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Research Article

The Finite Discrete KP Hierarchy and the Rational Functions

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The set of all rational functions with any fixed denominator that simultaneously nullify in the infinite point is parametrized by means of a well-known integrable system: a finite dimensional version of the discrete KP hierarchy. This type of study was originated in Y. Nakamura's works who used others integrable systems. Our work proves that the finite discrete KP hierarchy completely parametrizes the space $\mathrm{Rat}_{\Lambda}(n)$ of rational functions of the form $f(x) = q(x)/z^n$, where $q(x)$ is a polynomial of order $n-1$ with nonzero independent coefficent. More exactly, it is proved that there exists a bijection from $\mathrm{Rat}_{\Lambda}(n)$ to the moduli space of solutions of the finite discrete KP hierarchy and a compatible linear system.

1. Introduction

Few decades ago, an unexpected relation between the control theory and the integrable systems was revealed. Papers [1, 2] which deal with problems related to those discussed here are examples of these researches.

A large part of this research activity shows that some nonlinear integrable systems have rich information about the moduli space of certain classes of solutions of linear dynamical systems. In particular, they have relation with spaces of certain classes of rational functions. Also according to the state space realization theory, some rational functions can be associated with a controllable and observable linear dynamical system.

A convenient property of these spaces of rational functions is that they can be considered as varieties, thus a question arises about the study of its moduli space. Maybe, in this context, the fundamental role of the integrable systems is its compatibility with families of controllable and observable linear dynamical systems.

During the last ten years, the subject of integrable system has been enriched in a remarkable way by its extensions to the other setting, notably, to discrete case of the KP hierarchy. These developments have originated in the mathematical physics. In the new settings, many of the classical tools are available, for example, we point out one of them which is basic to this paper, the Gauss-Borel decomposition for the discrete KP hierarchy, proposed by Felipe and Ongay [3] as an extension of the Mulase's algebraic geometric approach to the KP hierarchy [4]. This decomposition allows us to consider on an almost equal footing the cases of semi-infinite and bi-infinite matrices.

In this paper, we use the well-known theory of the discrete KP hierarchy studied, for instance, in [3], restricted to finite matrices to characterize $\mathrm{Rat}_{\Lambda}(n)$. Thus we will give another example of nonlinear integrable system that also has the property of completely parameterizing some kind of rational function space.

We must observe that an interesting property of the finite discrete KP hierarchy is that it contains the full Kostant-Toda equation.

2. An algebraic geometric approach for the finite discrete KP hierarchy

The goals of this section are as follows.

(1) The first goal is to introduce a natural commuting finite hierarchy of flows. We make three comments of this hierarchy. First, it can be defined by a Lax-type operator (matrix) with respect to the shift matrix and its transpose. Second, the Lax matrix introduced admits in certain cases a dressing matrix, in terms of which the hierarchy can be rewritten. Third, there is a Sato-Wilson matrix "to dress" the shift matrix. We mention that the situation is similar to the Sato theory and his dressing technique (pseudodifferential theory).

(2) The second is to review the integrability in the sense of Frobenius for the hierarchy introduced which turns out very simple in this context: the key point in our method is the so-called Gauss-Borel decomposition. It also verifies that the finite discrete KP hierarchy like any integrable system is always related to some kind of group factorization.

Next, we describe the corresponding Mulase's approach associated to the finite discrete KP hierarchy as it is considered in [3]. We omit the proofs because in [3] these were given in a more general background.

Let Λ be the n by n-matrix with ones (the matrix entry equals to 1) in the first upper diagonal and zero in the remaining entries

$$\Lambda = \begin{pmatrix} 0 & 1 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix} \tag{2.1}$$

and Λ^T its transpose. The matrix Λ is like a shift operator of coordinates for vectors in R^n. The Λ^k matrix, $2 \leq k \leq n-1$, is a zero matrix except in the kth upper diagonal where it has ones. Note that $\Lambda^n = \Lambda^{T^n} = O$, where O is the zero matrix.

