetilde{\Omega}_h)} = \left(\sum_{x\in\overline{\Omega_h}} |\varphi^h(x)|^2 h_1 \cdots h_n\right)^{1/2}. \tag{2.42}$$

Theorem 2.5. *Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small numbers. Then the solutions of the difference scheme* (2.41) *satisfy the following stability and almost coercivity estimates:*

$$\begin{aligned}
&\left\|\{u^h_k\}_{-N}^{N-1}\right\|_{C([-1,1]_\tau,L_{2h})} \le M\left[\left\|\{f^h_k\}_{-N+1}^{-1}\right\|_{C([-1,0]_\tau,L_{2h})} + \left\|\{g^h_k\}_1^{N-1}\right\|_{C([0,1]_\tau,L_{2h})} + \|\mu^h\|_{L_{2h}}\right], \\
&\left\|\{\tau^{-2}(u^h_{k+1} - 2u^h_k + u^h_{k-1})\}_1^{N-1}\right\|_{C([0,1]_\tau,L_{2h})} \\
&\qquad + \left\|\{\tau^{-1}(u^h_k - u^h_{k-1})\}_{-N+1}^{0}\right\|_{C([-1,0]_\tau,L_{2h})} + \left\|\{u^h_k\}_{-N}^{N-1}\right\|_{C([-1,1]_\tau,W^2_{2h})} \\
&\le M\left[\|\mu^h\|_{W^2_{2h}} + \tau\|f^h_0\|_{W^1_{2h}} + \ln\frac{1}{\tau+|h|}\left[\left\|\{f^h_k\}_{-N+1}^{-1}\right\|_{C([-1,0]_\tau,L_{2h})} + \left\|\{g^h_k\}_1^{N-1}\right\|_{C([0,1]_\tau,L_{2h})}\right]\right],
\end{aligned} \tag{2.43}$$

where M is independent of τ, h, $\mu^h(x)$, and $g^h_k(x)$, $1 \le k \le N-1$, f^h_k, $-N+1 \le k \le 0$.

The proof of Theorem 2.5 is based on the abstract Theorems 2.2, 2.3, on the estimate

$$\min\left\{ \ln\frac{1}{\tau},\ 1+|\ln\|A_h^x\|_{L_{2h}\to L_{2h}}|\right\} \le M\ \ln\frac{1}{\tau+|h|} \tag{2.44}$$

as well as the symmetry properties of the difference operator A_h^x defined by formula (2.39) in L_{2h}, along with the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 2.6. *For the solution of the elliptic difference problem,*

$$A_h^x u^h(x) = \omega^h(x), \quad x \in \Omega_h, \tag{2.45}$$

$$u^h(x) = 0, \quad x \in S_h, \tag{2.46}$$

the following coercivity inequality holds [7]:

$$\sum_{r=1}^{n}\left\|(u^h)_{\overline{x}_r x_r, j_r}\right\|_{L_{2h}} \le M\|\omega^h\|_{L_{2h}}. \tag{2.47}$$

Theorem 2.7. *Let τ and $|h|$ be sufficiently small numbers. Then the solutions of the difference scheme* (2.41) *satisfy the following coercivity stability estimates:*

$$\begin{aligned}
&\left\|\left\{\tau^{-2}\left(u_{k+1}^h - 2u_k^h + u_{k-1}^h\right)\right\}_1^{N-1}\right\|_{C_{0,1}^{\alpha}([0,1]_\tau, L_{2h})} \\
&\quad + \left\|\left\{\tau^{-1}\left(u_k^h - u_{k-1}^h\right)\right\}_{-N+1}^{0}\right\|_{C_0^{\alpha}([-1,0]_\tau, L_{2h})} + \left\|\left\{u_k^h\right\}_{-N}^{N-1}\right\|_{C_{0,1}^{\alpha}([-1,1]_\tau, W_{2h}^2)} \\
&\le M\left[\|\mu^h\|_{W_{2h}^2} + \tau\|f_0^h\|_{W_{2h}^1} + \frac{1}{\alpha(1-\alpha)}\left[\left\|\left\{f_k^h\right\}_{-N+1}^{-1}\right\|_{C_0^{\alpha}([-1,0]_\tau, L_{2h})} + \left\|\left\{g_k^h\right\}_1^{N-1}\right\|_{C_{0,1}^{\alpha}([0,1]_\tau, L_{2h})}\right]\right],
\end{aligned} \tag{2.48}$$

where M is independent of $\tau,\ h,\ \mu^h(x)$, and $g_k^h(x),\ 1 \le k \le N-1,\ f_k^h, -N+1 \le k \le 0$.

The proof of Theorem 2.7 is based on the abstract Theorem 2.4, the symmetry properties of the difference operator A_h^x defined by formula (2.39), and on Theorem 2.6 on the coercivity inequality for the solution of the elliptic difference equation (2.45) in L_{2h}.

Note that in a similar manner the difference schemes of the first order of accuracy with respect to one variable for approximate solutions of the boundary value problem (1.12) can be constructed. Abstract theorems given above permit us to obtain the stability, the almost stability, and the coercive stability estimates for the solution of these difference schemes.

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References

[1] D. Bazarov and H. Soltanov, *Some Local and Nonlocal Boundary Value Problems for Equations of Mixed and Mixed-Composite Types*, Ylim, Ashgabat, Turkmenistan, 1995.

[2] S. N. Glazatov, "Nonlocal boundary value problems for linear and nonlinear equations of variable type," Sobolev Institute of Mathematics SB RAS, no. 46, pp. 26, 1998.

[3] S. G. Krein, *Linear Differential Equations in a Banach Space*, Nauka, Moscow, Russia, 1967.

[4] M. S. Salakhitdinov, *Equations of Mixed-Composite Type*, Fan, Tashkent, Uzbekistan, 1974.

[5] A. Ashyralyev and H. Soltanov, "On elliptic-parabolic equations in a Hilbert space," in *Proceeding of the IMM of CS of Turkmenistan*, pp. 101–104, Ashgabat, Turkmenistan, 1995.

[6] A. Ashyralyev, "A note on the nonlocal boundary value problem for elliptic-parabolic equations," *Nonlinear Studies*, vol. 13, no. 4, pp. 327–333, 2006.

[7] P. E. Sobolevskiĭ, *On Difference Methods for the Approximate Solution of Differential Equations*, Voronezh State University Press, Voronezh, Russia, 1975.

[8] P. E. Sobolevskiĭ, "The theory of semigroups and the stability of difference schemes," in *Operator Theory in Function Spaces (Proc. School, Novosibirsk, 1975)*, pp. 304–337, Nauka, Novosibirsk, Russia, 1977.

[9] P. E. Sobolevskiĭ, "The coercive solvability of difference equations," *Doklady Akademii Nauk SSSR*, vol. 201, no. 5, pp. 1063–1066, 1971.