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Research Article

Convergence of Weighted Linear Process for ρ-Mixing Random Variables

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A central limit theorem and a functional central limit theorem are obtained for weighted linear process of ρ-mixing sequences for the $X_t = \sum_{i=0}^{\infty} a_i Y_{t-i}$, where $\{Y_i, 0 \leq i < \infty\}$ is a sequence of ρ-mixing random variables with $EY_i = 0$, $0 < EY_i^2 < \infty$, $\sum_{i=1}^{\infty} \rho(2^i) < \infty$. The results obtained generalize the results of Liang et al. (2004) to ρ-mixing sequences.

1. Introduction and main results

A sequence $\{X_n, n \geq 1\}$ is said to be a ρ-mixing sequence, if $n \to \infty$, we have

$$\rho(n) = \sup_{k \geq 1, X \in L^2(F_1^k), Y \in L^2(F_{k+n}^{\infty})} \left| \operatorname{cov}(X, Y) \right| / \|X\|_2 \|Y\|_2 \longrightarrow 0, \tag{1.1}$$

where F_n^m is a σ-field that is generated by the random variables $X_n, X_{n+1}, \dots, X_m$. Here $\|X\|_p = (E|X|^p)^{1/p}$.

We assume that $\{Y_i, 0 \leq i < \infty\}$ is a ρ-mixing sequence. Let X_t be a linear process generated by Y_t, that is,

$$X_t = \sum_{i=0}^{\infty} a_i Y_{t-i}, \tag{1.2}$$

where

$$\sum_{i=0}^{\infty} |a_i| < \infty. \tag{1.3}$$

And let $EY_i = 0$, $0 < EY_i^2 < \infty$, that is,

$$\sigma^2 = \lim_{n\to\infty} E\frac{\left(\sum_{i=1}^n Y_i\right)^2}{n} > 0. \tag{1.4}$$

Let $S_n = \sum_{i=1}^n X_i, \tau^2 = \sigma^2(\sum_{i=0}^\infty a_i)^2, \xi_n(u) = (S_{[nu]})/(\tau\sqrt{n})$.

Let $\widetilde{X}_t = (\sum_{i=0}^\infty a_i)Y_t$, $\widetilde{S}_n = \sum_{t=1}^n \widetilde{X}_t$, $\widetilde{\xi}_n(u) = (\widetilde{S}_{[nu]})/(\tau\sqrt{n})$.

For the linear process, Ho and Hsing [1], Phillips and Solo [2] and Wang et al. [3] got central limit theorems (functional central limit theorems) for linear process under independent assumptions. Kim and Baek [4] got a central limit theorem (functional central limit theorem) for strongly stationary linear process under linear positive quadrant-dependent assumptions.

As for NA random variable, Liang et al. [5] obtained the following result.

Theorem 1.1. *Let $\{Y_n, n \ge 0\}$ be a sequence of NA random variables with $EY_i = 0$ and $\sum_{j:|k-j|\ge u} |\operatorname{cov}(Y_k, Y_j)| \to 0$ as $n \to \infty$ uniformly for $k \ge 1$. Assume that $\{b_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying $\max_{1\le i\le n} |b_{ni}| = O(n^{-1/2})$, satisfying $\sum_{i=1}^n b_{ni}^2 = O(1)$, $\max_{1\le i\le n} |b_{ni}| \to 0$, as $n \to \infty$ and that $\operatorname{Var}(\sum_{i=1}^n b_{ni}X_i) \to 1$. $\sum_{i=1}^\infty \rho(2^i) < \infty$, $\{Y_n^2, n \ge 0\}$ is uniformly integrable, (1.4) holds, and let $\{Y_n, n \ge 0\}$ be a linear process defined by (1.2). Suppose that (1.3) holds, then*

$$\sum_{i=1}^n b_{ni}X_i \xrightarrow{d} N\left(0, \left(\sum_{i=0}^\infty |a_i|\right)^2\right). \tag{1.5}$$

We are inspired by Wang et al. [3] and Salvadori [6]. Salvadori [3] have obtained Linear combinations of order statistics to estimate the quantiles of generalized Pareto and extreme values distributions. In this paper, we obtain a central limit theorem (functional central limit theorem) for linear process under ρ-mixing sequence assumptions. The results obtained generalize the results of Liang et al. [5] to ρ-mixing sequences. More precisely, we will prove the following theorem.

