STABILITY OF DIFFERENCE ANALOGUE OF LINEAR MATHEMATICAL INVERTED PENDULUM

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A sufficient condition to preserve the property of asymptotic stability for a difference analogue of the linear mathematical inverted pendulum is obtained.

1. Statement of the problem

To use numerical investigation of functional differential equations it is very important to know if difference analogues of the considered differential equations have the reliability to preserve some general properties of these equations, in particular, property of stability. This problem is considered here by investigation of a difference analogue of the linear mathematical inverted pendulum.

The problem of stabilization of the mathematical inverted pendulum is very popular among the researches (see, for instance [1, 2, 3, 5, 13, 14]). The linearized mathematical model of the controlled inverted pendulum can be described by the following linear differential equation of second order

$$\ddot{x}(t) - ax(t) = u(t), \quad a > 0,\ t \geq 0. \tag{1.1}$$

The classical way of stabilization of system (1.1) uses the control $u(t) = -b_1 x(t) - b_2 \dot{x}(t)$, where $b_1 > a$, $b_2 > 0$. But this type of control which represents an instantaneous feedback is quite difficult to realize because usually we need some finite time to make measurements of the coordinates and velocities, to treat the results of the measurements and to implement them in the control action.

Unlike of the classical way of stabilization in which the stabilized control is a linear combination of the state and velocity of the pendulum another way of stabilization was proposed in [4]. There it was supposed that only the trajectory of the pendulum can be observed and stabilized control depends on the whole trajectory of the pendulum, that is

$$u(t) = \int_0^\infty dK(\tau) x(t - \tau), \tag{1.2}$$

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where the kernel $K(\tau)$ is a function of bounded variation on $[0,\infty]$ and the integral is understood in the Stiltjes sense. It means, in particular, that both distributed and discrete delays can be used depending on the concrete choice of the kernel $K(\tau)$. The initial condition for the system (1.1), (1.2) has the form

$$x(s) = \varphi(s), \qquad \dot{x}(s) = \dot{\varphi}(s), \quad s \le 0, \tag{1.3}$$

where $\varphi(s)$ is a given continuously differentiable function.

Definition 1.1. The trivial solution of system (1.1)–(1.3) is called stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\max\{|x(t)|, |\dot{x}(t)|\} < \epsilon$ for all $t \ge 0$ if $\|\varphi\| = \sup_{s\le 0}(|\varphi(s)| + |\dot{\varphi}(s)|) < \delta$. If, besides, $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} \dot{x}(t) = 0$ for every initial function φ, then the trivial solution of system (1.1)–(1.3) is called asymptotically stable.

Put

$$a_1 = -a - k_0, \qquad k_i = \int_0^\infty \tau^i dK(\tau), \quad i = 0,1, \qquad \hat{k}_2 = \int_0^\infty \tau^2 \,|\, dK(\tau) \,|\,. \tag{1.4}$$

Theorem 1.2 (see [4]). *Let*

$$a_1 > 0, \qquad k_1 > 0, \tag{1.5}$$

$$\hat{k}_2 < k_m = \frac{4}{1 + \sqrt{1 + \left((1 + a_1)/k_1\right)^2}}. \tag{1.6}$$

Then the trivial solution of system (1.1)–(1.3) is asymptotically stable.

It is shown also [4] that inequalities (1.5) are necessary conditions for asymptotic stability of the trivial solution of system (1.1)–(1.3) but inequality (1.6) is only sufficient one.

Below the mathematical model of the controlled inverted pendulum (1.1)–(1.3) is considered in the following simple form

$$\ddot{x}(t) - ax(t) = b_1 x(t - h_1) + b_2 x(t - h_2), \quad t \ge 0. \tag{1.7}$$

Here $a > 0$, b_1, b_2, $h_1 > 0$, $h_2 > 0$ are given arbitrary numbers. From (1.4) it follows that for equation (1.7)

$$k_0 = b_1 + b_2, \qquad k_1 = b_1 h_1 + b_2 h_2, \qquad \hat{k}_2 = |\, b_1 \,| h_1^2 + |\, b_2 \,| h_2^2. \tag{1.8}$$

The main conclusion of our investigation here can be formulated in the following way: if conditions (1.5), (1.6) hold then the trivial solution of equation (1.7) is asymptotically stable and there exists enough small step of discretization of this equation that the trivial solution of the corresponding difference equation is asymptotically stable too.