Let L be a matrix

$$L = \Lambda + D_0 + \sum_{i=1,\ldots,n-1} D_i \left(\Lambda^T\right)^i, \tag{2.2}$$

where D_i are diagonal matrices. The entries of L are assumed to be functions depending on parameters $t = (t_1, \ldots, t_{n-1})$.

Definition 2.1. The finite discrete KP hierarchy is the Lax system

$$\frac{\partial L}{\partial t_k} = \left[L_{\geq}^k, L\right], \quad k = 1, \ldots, n-1, \tag{2.3}$$

where $L_{\geq}$ $(L_{>})$ denotes the (strictly) upper triangular part of L, analogously $L_{\leq}$ $(L_{<})$ denotes the (strictly) lower triangular part of L.

Now, let us assume that for an operator defined as in (2.2), one can find an invertible matrix S:

$$S = I + S_1\Lambda^T + S_2\left(\Lambda^T\right)^2 + \cdots + S_{n-1}\left(\Lambda^T\right)^{n-1} \tag{2.4}$$

such that

$$L = S\Lambda S^{-1}, \tag{2.5}$$

$$S^{-1} = I + W_1\Lambda^T + W_2\left(\Lambda^T\right)^2 + \cdots + W_{n-1}\left(\Lambda^T\right)^{n-1} \tag{2.6}$$

where in (2.4) and (2.6), S_i and W_i are diagonal matrices, then L will be of Lax type.

From now, we can only consider solutions of (2.3) which are of the Lax type. The operator S is called a dressing operator, and the decomposition (2.5) is unique, up to right multiplication by an invertible matrix, taking the form of (2.4) that commutes with Λ. Note that $L = \Lambda$ is the most simple solution of (2.3) and for it $S = I$.

If there is a dressing operator such that

$$\frac{\partial S}{\partial t_k} = -L_{<}^k S, \quad k = 1, \ldots, n-1, \tag{2.7}$$

where $L = S\Lambda S^{-1}$, then L is of the Lax type, and moreover it satisfies (2.3). Conversely, if L is a solution of (2.3) which is of the Lax type, then there exists a dressing operator S of L, such that S is solution of (2.7). This operator S is called the Sato-Wilson matrix (see [3] for more details).

In this point, it is very important to observe that for any given order n, it is always possible to find matrices L of the form (2.2) which are not of Lax type. It is explained bellow with more details.

It was shown in [3] that from (2.3) the Zakharov-Shabat equations follow

$$\frac{\partial L_{\geq}^i}{\partial t_j} - \frac{\partial L_{\geq}^j}{\partial t_i} = \left[L_{\geq}^i, L_{\geq}^j\right], \quad i \neq j, \; i, j = 1, \ldots, n-1. \tag{2.8}$$

We say that a matrix M admits a Gauss-Borel decomposition if M can be written in the form

$$M = M_{\leq} M_{\geq}, \tag{2.9}$$

where $M_{\leq}$ has ones on the principal diagonal and $M_{\geq}$ has nonzero elements on the principal diagonal. Decomposition (2.9) is equivalent to

$$M = \left(I + G_{<} G_0^{-1}\right)\left(I + G_0 + G_{>}\right), \tag{2.10}$$

where G_0 is a diagonal matrix with nonzero elements. It can be proved that the Gauss-Borel decomposition is unique. M admits the Gauss-Borel decomposition if and only if $M_k \neq 0,\ k = 1, \dots, n$, where M_k is the determinant of the k-order principal submatrix. Of particular interest will be the matrix space M^* of matrices M depending on t, admitting a Gauss-Borel decomposition and for which $M_{\geq}(0) = I$, where I is the identity matrix.

