

Research Article

Monotone Iterative Technique for Partial Dynamic Equations of First Order on Time Scales

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This work is concerned with the monotone iterative technique for partial dynamic equations of first order on time scales and for this purpose, the existence, uniqueness, and comparison results are also established.

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1. Introduction

The study of dynamic equations on time scales is an area of research that has received a lot of attention. Many authors have been devoted to the qualitative research of ordinary differential equations on time scales. We can refer to monographs by Bohner and Peterson [1], Lakshmikantham et al. [2], and the references cited therein. With the development of technology, in recent years, some authors have explored the partial differential equations on time scales. Ahlbrandt and Morian [3] have introduced definitions of time scale derivatives and integrals for functions of two variables and obtained the results that include an Euler-Lagrange equation for double integral variational problems on time scales and a Picone identity which implies a Sturm-Picone comparison theorem for second-order elliptic partial differential equations on time scales. Jackson [4] has extended the existing ideas of the univariate case of the time scales calculus to the multivariate case.

Lakshmikantham et al. [5] investigated the monotone iterative technique for partial differential equations of first order in 1984. Nevertheless, up till now, we have not yet seen any research about monotone iterative technique for partial dynamic equations on time scales. In this paper, we will investigate the monotone iterative technique for partial dynamic equations of

first order on time scales. Firstly, in Sections 2 and 3, we will introduce the univariable calculus and multivariable calculus on time scales that will be the preliminary work in order to define the partial dynamic equations on time scales. Secondly, in Sections 4 and 5, we will present existence, uniqueness, and comparison results and provide the monotone iterative technique for such partial dynamic equations.

2. Univariable calculus on time scales

For the detail of basic notions concerned with time scales, we refer to [1, 2]. To meet the requirements in the next sections, we introduce some notions and lemma here.

A time scale is a nonempty closed subset of the real number \mathbb{R} , and we denote it by the symbol \mathbb{T} . We define the forward and backward jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ by

$$\sigma(t) = \inf \{s \in \mathbb{T} \mid s > t\}, \quad \rho(t) = \sup \{s \in \mathbb{T} \mid s < t\} \quad (2.1)$$

(supplemented by $\inf \emptyset = \sup \mathbb{T}$). A point $t \in \mathbb{T}$ is called right-scattered, right-dense, left-scattered, left-dense if $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) < t$, $\rho(t) = t$ hold, respectively. The set \mathbb{T}^k is defined to be \mathbb{T} if \mathbb{T} does not have a left-scattered maximum; otherwise it is \mathbb{T} without this left-scattered maximum. The graininess $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$.

If $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function and $t \in \mathbb{T}^k$, then the "delta-derivative" of f at the point t is defined to be the number $f^\Delta(t)$ (provided it exists) with the property that for each $\epsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon |\sigma(t) - t| \quad \forall s \in U. \quad (2.2)$$

Lemma 2.1 (see [1]). *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuously differentiable and suppose that $g : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^\Delta(t)) dh \right\} g^\Delta(t) \quad (2.3)$$

holds.

3. Multivariable calculus on time scales

In this section, we generalize existing ideas of the time scales calculus to the multivariate case.

Firstly, consider the product $\mathbb{T}^n = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$, where \mathbb{T}_i is a time scale for all $1 \leq i \leq n$, so \mathbb{T}^n is a time scale too. Then for any $x \in \mathbb{T}^n$ with $x = (x_1, x_2, \dots, x_n)$, in which $x_i \in \mathbb{T}_i$, $1 \leq i \leq n$, define the following:

- (i) the forward jump operator $\sigma : \mathbb{T}^n \rightarrow \mathbb{T}^n$ by $\sigma(x) = (\sigma(x_1), \sigma(x_2), \dots, \sigma(x_n))$, where $\sigma(x_i)$ represents the forward jump operator of $x_i \in \mathbb{T}_i$ for all $1 \leq i \leq n$. Hereafter, the forward jump operator of $x_i \in \mathbb{T}_i$ will be denoted by $\sigma(x_i) = \sigma_i(x)$;
- (ii) the back jump operator $\rho : \mathbb{T}^n \rightarrow \mathbb{T}^n$ by $\rho(x) = (\rho(x_1), \rho(x_2), \dots, \rho(x_n))$, where $\rho(x_i)$ represents the backward jump operator of $x_i \in \mathbb{T}_i$ for all $1 \leq i \leq n$. The backward jump operator of $x_i \in \mathbb{T}_i$ will be denoted by $\rho(x_i) = \rho_i(x)$;

- (iii) the graininess function $\mu : \mathbb{T}^n \rightarrow \mathbb{R}^n$ by $\mu(x) = (\mu(x_1), \mu(x_2), \dots, \mu(x_n))$, where $\mu(x_i)$ represents the graininess function of $x_i \in \mathbb{T}_i$ on the time scale \mathbb{T}_i for all $1 \leq i \leq n$. Again, from this point on the graininess function of the time scale \mathbb{T}_i for $x_i \in \mathbb{T}_i$ will be denoted by $\mu(x_i) = \mu_i(x)$;
- (iv) $\mathbb{T}^{kn} = \mathbb{T}_1^k \times \mathbb{T}_2^k \cdots \times \mathbb{T}_n^k$.

Secondly, we introduce the definitions of time-scale derivatives for the function $u(t, x)$, where $t \in \mathbb{T}$ and $x \in \mathbb{T}^n$. The forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ and $\tau : \mathbb{T}^n \rightarrow \mathbb{T}^n$ are defined by $\sigma(t) = \inf \{s_1 \in \mathbb{T} \mid s_1 > t\}$ and $\tau(x) = \inf \{s_2 \in \mathbb{T}^n \mid s_2 > x\}$. The graininess $h : \mathbb{T} \rightarrow \mathbb{R}$ is defined as $h(t) = \sigma(t) - t$. The graininess $k : \mathbb{T}^n \rightarrow \mathbb{T}^n$ is defined as $k(x) = \tau(x) - x$. We will use the notation $u^\sigma(t, x) = u(\sigma(t), x)$, $u^\tau(t, x) = u(t, \tau(x))$ and $u^{\sigma\tau}(t, x) = u(\sigma(t), \tau(x))$.

Because we will need notation for partial derivatives with respect to time-scale variables t and x , we employ lexicographic ordering for consistency. Let $u^\Gamma(t, x)$ denote the time-scale partial derivative with respect to t , in order to distinguish for the sign of Δ -derivative in Section 2, let $u^{\Delta_2}(t, x)$ denote the time-scale partial derivative with respect to x .

The following definitions of these partial derivatives are now given.

Definition 3.1. Let u be a real-valued function on $\mathbb{T} \times \mathbb{T}^n$. At $(t, x) \in \mathbb{T}^k \times \mathbb{T}^{kn}$, we say that u has a “ Γ partial derivative” $u^\Gamma(t, x)$, with respect to t , if for each $\varepsilon > 0$, there exists a neighborhood \mathcal{M}_t (open in the relative topology of \mathbb{T} of t) such that

$$|u(\sigma(t), x) - u(s_1, x) - u^\Gamma(t, x)[\sigma(t) - s_1]| \leq \varepsilon |\sigma(t) - s_1| \quad (3.1)$$

for all $s_1 \in \mathcal{M}_t$. Similarly, we say that u has a “ Δ partial derivative” denoted by $u^{\Delta_2}(t, x) \in \mathbb{R}$, with respect to x , if for each $\varepsilon > 0$ there exists a neighborhood \mathcal{M}_x of x such that

$$|u(t, \tau(x)) - u(t, s_2) - u^{\Delta_2}(t, x)[\tau(x) - s_2]| \leq \varepsilon |\tau(x) - s_2| \quad (3.2)$$

for all $s_2 \in \mathcal{M}_x$.