Note, that the conditions for asymptotic stability are obtained here by virtue of Kolmanovskii and Shaikhet's general method of Lyapunov functionals construction [6, 7, 8, 9, 10, 11, 12, 15] which is applicable for both differential and difference equations, both for deterministic and stochastic systems with delay.

2. Construction of difference analogue

Transform equation (1.7) to a system of the equations

$$\dot{x}(t) = y(t), \qquad \dot{y}(t) = ax(t) + \sum_{l=1}^{2} b_l x(t-h_l). \tag{2.1}$$

To construct a difference analogue of system (2.1) put

$$x_i = x(t_i), \quad t_i = i\tau, \qquad h_1 = m_1\tau, \quad h_2 = m_2\tau, \quad \tau > 0. \tag{2.2}$$

A difference analogue of system (2.1) can be considered in the form

$$x_{i+1} = x_i + \tau y_i, \quad y_{i+1} = y_i + \tau\Big(ax_i + \sum_{l=1}^{2} b_l x_{i-m_l}\Big). \tag{2.3}$$

From the first equation of system (2.3) we have

$$x_i = x_{i-m_l} + \tau \sum_{j=i-m_l}^{i-1} y_i, \quad l = 1,2. \tag{2.4}$$

From here and (1.8) it follows

$$\sum_{l=1}^{2} b_l x_{i-m_l} = k_0 x_i - \tau \sum_{l=1}^{2} b_l \sum_{j=i-m_l}^{i-1} y_i. \tag{2.5}$$

Substituting (2.5) into the second equation of system (2.3) and using (1.4) we obtain

$$y_{i+1} = y_i - \tau a_1 x_i - \tau^2 \sum_{l=1}^{2} b_l \sum_{j=i-m_l}^{i-1} y_j. \tag{2.6}$$

Put

$$F_i = \tau^2 \sum_{l=1}^{2} b_l \sum_{j=i-m_l}^{i-1} (j-i+1+m_l)\, y_j, \qquad q_1 = b_1 m_1 + b_2 m_2 = \tau^{-1} k_1. \tag{2.7}$$

Calculating $\Delta F_i = F_{i+1} - F_i$, we have

$$\begin{aligned} \Delta F_i &= \tau^2 \sum_{l=1}^{2} b_l \Bigg[\sum_{j=i+1-m_l}^{i} (j-i+m_l)\, y_j - \sum_{j=i-m_l}^{i-1} (j-i+1+m_l)\, y_j \Bigg] \\ &= \tau^2 \sum_{l=1}^{2} b_l \Big(m_l y_i - \sum_{j=i-m_l}^{i-1} y_i \Big) = \tau \Big(k_1 y_i - \tau \sum_{l=1}^{2} b_l \sum_{j=i-m_l}^{i-1} y_j \Big). \end{aligned} \tag{2.8}$$

From here and (2.6) it follows

$$y_{i+1} = -\tau a_1 x_i + (1 - \tau k_1)\, y_i + \Delta F_i. \tag{2.9}$$

So, system (2.3) can be written in the matrix form

$$z(i+1) = Az(i) + \Delta F(i), \tag{2.10}$$

where

$$z(i) = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \qquad F(i) = \begin{pmatrix} 0 \\ F_i \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & \tau \\ -\tau a_1 & 1 - \tau k_1 \end{pmatrix}. \tag{2.11}$$