Theorem 1.2. *Let $\{Y_n, n \ge 0\}$ be a ρ-mixing sequence of indentically distributed random variables with $EY_i = 0$, $0 < EY_i^2 < \infty$, $\sum_{i=1}^\infty \rho(2^i) < \infty$, $\{Y_n^2, n \ge 0\}$ is uniformly integrable, (1.4) holds, and let $\{Y_n, n \ge 0\}$ be a linear process defined by (1.2). Suppose that (1.3) holds, let $\{b_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying $\max_{1\le i\le n} |b_{ni}| = O(n^{-1/2})$, then*

$$\frac{\sum_{i=1}^n b_{ni}X_i}{\tau} \xrightarrow{d} N(0,1), \qquad W_n \Longrightarrow W, \tag{1.6}$$

where $W_n(t) = \sum_{i=1}^{[nt]} Y_i/\sigma\sqrt{n}$ and $\{W(t); t \ge 0\}$ is a standard Brownian motion.

The weak convergence of Theorem 1.2 is quite useful in characterizing the limit distribution of various statistics arising from the inference of econometric theory, when the economic time series $\{y_t\}$ defined as

$$y_t = \alpha y_{t-1} + X_t, \quad t = 1, 2, \ldots, \tag{1.7}$$

where y_0 is a constant with probability one. The least squares estimator of α is given by

$$\widehat{\alpha}_n = \frac{\sum_{t=1}^{n} y_t y_{t-1}}{\sum_{t=1}^{n} y_t^2}. \tag{1.8}$$

To test $\alpha = 1$ against $\alpha < 1$, that is, the unit root test, a key step is to derive the limit distribution of the DF (Dickey-Feller) test statistic

$$n(\widehat{\alpha}_n - 1) = \frac{\sum_{t=1}^{n} y_{t-1}(y_t - y_{t-1})}{n^{-2}\sum_{t=1}^{n} y_{t-1}^2}. \tag{1.9}$$

As shown by Phillips [7] and Wang et al. [3] obtained the limit distribution of $n(\widehat{\alpha}_n - 1)$ when $\{X_t\}$ is a linear process generated by an i.i.d. sequence under some conditions. In this paper, we obtained the limit distribution of $n(\widehat{\alpha}_n - 1)$ when $\{X_t\}$ is a linear process under ρ-mixing random variables assumptions.

2. Proof of main theorem

In order to proof Theorem 1.2, we need the following lemmas.

Lemma 2.1 (see [8]). *Let $\{Y_n, n \geq 1\}$ be a centered ρ-mixing sequence, $E|Y_i|^p < \infty$, for some $p \geq 2$, then there exists a positive constant $C = C(p,\rho(\cdot))$, such that*

$$\begin{aligned} E\max_{1\leq m\leq n}\left|\sum_{m=1}^{n} Y_m\right|^p \leq C\Bigg\{&\exp\left(C\sum_{i=0}^{[\log n]}\rho(2^i)\right)\left(n\max_{1\leq m\leq n}E|Y_m|^2\right)^{p/2} \\ &+ n\exp\left(C\sum_{i=0}^{[\log n]}\rho^{2/p}(2^i)\right)\left(\max_{1\leq m\leq n}E|Y_m|^p\right)\Bigg\}. \end{aligned} \tag{2.1}$$

Lemma 2.2 ([9]). *Let $\{Y_n, n\geq 1\}$ be a ρ-mixing sequence with $EY_i=0$, $EY_i^2<\infty$, $\sum_{i=1}^{\infty}\rho(2^i)<\infty$, $\{Y_n^2, n \geq 0\}$ is uniformly integrable, and (1.4)holds, then*

$$\frac{\sum_{i=1}^{n} Y_i}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1), \qquad W_n \Longrightarrow W, \tag{2.2}$$

where $W_n(t) = \sum_{i=1}^{[nt]} Y_i/\sigma\sqrt{n}$ and $\{W(t); t \geq 0\}$ is a standard Brownian motion.

3. Proof of Theorem 1.2

It is clear that

$$\begin{aligned} \widetilde{S}_k = \sum_{t=1}^{k}\widetilde{X}_t &= \sum_{t=1}^{k}\left(\sum_{i=0}^{k-t} a_i\right)Y_t + \sum_{t=1}^{k}\left(\sum_{i=k-t+1}^{\infty} a_i\right)Y_t \\ &= \sum_{t=1}^{k}\left(\sum_{i=0}^{t-1} a_i Y_{t-i}\right) + \sum_{t=1}^{k}\left(\sum_{i=k-t+1}^{\infty} a_i\right)Y_t. \end{aligned} \tag{3.1}$$

Then

$$\widetilde{S}_k - S_k = -\sum_{t=1}^{k}\left(\sum_{i=t}^{\infty} a_i Y_{t-i}\right) + \sum_{t=1}^{k}\left(\sum_{i=k-t+1}^{\infty} a_i\right) Y_t =: A + B. \tag{3.2}$$

First we proof

$$n^{-1/2} \max_{1\le k\le n} \left| \widetilde{S}_k - S_k \right| \xrightarrow{P} 0. \tag{3.3}$$