Let Z be the formal 1-form of L given by

$$Z = \sum_{k=1}^{n-1} L_{\geq}^{k} dt_k. \tag{2.11}$$

If L satisfies (2.3), then Z satisfies

$$dZ = \frac{1}{2}[Z, Z] \tag{2.12}$$

which is equivalent to the Zakharov-Shabat equations, where by definition

$$[Z, Z] = \sum_{i,j=1}^{n-1} \left[L_{\geq}^{i}, L_{\geq}^{j}\right] dt_i\, dt_j. \tag{2.13}$$

Let

$$\Omega = \sum_{k=1}^{n-1} \Lambda^k dt_k \tag{2.14}$$

which is a trivial solution of (2.12). Each solution of (2.3) yields a solution of the linear equation

$$dU = \Omega U \tag{2.15}$$

in M^* and conversely for a solution of (2.15) in M^* we can build a matrix L which will be a solution of (2.3). The solutions of (2.15) take the form

$$U = e^{\sum_{k=1}^{n-1} \Lambda^k t_k} U_0. \tag{2.16}$$

Let us consider the Gauss-Borel factorization of U:

$$U = S^{-1} Y, \tag{2.17}$$

where S^{-1} is a lower triangular matrix with ones on the principal diagonal and Y is an upper triangular matrix with nonzero elements on the principal diagonal.

For $U \in M^*$ we have that

$$U_0 = U(0) = S^{-1}(0)Y(0) = S^{-1}(0)I = S_0^{-1}, \tag{2.18}$$

then

$$U = e^{\sum_{k=1}^{n-1}\Lambda^k t_k}U_0 = e^{\sum_{k=1}^{n-1}\Lambda^k t_k}S_0^{-1}, \tag{2.19}$$

where S_0^{-1} is a matrix, which takes the form of (2.4).

It is easy to show that if $U \in M^*$, then in (2.17) S is a Sato-Wilson operator, therefore using the Gauss-Borel factorization of U and doing $L = S\Lambda S^{-1}$, we obtain a solution of (2.3) for which

$$L(0) = S_0\Lambda S_0^{-1} = L_0, \tag{2.20}$$

(see [3]).

3. Rational functions induced by the finite discrete KP hierarchy solutions

Let $\mathrm{rat}_p(n)$ be the space of rational functions of grade n and fixed denominator $p(z) = z^n + p_{n-1}z^{n-1} + \cdots + p_0$. It is possible to see that an element $f(z) \in \mathrm{rat}_p(n)$ always admits a unique factorization

$$f(z) = \frac{q(z)}{p(z)} = C_0^T\left(zI_n - L_p\right)^{-1}B_0, \tag{3.1}$$

where $C_0^T = (1, 0, \ldots)$, $B_0^T = (b_{n-1} \cdot b_{n-2}, \ldots, b_0)$, $q(z) = b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \cdots + b_0$, and

$$L_p = \begin{pmatrix} -p_{n-1} & 1 & 0 & \cdots & 0 \\ -p_{n-2} & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -p_1 & 0 & \cdots & \ddots & 1 \\ -p_0 & 0 & \cdots & \cdots & 0 \end{pmatrix}, \tag{3.2}$$

this matrix L_p will be denominated by p-operator (note that L_p is of the form (2.2)). Observe that the characteristic polynomials of L_p is equal to the given $p(z)$.

At this point, it is convenient to inspect the Lax operator as function of its dressing operators. Let us see it for $n = 2$ and $n = 3$:

$$\begin{array}{ccc} & n=2 & n=3 \\ S^{-1} & \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix} \\ S & \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ab-c & -b & 1 \end{pmatrix} \\ L & \begin{pmatrix} -a & 1 \\ -a^2 & a \end{pmatrix} & \begin{pmatrix} a & 1 & 0 \\ -a^2+c & -a+b & 1 \\ (a^2-c)b-ac & (ab-c)-b^2 & -b \end{pmatrix}. \end{array} \tag{3.3}$$

From these simple examples, it follows that the unique p-operator that is also a Lax operator is justly $L_p = \Lambda$. This assertion holds also for any arbitrary dimension of matrix L_p and it shows that $\mathrm{rat}(n)$ (its definition is given next) is the unique space that could be characterized by this hierarchy, using the Gauss-Borel factorization.