3. Stability conditions of the auxiliary equation

Following the general method of Lyapunov functionals construction [7] at first consider the auxiliary equation

$$z(i+1) = Az(i) \tag{3.1}$$

which can be written in a scalar form

$$x_{i+2} = A_0 x_{i+1} + A_1 x_i \tag{3.2}$$

with

$$A_0 = 2 - \tau k_1, \qquad A_1 = \tau(k_1 - \tau a_1) - 1. \tag{3.3}$$

It is well known [15] that necessary and sufficient conditions for asymptotic stability of the trivial solution of equation (3.2) have the form

$$|A_1| < 1, \qquad |A_0| < 1 - A_1. \tag{3.4}$$

For A_1 from (3.3), (3.4) it follows $0 < \tau(k_1 - \tau a_1) < 2$. It means that

$$\begin{gathered} \tau \in \left(0, \frac{k_1}{a_1}\right), \quad \text{if} \quad k_1^2 < 8a_1, \\ \tau \in \left(0, \frac{k_1 - \sqrt{k_1^2 - 8a_1}}{2a_1}\right) \cup \left(\frac{k_1 + \sqrt{k_1^2 - 8a_1}}{2a_1}, \frac{k_1}{a_1}\right), \quad \text{if } k_1^2 \geq 8a_1. \end{gathered} \tag{3.5}$$

For A_0 from (3.3), (3.4) it follows $a_1\tau^2 - 2k_1\tau + 4 > 0$. It means that

$$\begin{gathered} \tau \in (0, \infty), \quad \text{if } k_1^2 < 4a_1, \\ \tau \in \left(0, \frac{k_1 - \sqrt{k_1^2 - 4a_1}}{a_1}\right) \cup \left(\frac{k_1 + \sqrt{k_1^2 - 4a_1}}{a_1}, \infty\right), \quad \text{if } k_1^2 \geq 4a_1. \end{gathered} \tag{3.6}$$

As a result we obtain necessary and sufficient conditions for asymptotic stability of the trivial solution of auxiliary equation (3.2) in the form

$$0 < \tau < \begin{cases} a_1^{-1} k_1, & k_1^2 < 4a_1, \\ a_1^{-1}\left(k_1 - \sqrt{k_1^2 - 4a_1}\right), & k_1^2 \geq 4a_1. \end{cases} \tag{3.7}$$

Note that if for arbitrary positive definite matrix C the matrix equation

$$A'DA - D = -C. \tag{3.8}$$

has a positive definite solution D then the function $v(i) = z'(i)Dz(i)$ is a Lyapunov function for equation (3.1), that is $\Delta v(i) = -z'(i)Cz(i)$.

Let matrix C be a diagonal matrix with positive elements c_1 and c_2. Then the elements d_{ij} of the matrix D satisfy the system of the equation

$$\begin{aligned} \tau^2 a_1^2 d_{22} - 2\tau a_1 d_{12} &= -c_1, \\ d_{11} - (\tau a_1 + k_1) d_{12} - a_1(1 - \tau k_1) d_{22} &= 0, \\ \tau^2 d_{11} + 2\tau(1 - \tau k_1) d_{12} - \tau k_1(2 - \tau k_1) d_{22} &= -c_2, \end{aligned} \tag{3.9}$$

with the solution

$$\begin{aligned} d_{11} &= \frac{c_1(a_1\tau + k_1)}{2a_1\tau} + \frac{2 - \tau(k_1 - a_1\tau)}{2} a_1 d_{22}, \\ d_{12} &= \frac{c_1}{2\tau a_1} + \frac{\tau a_1}{2} d_{22}, \quad d_{22} = \frac{c_1[2 - \tau(k_1 - a_1\tau)] + 2a_1 c_2}{\tau a_1(k_1 - a_1\tau)[4 - \tau(2k_1 - a_1\tau)]}. \end{aligned} \tag{3.10}$$

Remark 3.1. Note that without loss of generality in (3.10) we can put $c_1 = 1$, $c_2 = c$. Really, if it is not so we can divide matrix equation (3.8) on c_1. As a result we obtain a new diagonal matrix C with the elements 1 and $c = c_2/c_1$ and a new matrix D with the elements

$$\begin{aligned} d_{11} &= \frac{a_1\tau + k_1}{2a_1\tau} + \frac{2 - \tau(k_1 - a_1\tau)}{2} a_1 d_{22}, \\ d_{12} &= \frac{1}{2\tau a_1} + \frac{\tau a_1}{2} d_{22}, \quad d_{22} = \frac{2 - \tau(k_1 - a_1\tau) + 2a_1 c}{\tau a_1(k_1 - a_1\tau)[4 - \tau(2k_1 - a_1\tau)]}. \end{aligned} \tag{3.11}$$

Remark 3.2. It is easy to check that by condition (3.7) the matrix D with elements (3.11) is a positive definite one.