In order to proof (3.3), we need only to show

$$n^{-1/2} \max_{1\le k\le n} |A| \xrightarrow{P} 0, \tag{3.4}$$

$$n^{-1/2} \max_{1\le k\le n} |B| \xrightarrow{P} 0. \tag{3.5}$$

Using the Minkowsky inequality, Lemma 2.1 with $p = 2$ and the dominated convergence theorem, then

$$\begin{aligned} n^{-1} E \max_{1\le k\le n} \left| \sum_{t=1}^{k}\sum_{i=t}^{\infty} a_i Y_{t-i} \right|^2 &= n^{-1} E \max_{1\le k\le n} \left| \sum_{i=1}^{\infty}\sum_{t=1}^{i\wedge k} a_i Y_{t-i} \right|^2 \\ &\le n^{-1} E\left(\sum_{i=1}^{\infty} |a_i| \max_{1\le k\le n} \left| \sum_{t=1}^{i\wedge k} Y_{t-i} \right| \right)^2 \\ &\le n^{-1} \left(\sum_{i=1}^{\infty} |a_i| \left(E \max_{1\le k\le n} \left| \sum_{t=1}^{i\wedge k} Y_{t-i} \right|^2 \right)^{1/2} \right)^2 \\ &\le n^{-1} \left(\sum_{i=1}^{\infty} |a_i| \right)^2 C \exp\left(C \sum_{i=0}^{[\log n]} \rho(2^i) \right) (i\wedge n) E Y_1^2 \\ &\le C n^{-1} \left(\sum_{i=1}^{\infty} |a_i| (i\wedge n)^{1/2} \right)^2 = o(1). \end{aligned} \tag{3.6}$$

By (3.6), we have (3.4).

Because

$$\begin{aligned} B &= \sum_{t=1}^{k}\left(\sum_{i=k-t+1}^{\infty} a_i \right) Y_t \\ &= \sum_{i=1}^{k} a_i \sum_{t=k-i+1}^{k} Y_t + \sum_{i=k+1}^{\infty} a_i \sum_{t=1}^{k} Y_t =: B_1 + B_2. \end{aligned} \tag{3.7}$$

Let $\{p_n\}$ be a positive integers $\{p_n\}$ such that $p_n \to \infty$ and $p_n/n \to 0$, we have

$$\begin{aligned} n^{-1/2} \max_{1 \le k \le n} |B_2| &\le \left(\sum_{i=0}^{\infty} |a_i|\right) n^{-1/2} \max_{1 \le k \le p_n} \left|\sum_{i=1}^{k} Y_i\right| \\ &\quad + \left(\sum_{i=p_n+1}^{\infty} |a_i|\right) n^{-1/2} \max_{1 \le k \le n} \left|\sum_{i=1}^{k} Y_i\right| \\ &=: B_{21} + B_{22}. \end{aligned} \tag{3.8}$$

Using Lemma 2.1 with $p = 2$, we have

$$\begin{aligned} E(B_{21})^2 &= \left(\sum_{i=0}^{\infty} |a_i|\right)^2 n^{-1} E \max_{1 \le k \le p_n} \left|\sum_{i=1}^{k} Y_i\right|^2 \\ &\le \left(\sum_{i=0}^{\infty} |a_i|\right)^2 n^{-1} C \exp\left(C \sum_{i=0}^{[\log n]} \rho(2^i)\right) p_n E Y_1^2 \\ &\le C \left(\sum_{i=0}^{\infty} |a_i|\right)^2 \left(\frac{p_n}{n}\right) = o(1). \end{aligned} \tag{3.9}$$

Using Lemma 2.1 with $p = 2$, we have

$$\begin{aligned} E(B_{22})^2 &= \left(\sum_{i=p_n+1}^{\infty} |a_i|\right)^2 n^{-1} E \max_{1 \le k \le n} \left|\sum_{i=1}^{k} Y_i\right|^2 \\ &\le \left(\sum_{i=p_n+1}^{\infty} |a_i|\right)^2 n^{-1} C \exp\left(C \sum_{i=0}^{[\log n]} \rho(2^i)\right) n E Y_1^2 \\ &\le C \left(\sum_{i=p_n+1}^{\infty} |a_i|\right)^2 = o(1). \end{aligned} \tag{3.10}$$