Finally, we define the partial derivative of $u(t, x)$ at x with respect to the variable x_i , where $t \in \mathbb{T}$ and $x = (x_1, x_2, \dots, x_n) \in \mathbb{T}^n$. Having defined the multivariable calculus as earlier, we set

$$\begin{aligned} u^{\sigma_i}(t, x) &= u(t, x_1, x_2, \dots, x_{i-1}, \sigma_i(x), x_{i+1}, \dots, x_n), \\ u_i^{\Delta_2}(t, x) &= u(t, x_1, x_2, \dots, x_{i-1}, s_3, x_{i+1}, \dots, x_n). \end{aligned} \quad (3.3)$$

Definition 3.2. Let $u : \mathbb{T} \times \mathbb{T}^n \rightarrow \mathbb{R}$ be a real-valued function and let $x = (x_1, x_2, \dots, x_n) \in \mathbb{T}^{kn}$. Then define $u^{\Delta_2^i}(t, x)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there exists a neighborhood \mathbb{U} of x_i , with $\mathbb{U} = (x_i - \delta, x_i + \delta) \cap \mathbb{T}_i$ for $\delta > 0$ such that

$$|[u^{\sigma_i}(t, x) - u_i^{\Delta_2}(t, x)] - u^{\Delta_2^i}(t, x)[\sigma_i(x) - s_3]| \leq \varepsilon |\sigma_i(x) - s_3| \quad (3.4)$$

for all $s_3 \in \mathbb{U}$. $u^{\Delta_2^i}(t, x)$ is called the partial delta derivative of $u(t, x)$ at x with respect to variable x_i .

4. Existence, uniqueness, and comparison result

In this section, we consider the initial value problem (IVP) for partial dynamic equation of first order on time scales

$$u^\Gamma + f(t, x)u^{\Delta_2} = g(t, x, u), \quad u(0, x) = \phi(x), \quad (4.1)$$

where $f \in C(\Omega, \mathbb{R}^n)$, $g \in C(\Omega \times \mathbb{R}, \mathbb{R})$, $\Omega = \{(t, x) : 0 \leq t \leq T, a \leq x \leq b, t \in \mathbb{T}^k, a, b, x \in \mathbb{T}^{kn}\}$, $f u^{\Delta_2} = \sum_{i=1}^n f_i(t, x)u^{\Delta_2^i}$ and $\phi \in C^1([a, b], \mathbb{R})$.

We define the set $\mathbb{D} = \{u \in C[\Omega_1, \mathbb{R}], u^\Gamma, u^{\Delta_2}$, and $u^{\Delta_2^i}$ are continuous on $\mathbb{T}^k \times \mathbb{T}^{kn}\}$, $\Omega_1 = \{(t, x), 0 \leq t \leq T, a \leq x \leq \sigma(b), t \in \mathbb{T}^k, a, b$, and $x \in \mathbb{T}^{kn}\}$.

Now, we begin to prove the following comparison result.

Theorem 4.1. *Assume that*

(A₀) $\alpha^\Gamma + f(t, x)\alpha^{\Delta_2} \leq g(t, x, \alpha)$, $\alpha(0, x) \leq \phi(x)$, and $\beta^\Gamma + f(t, x)\beta^{\Delta_2} \geq g(t, x, \beta)$, $\beta(0, x) \geq \phi(x)$ for $(t, x) \in \Omega_1$, where $\alpha, \beta \in \mathbb{D}$.

(A₁) $f(t, x)$ is quasimonotone nonincreasing in x for each i and $f(t, a) \leq 0$, $f(t, b) \geq 0$.

(A₂) $g(t, x, u_1) - g(t, x, u_2) \leq L(u_1 - u_2)$ whenever $u_1 \geq u_2$ for some $L \geq 0$.

Then $\alpha(t, x) \leq \beta(t, x)$ on Ω_1 .