4. Stability conditions of the difference analogue

Let us obtain now a sufficient condition for asymptotic stability of the trivial solution of (2.10). Transform this equation to the form

$$z(i+1) - F(i+1) = Az(i) - F(i). \tag{4.1}$$

Following the general method of Lyapunov functionals construction [7] we will construct Lyapunov functional V_i for equation (2.10) in the form $V_i = V_{1i} + V_{2i}$, where

$$V_{1i} = (z(i) - F(i))' D(z(i) - F(i)) \tag{4.2}$$

and the matrix D is a positive definite solution of matrix equation (3.8).

Calculating ΔV_{1i} via (4.2), (4.1), (3.8) we have

$$\begin{aligned}\Delta V_{1i} &= \big(z(i+1)-F(i+1)\big)'D\big(z(i+1)-F(i+1)\big)\\ &\quad -\big(z(i)-F(i)\big)'D\big(z(i)-F(i)\big)\\ &= \big(Az(i)-F(i)\big)'D\big(Az(i)-F(i)\big)-\big(z(i)-F(i)\big)'D\big(z(i)-F(i)\big)\\ &= -z'(i)Cz(i)-2F'(i)D(A-I)z(i).\end{aligned} \tag{4.3}$$

Note that

$$\begin{aligned}&2F'(i)D(A-I)z(i)\\ &\quad= 2\begin{pmatrix}0 & F_i\end{pmatrix}\begin{pmatrix}d_{11} & d_{12}\\ d_{12} & d_{22}\end{pmatrix}\begin{pmatrix}0 & \tau\\ -\tau a_1 & -\tau k_1\end{pmatrix}\begin{pmatrix}x_i\\ y_i\end{pmatrix}\\ &\quad= 2F_i\begin{pmatrix}d_{12} & d_{22}\end{pmatrix}\begin{pmatrix}\tau y_i\\ -\tau(a_1x_i+k_1y_i)\end{pmatrix}\\ &\quad= 2F_i\big(-\tau a_1d_{22}x_i+\tau(d_{12}-k_1d_{22})y_i\big)\\ &\quad= -2\tau a_1d_{22}x_iF_i+2\tau(d_{12}-k_1d_{22})y_iF_i.\end{aligned} \tag{4.4}$$

Put

$$\alpha=\frac{2-\tau(k_1-a_1\tau)+2a_1c}{(k_1-a_1\tau)\big[4-\tau(2k_1-a_1\tau)\big]}. \tag{4.5}$$

Then from (3.11), (4.5) it follows

$$d_{22}=\frac{\alpha}{\tau a_1}. \tag{4.6}$$

Using (3.11), (4.5), (4.6) we obtain

$$\begin{aligned}\tau(d_{12}-k_1d_{22}) &= \frac{1}{2a_1}\big(1-\alpha(2k_1-a_1\tau)\big)\\ &= \frac{1}{2a_1}\left(1-\frac{(2k_1-a_1\tau)\big[2-\tau(k_1-a_1\tau)+2a_1c\big]}{(k_1-a_1\tau)\big[4-\tau(2k_1-a_1\tau)\big]}\right)=-\beta,\end{aligned} \tag{4.7}$$

where

$$\beta=\frac{\tau+c(2k_1-a_1\tau)}{(k_1-a_1\tau)\big[4-\tau(2k_1-a_1\tau)\big]}. \tag{4.8}$$

So, via Remark 3.1, (4.3), (4.4), (4.6), (4.7)

$$\Delta V_{1i}=-x_i^2-cy_i^2-2\alpha x_iF_i-2\beta y_iF_i. \tag{4.9}$$

Put now

$$q_2=\frac{1}{2}\sum_{l=1}^{2}|b_l|\,m_l(m_l+1),\qquad S_i=\sum_{l=1}^{2}|b_l|\sum_{j=i-m_l}^{i-1}(j-i+1+m_l)\,y_j^2. \tag{4.10}$$