Let $\mathrm{rat}(n)$ be a space of rational functions $f(z) \in \mathrm{rat}_p(n)$ such that

$$f(z) = \frac{a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n}{z^n} = \frac{q(z)}{z^n} \tag{3.4}$$

with $a_n \neq 0$ and $a_i \in \mathbb{R}$ for any i. On this space, it is usual to introduce the equivalence relation

$$f \sim \alpha f \quad \alpha \in R \setminus \{0\}. \tag{3.5}$$

We denote the equivalence class to which belongs $f(z)$ and by $[f]$ and the set $\mathrm{rat}(n) \setminus \sim\, = \{[f] : f \in \mathrm{rat}(n)\}$ by $\mathrm{Rat}_\Lambda(n)$. As mentioned before, any element of $\mathrm{rat}(n)$ can be written in the form

$$f(z) = C_0^T (zI - \Lambda)^{-1} B_0, \tag{3.6}$$

where $C_0^T = (1, 0, \dots, 0), B_0^T = (a_1, a_2, \dots, a_n)$ are constant vectors in R^n, and a_i are the coefficients of $f(z)$.

Note that

$$\alpha f(z) = C_0^T (zI - \Lambda)^{-1} (\alpha B_0). \tag{3.7}$$

The functions $f \in \mathrm{rat}(n)$ expressed according to (3.6) can be considered as the transfer functions of the linear dynamical system

$$\frac{dx}{d\tau} = \Lambda x(\tau) + B_0 u(\tau), \qquad y(\tau) = C_0^T x(\tau). \tag{3.8}$$

Proposition 3.1. *The linear dynamical system* (3.8) *associated to* $f(z) = C_0^T (zI - \Lambda)^{-1} B_0 \in \mathit{rat}(n)$ *is controllable and observable.*

Proof. Indeed,

$$\mathrm{rang}\big((B_0)(\Lambda B_0) \cdots (\Lambda^{n-1} B_0)\big) = n \tag{3.9}$$

because

$$\begin{vmatrix} a_1 & a_2 & & a_n \\ a_2 & a_3 & & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n-1} & a_n & & 0 \\ a_n & 0 & & 0 \end{vmatrix} = -\left(a_n^n\right) \neq 0 \tag{3.10}$$

if $a_n \neq 0$. This fact implicates the controllability of (3.8). Also

$$\mathrm{rang}\big((C_0)(\Lambda^T C_0) \cdots (\Lambda^{T^{n-1}} C_0)\big) = n \tag{3.11}$$

because $(\Lambda^T)^i C_0$, $i = 0, \dots, n-1$, is the canonical base on R^n. This fact implies that (3.8) is observable. □

Proposition 3.2. *Let $U \in M^*$, $U = S^{-1}Y$ and B, C be defined as follows:*

$$B(t_1,\ldots,t_{n-1}) = Y(t_1,\ldots,t_{n-1})B_0, \quad C(t_1,\ldots,t_{n-1}) = \left(S^{-1}(t_1,\ldots,t_{n-1})\right)^T C_0, \tag{3.12}$$

then B and C satisfy the linear equations

$$\frac{\partial B}{\partial t_k} = L^k_{\geq} B, \quad \frac{\partial C}{\partial t_k} = \left(L^k_{<}\right)^T C, \quad k = 1,\ldots,n-1, \ B(0) = B_0, \ C(0) = C_0, \tag{3.13}$$

where $L = S\Lambda S^{-1}$.

Proof. We have that

$$\frac{\partial B}{\partial t_k} = \frac{\partial Y}{\partial t_k} B_0 \tag{3.14}$$

for $k = 1,\ldots,n-1$. Being $U = S^{-1}Y$, we obtain

$$\frac{\partial U}{\partial t_k} = \frac{\partial S^{-1}}{\partial t_k} Y + S^{-1}\frac{\partial Y}{\partial t_k} \tag{3.15}$$

and since

$$S^{-1}S = I \quad \frac{\partial S^{-1}}{\partial t_k} S + S^{-1}\frac{\partial S}{\partial t_k} = 0, \quad \frac{\partial U}{\partial t_k} = -S^{-1}\frac{\partial S}{\partial t_k} S^{-1} Y + S^{-1}\frac{\partial Y}{\partial t_k}. \tag{3.16}$$