Proof. Let us prove the theorem for strict inequalities firstly. For example, we suppose that $\alpha^\Gamma + f(t, x)\alpha^{\Delta_2} < g(t, x, \alpha)$ and $\alpha(0, x) < \phi(x)$ on Ω_1 , and prove that $\alpha(t, x) < \beta(t, x)$ on Ω_1 . If this conclusion is not true, then consider the set

$$\mathbb{Z} = \{(t, x) \in \Omega_1 \mid \alpha(t, x) \geq \beta(t, x)\}. \quad (4.2)$$

Let \mathbb{Z}_t be the projection of \mathbb{Z} on the t -axis and let $t_0 = \inf \mathbb{Z}_t$. Clearly, $t_0 > 0$ and there exists an $x_0 \in [a, \sigma(b)]$ such that

$$\begin{aligned} \alpha(t_0, x_0) &= \beta(t_0, x_0), & \alpha(\sigma(t_0), x_0) &\geq \beta(\sigma(t_0), x_0), \\ \alpha(t_0, x) &\leq \beta(t_0, x) \quad \text{for } a \leq x \leq \sigma(b). \end{aligned} \quad (4.3)$$

It then follows that $\alpha^\Gamma(t_0, x_0) \geq \beta^\Gamma(t_0, x_0)$ and if $a \leq x \leq \sigma(b)$, we also have $\alpha^{\Delta_2^i}(t_0, x_0) = \beta^{\Delta_2^i}(t_0, x_0)$, $i = 1, 2, \dots, n$. In this case, we get the following contradiction:

$$g(t_0, x_0, \alpha(t_0, x_0)) > \alpha^\Gamma + f(t_0, x_0)\alpha^{\Delta_2} \geq \beta^\Gamma + f(t_0, x_0)\beta^{\Delta_2} \geq g(t_0, x_0, \beta(t_0, x_0)). \quad (4.4)$$

If, on the other hand, for some j , $x_{0,j} = b_j$ and $x_{0,i} < b_i$, $i \neq j$, and $a < x_0$, then we have $\alpha^{\Delta_2^i}(t_0, x_0) = \beta^{\Delta_2^i}(t_0, x_0)$, $i \neq j$, and $\alpha(t_0, \sigma(x_0)) \geq \beta(t_0, \sigma(x_0))$, then we have $\alpha^{\Delta_2^j}(t_0, x_0) \geq \beta^{\Delta_2^j}(t_0, x_0)$. Hence using the assumption (A₁), we obtain $f_j(t_0, x_0) \geq f_j(t_0, b) \geq 0$ and $f_j(t_0, x_0)\alpha^{\Delta_2^j}(t_0, x_0) \geq f_j(t_0, x_0)\beta^{\Delta_2^j}(t_0, x_0)$. Consequently, we get the inequality

$$g(t_0, x_0, \alpha(t_0, x_0)) > \alpha^\Gamma + f(t_0, x_0)\alpha^{\Delta_2} \geq \beta^\Gamma + f(t_0, x_0)\beta^{\Delta_2} \geq g(t_0, x_0, \beta(t_0, x_0)), \quad (4.5)$$

which is a contradiction as before. This proves that $\alpha(t, x) < \beta(t, x)$ on Ω_1 . If one of the inequalities in (A₀) is not strict, and on one hand, if $\mu(t) = 0$, we set $\tilde{\alpha}(t, x) = \alpha(t, x) - \varepsilon e^{2Lt}$ and note that $\tilde{\alpha} < \alpha$. Then using (A₀) and (A₂), we have

$$\begin{aligned} \tilde{\alpha}^\Gamma + f(t, x)\tilde{\alpha}^{\Delta_2} &= \alpha^\Gamma + f(t, x)\alpha^{\Delta_2} - 2L\varepsilon e^{2Lt} \\ &\leq g(t, x, \alpha) - 2L\varepsilon e^{2Lt} \leq g(t, x, \tilde{\alpha}) + L\varepsilon e^{2Lt} - 2L\varepsilon e^{2Lt} < g(t, x, \tilde{\alpha}), \end{aligned} \quad (4.6)$$

and $\tilde{\alpha}(0, x) < \phi(x)$ on Ω_1 . On the other hand, if $\mu(t) \neq 0$, we set $\tilde{\alpha}(t, x) = \alpha(t, x) - \varepsilon e^{Lt}$ and note that $\tilde{\alpha} < \alpha$. Then using (A₀), (A₂), and Lemma 2.1, we have