By (3.8), (3.9), and (3.10), when $n \to \infty$, we have

$$n^{-1/2} \max_{1 \le k \le n} |B_2| \xrightarrow{P} 0. \tag{3.11}$$

Next, when $n \to \infty$, we want to proof

$$L_n = n^{-1/2} \max_{1 \le k \le n} |B_1| \xrightarrow{P} 0. \tag{3.12}$$

For each $m \ge 1$, let

$$B_{1,m} = \sum_{i=1}^{k} b_i \sum_{t=k-i+1}^{k} Y_t, \tag{3.13}$$

where $b_i = a_i I\ (i \le m)$. Let

$$L_{n,m} = n^{-1/2} \max_{1 \le k \le n} |B_{1,m}|, \tag{3.14}$$

for each $m \geq 1$, when $n \to \infty$, then

$$L_{n,m} \leq (|a_1| + \cdots + |a_m|)n^{-1/2}(|Y_1| + \cdots + |Y_m|) \xrightarrow{P} 0, \tag{3.15}$$

for all $\varepsilon > 0$, by Lemma 2.1, we have

$$\begin{aligned} P(|L_n - L_{n,m}| > \varepsilon) &\leq \varepsilon^{-2}(L_n - L_{n,m})^2 \\ &\leq \varepsilon^{-2}n^{-1}E\max_{m\leq k\leq n}\left|\sum_{i=1}^{k}(a_i - b_i)(Y_k + \cdots + Y_{k-i+1})\right|^2 \\ &\leq \varepsilon^{-2}n^{-1}E\max_{m\leq k\leq n}\left(\sum_{i=m+1}^{k}|a_i|\left(\sum_{i=1}^{k}Y_i - \sum_{i=1}^{k-i}Y_i\right)\right)^2 \\ &\leq 4\varepsilon^{-2}\left(\sum_{i=m+1}^{\infty}|a_i|\right)^2 n^{-1}E\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}Y_i\right|^2 \\ &\leq 4\varepsilon^{-2}\left(\sum_{i=m+1}^{\infty}|a_i|\right)^2 n^{-1}C\exp\left(C\sum_{i=0}^{[\log n]}\rho(2^i)\right)nEY_1^2 \\ &\leq C\left(\sum_{i=m+1}^{\infty}|a_i|\right)^2 \longrightarrow 0, \end{aligned} \tag{3.16}$$

when $m \to \infty$. By (3.16), we have

$$|L_n - L_{n,m}| \xrightarrow{P} 0. \tag{3.17}$$

Using (3.15) and (3.17), we have (3.12). By (3.11), (3.12), and (3.7), we have (3.5). Therefore we have (3.3). By $\max_{1\leq i\leq n}|b_{ni}| = O(n^{-1/2})$, and (3.3), then

$$\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}b_{ni}\widetilde{X}_i - \sum_{i=1}^{k}b_{ni}X_i\right| \xrightarrow{P} 0. \tag{3.18}$$

By Lemma 2.2 and (1.4), we have

$$\frac{\widetilde{S}_n}{\tau} \xrightarrow{d} N(0,1), \qquad \widetilde{\xi}_n \Longrightarrow W, \tag{3.19}$$

and by (3.3), we have

$$\frac{S_n}{\tau} \xrightarrow{d} N(0,1), \qquad \xi_n \Longrightarrow W. \tag{3.20}$$

Now, we complete the proof of Theorem 1.2.

Remark 3.1. Theorem 1.2 generalizes Theorem A to ρ-mixing sequences.

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References

[1] H.-C. Ho and T. Hsing, "Limit theorems for functionals of moving averages," *The Annals of Probability*, vol. 25, no. 4, pp. 1636–1669, 1997.

[2] P. C. B. Phillips and V. Solo, "Asymptotics for linear processes," *The Annals of Statistics*, vol. 20, no. 2, pp. 971–1001, 1992.

[3] Q. Wang, Y.-X. Lin, and C. M. Gulati, "The invariance principle for linear processes with applications," *Econometric Theory*, vol. 18, no. 1, pp. 119–139, 2002.

[4] T.-S. Kim and J.-I. Baek, "A central limit theorem for stationary linear processes generated by linearly positively quadrant-dependent process," *Statistics & Probability Letters*, vol. 51, no. 3, pp. 299–305, 2001.

[5] H.-Y. Liang, D.-X. Zhang, and J.-I. Baek, "Convergence of weighted sums for dependent random variables," *Journal of the Korean Mathematical Society*, vol. 41, no. 5, pp. 883–894, 2004.

[6] G. Salvadori, "Linear combinations of order statistics to estimate the quantiles of generalized pareto and extreme values distributions," *Stochastic Environmental Research and Risk Assessment*, vol. 17, no. 1-2, pp. 116–140, 2003.

[7] P. C. B. Phillips, "Time series regression with a unit root," *Econometrica*, vol. 55, no. 2, pp. 277–301, 1987.

[8] Q. M. Shao, "Maximal inequalities for partial sums of ρ-mixing sequences," *The Annals of Probability*, vol. 23, no. 2, pp. 948–965, 1995.

[9] Z. Lin and C. Lu, *Limit Theory for Mixing Dependent Random Variables*, vol. 378 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

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