Having in mind that S is a Sato-Wilson operator and $\partial U/\partial t_k = \Lambda^k U$,

$$\begin{gathered}
\Lambda^k U = \frac{\partial U}{\partial t_k} = -S^{-1}\frac{\partial S}{\partial t_k} S^{-1} Y + S^{-1}\frac{\partial Y}{\partial t_k}, \qquad S\Lambda^k U + \frac{\partial S}{\partial t_k} S^{-1} Y = \frac{\partial Y}{\partial t_k}, \\
\left(S\Lambda^k S^{-1}\right) Y - L^k_{<} Y = \frac{\partial Y}{\partial t_k}, \qquad L^k_{\geq} Y = \left(L^k - L^k_{<}\right) Y = \frac{\partial Y}{\partial t_k},
\end{gathered} \tag{3.17}$$

then

$$\frac{\partial B}{\partial t_k} = \frac{\partial Y}{\partial t_k} B_0 = L^k_{\geq} Y B_0 = L^k_{\geq} B. \tag{3.18}$$

On the other hand, we have that

$$\frac{\partial C}{\partial t_k} = \frac{\partial \left(S^{-1}\right)^T}{\partial t_k} B_0, \quad C = \left(S^{-1}\right)^T C_0 = \left(S^T\right)^{-1} C_0 \tag{3.19}$$

for $k = 1,\ldots,n-1$. Since S verifies (2.7), it is Sato-Wilson matrix then

$$\frac{\partial S^T}{\partial t_k} = -S^T L^k_{<}. \tag{3.20}$$

From (3.19) and (3.20), its follows that

$$\frac{\partial C}{\partial t_k} = \frac{\partial \left(S^{-1}\right)^T}{\partial t_k} C_0 = \frac{\partial \left(S^T\right)^{-1}}{\partial t_k} C_0 = \left(L^k_{<}\right)^T \left(S^T\right)^{-1} C_0 = \left(L^k_{<}\right)^T C. \tag{3.21}$$

Let us show that the initial conditions hold.

Since $B = YB_0$ and $Y(0) = I$, then $B(0) = B_0$. On other hand, due to $C = (S^{-1})^T C_0$, we have that $C(0) = (S^{-1}(0))^T C_0 = (S_0^{-1})^T C_0$. As $(S_0^{-1})^T$ is an upper triangular matrix with ones on the diagonal, then $(S_0^{-1})^T C_0 = C_0$. □

Remark 3.3. It is interesting to note that independently of the selection of S_0^{-1} as initial condition in the factorization of U, the flow for C always begins in C_0.

We restrict ourselves to consider a solution L of (2.3) to which $L_I(0) = \Lambda$ and we denote it by L_I. In this case, we must take $S_0 = I$ by (2.20).

Next, let us define a linear dynamical system of parameter $t = (t_1, \dots, t_{n-1})$ using $B(t)$, $C(t)$, and $L_I(t)$ in the following way

$$\frac{d}{d\tau}x(\tau,t) = L(\tau,t)x(\tau,t) + B(\tau,t)u(\tau,t), \qquad y(\tau,t_1) = C^T(\tau,t)x(\tau,t). \tag{3.22}$$

Note that for $t_1 = \cdots = t_{n-1} = 0$, we obtain (3.8).

Proposition 3.4. *The linear dynamical system* (3.22) *is controllable and observable.*

Proof. Indeed,

$$\begin{aligned} \operatorname{rang}((B)(LB)\cdots(L^{n-1}B)) &= \operatorname{rang}((YB_0)(S\Lambda S^{-1}YB_0)\cdots(S\Lambda^{n-1}S^{-1}YB_0)) \\ &= \operatorname{rang}(S((UB_0)(\Lambda UB_0)\cdots(\Lambda^{n-1}UB_0))). \end{aligned} \tag{3.23}$$