$$\begin{aligned} \tilde{\alpha}^\Gamma + f(t, x)\tilde{\alpha}^{\Delta_2} &= \alpha^\Gamma + f(t, x)\alpha^{\Delta_2} - \varepsilon \left(\frac{e^{Lt}}{\mu} (e^{L\mu} - 1) \right) \\ &\leq g(t, x, \alpha) - \frac{\varepsilon e^{Lt}}{\mu} (e^{L\mu} - 1) \leq g(t, x, \tilde{\alpha}) + L\varepsilon e^{Lt} - \frac{\varepsilon e^{Lt}}{\mu} (e^{L\mu} - 1) \\ &= g(t, x, \tilde{\alpha}) + \frac{L\mu + 1 - e^{L\mu}}{\mu} \varepsilon e^{Lt} < g(t, x, \tilde{\alpha}), \end{aligned} \quad (4.7)$$

and $\tilde{\alpha}(0, x) < \phi(x)$ on Ω_1 . Thus the foregoing arguments imply that $\alpha(t, x) < \beta(t, x)$ on Ω_1 . Taking limit as $\varepsilon \rightarrow 0$, we then get $\alpha(t, x) \leq \beta(t, x)$ on Ω_1 , and the proof is complete. \square

Theorem 4.2. *Assume that (A₁) and (A₂) hold. Suppose further that (A₃) for each $(t_0, x_0) \in \Omega$, there exists a unique solution $x(t_0, x_0)$ of*

$$x^\Delta = f(t, x), \quad x(t_0) = x_0, \quad (4.8)$$

on $0 \leq t \leq T$, $x(t, t_0, x_0)$ is continuously differentiable with respect to (t_0, x_0) . Assume (A₄) for each $x_0 \in [a, b]$ and $y_0 \in \mathbb{R}$, there exists a unique solution $y(t, 0, y_0; x_0)$ of

$$y^\Delta = g(t, x(t, 0, x_0), y), \quad y(0) = y_0, \quad (4.9)$$

on $0 \leq t \leq T$, where $x(t, 0, x_0)$ is the unique solution of (4.8) and $y(t, 0, y_0; x_0)$ is continuously differentiable with respect to (y_0, x_0) .

Then there exists a unique solution $u(t, x) = y(t, 0, \phi(x(0, t, x)); x(0, t, x))$ for the problem (4.1) on Ω_1 .

Proof. By (A₃) and (A₄), $x(t, t_0, x_0)$, $y(t, 0, y_0; x_0)$ are unique solutions of (4.8) and (4.9), respectively, on $0 \leq t \leq T$. Choose $y_0 = \phi(x_0)$ and note that if $x = x(t, 0, x_0)$, then because of uniqueness, $x_0 = x(0, t, x)$. Hence, $y(t, 0, \phi(x(0, t, x)); x(0, t, x))$ is a unique solution of (4.9). Then we have

$$y^\Delta = y^\Gamma + y^{\Delta_2} x^\Delta = y^\Gamma + f(t, x)y^{\Delta_2} = g(t, x, y), \quad y(0, x) = \phi(x), \quad (4.10)$$

that is, $u(t, x) = y(t, 0, \phi(x(0, t, x)); x(0, t, x))$ satisfies (4.1) and consequently, $u(t, x)$ is a solution of the problem (4.1).

To show uniqueness of solution of (4.1), suppose that $u_1(t, x)$, $u_2(t, x)$ are two solutions of (4.1) on Ω_1 . Then setting $\alpha = u_1$, $\beta = u_2$ and applying Theorem 4.1, we get $u_1(t, x) \leq u_2(t, x)$ on Ω_1 . Similarly we can prove that $u_1(t, x) \geq u_2(t, x)$ on Ω_1 . Hence the proof is complete. \square

5. Monotone iterative technique

we are now in a position to describe the monotone iterative technique which yields monotone sequences on time scales. We prove the following result specifically.

Theorem 5.1. *Assume that (A_0) , (A_1) , and (A_3) hold with $\alpha \leq \beta$ on Ω_1 . Suppose further (A_5) for some $M > 0$, $g(t, x, u_1) - g(t, x, u_2) \geq -M(u_1 - u_2)$ whenever $\alpha \leq u_2 \leq u_1 \leq \beta$ on Ω_1 . Assume (A_6) for each $x_0 \in [a, b]$ and $y_0 \in \mathbb{R}$, there exists a unique solution $y(t, 0, y_0; x_0)$ of*

$$y^\Delta = g(t, x(t, 0, x_0), y; \eta(t, x(t, 0, x_0))) - M(y - \eta), \quad y(0) = y_0, \quad (5.1)$$

whenever $\alpha \leq \eta \leq \beta$.

Then there exist monotone sequences $\{\alpha_n(t, x)\}$, $\{\beta_n(t, x)\}$, and the functions $\rho(t, x)$, $r(t, x)$ such that if u is any solution of (4.1), then

$$\alpha \leq \alpha_1 \leq \dots \leq \alpha_n \leq \rho \leq u \leq r \leq \beta_n \leq \dots \leq \beta_1 \leq \beta \quad \text{on } \Omega_1. \quad (5.2)$$

Proof. Consider the linear IVP

$$u^\Gamma + f(t, x)u^{\Delta_2} = G(t, x, u; \eta), \quad u(0, x) = \phi(x), \quad (5.3)$$

where $G(t, x, u; \eta) = g(t, x, \eta) - M(u - \eta)$ and $\eta \in C(\Omega_1, \mathbb{R})$ is such that $\alpha \leq \eta \leq \beta$ on Ω_1 .

By (A_5) , we have

$$g(t, x, \alpha) - G(t, x, \alpha; \eta) = g(t, x, \alpha) - g(t, x, \eta) + M(u - \eta) \leq M(\eta - \alpha) + M(\alpha - \eta) = 0, \quad (5.4)$$

so it follows that $\alpha^\Gamma + f(t, x)\alpha^{\Delta_2} \leq G(t, x, \alpha; \eta)$, $\alpha(0, x) \leq \phi(x)$ on Ω_1 . Similarly, we obtain that $\beta^\Gamma + f(t, x)\beta^{\Delta_2} \leq G(t, x, \beta; \eta)$, $\beta(0, x) \leq \phi(x)$ on Ω_1 . Hence (A_0) holds. Also, if $u_1 \geq u_2$,

$$G(t, x, u_1; \eta) - G(t, x, u_2; \eta) = -M(u_1 - u_2) \leq L(u_1 - u_2), \quad (5.5)$$

and therefore (A_2) is satisfied for G . Furthermore, by (A_6) , (A_4) is satisfied relative to

$$y^\Delta = G(t, x(t, 0, x_0), y; \eta(t, x(t, 0, x_0))), \quad y(0) = y_0. \quad (5.6)$$

As a result, by Theorem 4.2, there exists a unique solution $u(t, x)$ of (5.3) on Ω_1 for every $\eta \in C(\Omega_1, \mathbb{R})$ such that $\alpha \leq \eta \leq \beta$ on Ω_1 .

Defining a mapping A by $A\eta = u$, where u is the unique solution of (5.3) corresponding to η . Concerning this mapping A , we will show that (i) $\alpha \leq A\alpha$, $\beta \geq A\beta$, and (ii) A is monotone on the sector $[\alpha, \beta]$, namely, if $\alpha \leq \eta_1 \leq \eta_2 \leq \beta$, then $A\eta_1 \leq A\eta_2$.

Let $\eta = \alpha$ and let $A\alpha = \alpha_1$, where α_1 is the unique solution of (5.3). Then we have $\alpha^\Gamma + f(t, x)\alpha^{\Delta_2} \leq G(t, x, \alpha; \alpha)$, $\alpha(0, x) \leq \phi(x)$, and $\alpha_1^\Gamma + f(t, x)\alpha_1^{\Delta_2} = G(t, x, \alpha; \alpha)$, $\alpha_1(0, x) = \phi(x)$ on Ω_1 . By Theorem 4.1, we see that $\alpha \leq \alpha_1 = A\alpha$. Similarly, we can show that $\beta \geq A\beta$.