Since Λ and U commute, then

$$\begin{aligned} \operatorname{rang}((B)(LB)\cdots(L^{n-1}B)) &= \operatorname{rang}(SU(B_0)(\Lambda B_0)\cdots(\Lambda^{n-1}B_0)) \\ &= \operatorname{rang}(Y(B_0)(\Lambda B_0)\cdots(\Lambda^{n-1}B_0)). \end{aligned} \tag{3.24}$$

By (3.9) and due to $\det Y \neq 0$, we have that

$$\operatorname{rang}(Y(B_0)(\Lambda B_0)\cdots(\Lambda^{n-1}B_0)) = n \tag{3.25}$$

from where

$$\operatorname{rang}((B)(LB)\cdots(L^{n-1}B)) = n \tag{3.26}$$

then (3.22) is controllable.

Now we will show the observability of (3.32).

$$\begin{aligned} &\operatorname{rang}((C)(LC)\cdots(L^{n-1}C)) \\ &\quad = \operatorname{rang}(((S^T)^{-1}C_0)((S^T)^{-1}\Lambda^T S^T (S^T)^{-1}C_0)\cdots((S^T)^{-1}(\Lambda^T)^{n-1}S^T(S^T)^{-1}C_0)) \\ &\quad = \operatorname{rang}((S^T)^{-1}(C_0(\Lambda^T C_0)\cdots((\Lambda^T)^{n-1}C_0)). \end{aligned} \tag{3.27}$$

By (3.11) and since $\det (S^T)^{-1} \neq 0$, we have

$$\operatorname{rang}((S^T)^{-1}(C_0(\Lambda^T C_0)\cdots((\Lambda^T)^{n-1}C_0)) = n \tag{3.28}$$

from where

$$\operatorname{rang}((C)(LC)\cdots(L^{n-1}C)) = n\,. \tag{3.29}$$

□

Let us consider the function

$$\begin{aligned} F(z,t_1,\dots,t_{n-1}) &= C^T(zI - L_I)^{-1}B \\ &= C^T\left(S(zI-\Lambda)S^{-1}\right)^{-1}B \\ &= C^T S(zI-\Lambda)^{-1}S^{-1}B \\ &= \left((S^{-1})^T C_0\right)^T S(zI-\Lambda)^{-1}S^{-1}B \\ &= C_0^T S^{-1}S(zI-\Lambda)^{-1}S^{-1}YB_0 \\ &= C_0^T(zI-\Lambda)^{-1}U_I B_0. \end{aligned} \tag{3.30}$$

Since $U_I(0) = I$, then

$$F(z,0) = f(z) = C_0^T\left(zI - L_I(0)\right)^{-1}B_0\,. \tag{3.31}$$

We see that $F(z,t_1,\dots,t_{n-1})\,\epsilon\,\mathrm{rat}(n)$. From (3.30), we have that

$$F(z,t_1,\dots,t_{n-1}) = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_n}{z^n} = \frac{q_{n-1}(z,t_1,\dots,t_{n-1})}{z^n}. \tag{3.32}$$

As U_I is an upper triangular matrix with ones on the diagonal, then the independent terms of q_{n-1} and of the numerator of $f(z)$ coincide.

We can characterize the flow of (3.30). Taking derivate respect to t_k, we obtain

$$\frac{\partial F}{\partial t_k} = C_0^T(zI - L_I)^{-1}\frac{\partial U_I}{\partial t_k}B_0, \quad k = 1,\dots,n-1, \tag{3.33}$$

doing $V = U_I B_0$, we have

$$F = C_0^T(zI-\Lambda)^{-1}V. \tag{3.34}$$

The flows on $\mathrm{rat}(n)$ determined by L_I have the form

$$\frac{\partial V}{\partial t_k} = \Lambda^k V, \quad k = 1,\dots,n-1\,, \tag{3.35}$$

where V defines the numerator of F and it has the form

$$V = e^{\sum_{k=1}^{n-1}\Lambda^k t_k}B_0\,. \tag{3.36}$$

Let us notice that (3.35) is similar to the flows considered by Brockett and Faybusovich in [1].

Let $(L_I(t), B(t), C(t))$ be a triple, where $L_I(t)$ is the above fixed solution of (2.3); $B(t)$ and $C(t)$ are solutions of (3.13). This triple gives a flow of (3.30) on $\mathrm{rat}(n)$ defined by the initial conditions $L_I(0) = \Lambda$, $B(0) = B_0$, $C(0) = C_0$ that determine the triple (Λ, B_0, C_0).

In such way, we have the nontrivial flow $(\Lambda, B_0, C_0) \to (L_I, B, C)$ on R^{2n+n^2}.

The equivalence relation (3.5) induces an equivalence relation $\prec$ on the set of triples (Λ, B_0, C_0) such that

$$\left(\Lambda, B_0^1, C_0\right) \prec \left(\Lambda, B_0^2, C_0\right), \quad \text{if } B_0^1 = \alpha B_0^2\,, \ \alpha\epsilon\, R\setminus\{0\}. \tag{3.37}$$

Definition 3.5. The set of solutions $(L_I(t), B(t), C(t))$ of compatible systems (2.3) and (3.13) for any equivalence class $[(\Lambda, B_0, C_0)]$ that satisfies (3.6) is denominated by the moduli space of systems (2.3) and (3.13). We denote this moduli space as M_Λ.

Now, we will discuss the correspondence between $\mathrm{Rat}_\Lambda(n)$ and M_Λ and we will built a one-to-one mapping from $\mathrm{Rat}_\Lambda(n)$ to M_Λ.

We can obtain a flow on $\mathrm{Rat}_\Lambda(n)$ by means of (3.35), determined by $f(z)$ in the following way:

$$\begin{aligned} F(z, t_1, \ldots, t_{n-1}) &= C_0^T (zI - \Lambda)^{-1} V \\ &= C_0^T (zI - \Lambda)^{-1} e^{\sum_{k=1}^{n-1} \Lambda^k t_k} B_0 \\ &= C_0^T (zI - \Lambda)^{-1} S^{-1} Y B_0 \\ &= \left((S^{-1})^T C_0 \right)^T (zI - L_I)^{-1} Y B_0 \\ &= C^T (zI - L_I)^{-1} B, \end{aligned} \tag{3.38}$$

where C and B satisfy (3.13), L_I satisfy (2.3), and $L_I(0) = \Lambda$. Thus any $[f] \,\epsilon\, \mathrm{Rat}_\Lambda(n)$ defines a solution (L_I, B, C) determined by the initial conditions $[(\Lambda, B_0, C_0)]$. So we have a mapping

$$\beta : \mathrm{Rat}_\Lambda(n) \longrightarrow M_\Lambda. \tag{3.39}$$

Conversely, any solution $(L_I, B, C) \in M_\Lambda$ induces a flow (3.30) on $\mathrm{Rat}_\Lambda(n)$ which originates into $f(z)$ at $t = 0$. Furthermore, by (3.31) we obtain the class $[(\Lambda, B_0, C_0)]$. Thus β is an onto mapping form $\mathrm{Rat}_\Lambda(n)$ to M_Λ.

Theorem 3.6. *The mapping β defined in* (3.39) *is a bijection.*

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References

[1] R. W. Brockett and L. E. Faybusovich, “Toda flows, inverse spectral transform and realization theory,” *Systems & Control Letters*, vol. 16, no. 2, pp. 79–88, 1991.

[2] Y. Nakamura, “Geometry of rational functions and nonlinear integrable systems,” *SIAM Journal on Mathematical Analysis*, vol. 22, no. 6, pp. 1744–1754, 1991.

[3] R. Felipe and F. Ongay, “Algebraic aspects of the discrete KP hierarchy,” *Linear Algebra and Its Applications*, vol. 338, no. 1–3, pp. 1–17, 2001.

[4] M. Mulase, “Algebraic theory of the KP equations,” in *Perspectives in Mathematical Physics*, R. Penner and S. T. Yau, Eds., Lecture Notes in Mathematical Physics III, pp. 151–217, International Press, Cambridge, Mass, USA, 1994.