To prove (ii), let $\eta_1, \eta_2 \in C[\Omega_1, \mathbb{R}]$ be such that $\alpha \leq \eta_1 \leq \eta_2 \leq \beta$ and let $A\eta_1 = u_1$, $A\eta_2 = u_2$, where u_1, u_2 are the unique solutions of (5.3) corresponding to $\eta = \eta_1$, $\eta = \eta_2$, respectively. Then, $u_1^\Gamma + f(t, x)u_1^{\Delta_2} = G(t, x, u_1; \eta_1) \leq G(t, x, u_1; \eta_2)$ and $u_2^\Gamma + f(t, x)u_2^{\Delta_2} = G(t, x, u_2; \eta_2)$. Also, $u_1(0, x) = u_2(0, x)$. Hence by Theorem 4.1, we have $u_1 \leq u_2$ on Ω_1 . This proves $A\eta_1 \leq A\eta_2$.

Consider the sequence $\{\alpha_n(t, x)\}$ and note that $\alpha_n(t, x) = y_n[t, 0, \phi(x(0, t, x)); x(0, t, x)]$ on Ω_1 , where $y_n = y_n(t, 0, y_0; x_0)$ is the unique solution of (5.6) such that $y_0 = \phi(x_0)$. Thus $\alpha_n(t, x(t, 0, x_0)) = y_n(t, 0, \phi(x_0); x_0)$ and $\alpha \leq y_n \leq \beta$. Since $\{y_n\}$ is monotone sequence, it is easy to conclude that $y_n(t, 0, \phi(x_0); x_0)$ converges uniformly and monotonically as $n \rightarrow \infty$. Suppose that $\lim_{n \rightarrow \infty} y_n(t, 0, \phi(x_0); x_0) = y(t, 0, \phi(x_0); x_0)$ on $0 \leq t \leq T$. Then it is clear that $y^\Delta(t, 0, \phi(x_0); x_0) = g(t, x(t, 0, x_0), y(t, 0, \phi(x_0)); x_0)$, $y(0) = \phi(x_0)$. Consequently, we can now define $\rho(t, x) = y(t, 0, \phi(x(0, t, x)); x(0, t, x))$ on Ω_1 . Similar arguments hold relative to the sequence $\{\beta_n(t, x)\}$ and one defines $r(t, x) = \tilde{y}(t, 0, \phi(x(0, t, x)); x(0, t, x))$ on Ω_1 .

Finally, we show that $\alpha \leq \rho \leq u \leq r \leq \beta$ on Ω_1 , where u is any solution of (4.1) such that $\alpha \leq u \leq \beta$ on Ω_1 . For this, it is enough to show that $\alpha_n \leq u \leq \beta_n$ on Ω_1 and this we do by induction. Suppose that $\alpha_k \leq u \leq \beta_k$ for some k on Ω_1 . Then we have

$$\begin{aligned} \alpha_{k+1}^\Gamma + f(t, x)\alpha_{k+1}^{\Delta_2} &= G(t, x, \alpha_{k+1}; \alpha_k), & \alpha_{k+1}(0, x) &= \phi(x), \\ u^\Gamma + f(t, x)u^{\Delta_2} &= g(t, x, u) \geq G(t, x, u; \alpha_k), & u(0, x) &= \phi(x). \end{aligned} \quad (5.7)$$

By Theorem 4.1, we have $\alpha_{k+1} \leq u$ on Ω_1 . We can show similarly that $u \leq \beta_{k+1}$. Hence it follows that $\alpha_k \leq u \leq \beta_k$ on Ω_1 for all n , which prove the claim. \square

Remark 5.2. We note that if (A₄) holds, then ρ and r are actually solutions of (4.1).

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