Using (2.7) and $\lambda_1 > 0$ we have

$$\begin{aligned} 2x_iF_i &= 2\tau^2\sum_{l=1}^{2} b_l \sum_{j=i-m_l}^{i-1} (j-i+1+m_l)x_iy_j \\ &\le \tau^2\sum_{l=1}^{2} |b_l| \sum_{j=i-m_l}^{i-1} (j-i+1+m_l)\left(\lambda_1 x_i^2 + \frac{1}{\lambda_1}y_j^2\right) = \lambda_1\tau^2 q_2 x_i^2 + \frac{\tau^2}{\lambda_1}S_i \end{aligned} \tag{4.11}$$

and analogously

$$2y_iF_i \le \lambda_2\tau^2 q_2 y_i^2 + \frac{\tau^2}{\lambda_2}S_i, \quad \lambda_2 > 0. \tag{4.12}$$

As a result we obtain

$$\Delta V_{1i} \le -(1-\alpha\tau^2\lambda_1 q_2)x_i^2 - (c-\beta\tau^2\lambda_2 q_2)y_i^2 + \rho S_i, \tag{4.13}$$

where

$$\rho = \tau^2\left(\frac{\alpha}{\lambda_1} + \frac{\beta}{\lambda_2}\right). \tag{4.14}$$

To neutralize the positive component in the estimate for ΔV_{1i} choose V_{2i} in the form

$$V_{2i} = \frac{\rho}{2}\sum_{l=1}^{2} |b_l| \sum_{j=i-m_l}^{i-1} \left(j-i+\frac{3}{2}+m_l\right)^2 y_j^2, \qquad q_3 = \frac{1}{2}\sum_{l=1}^{2} |b_l| \left(\frac{1}{2}+m_l\right)^2. \tag{4.15}$$

Calculating ΔV_{2i}, we obtain

$$\begin{aligned} \Delta V_{2i} &= \frac{\rho}{2}\sum_{l=1}^{2} |b_l| \left[\sum_{j=i+1-m_l}^{i} \left(j-i+\frac{1}{2}+m_l\right)^2 y_j^2 - \sum_{j=i-m_l}^{i-1} \left(j-i+\frac{3}{2}+m_l\right)^2 y_j^2\right] \\ &= \frac{\rho}{2}\sum_{l=1}^{2} |b_l| \left[\left(\frac{1}{2}+m_l\right)^2 y_i^2 - \frac{1}{4}y_{i-m_l}^2 \right. \\ &\qquad \left. + \sum_{j=i-m_l}^{i-1} \left(\left(j-i+\frac{1}{2}+m_l\right)^2 - \left(j-i+\frac{3}{2}+m_l\right)^2\right) y_j^2\right] \\ &= \frac{\rho}{2}\sum_{l=1}^{2} |b_l| \left[\left(\frac{1}{2}+m_l\right)^2 y_i^2 - \frac{1}{4}y_{i-m_l}^2 - 2\sum_{j=i-m_l}^{i-1} (j-i+1+m_l) y_j^2\right] \\ &= \rho q_3 y_i^2 - \rho\sum_{l=1}^{2} |b_l| \left[\frac{1}{8}y_{i-m_l}^2 + \sum_{j=i-m_l}^{i-1} (j-i+1+m_l) y_j^2\right] \le \rho q_3 y_i^2 - \rho S_i. \end{aligned} \tag{4.16}$$

Thus, for the functional $V_i = V_{1i} + V_{2i}$ we have

$$\Delta V_i \le -(1-\alpha\tau^2\lambda_1 q_2)x_i^2 - (c-\beta\tau^2\lambda_2 q_2 - \rho q_3)y_i^2. \tag{4.17}$$

Using (4.14) we obtain the stability conditions in the form

$$\tau^2\alpha\lambda_1 q_2 < 1, \quad \tau^2\beta\lambda_2 q_2 + \tau^2 q_3\left(\frac{\alpha}{\lambda_1} + \frac{\beta}{\lambda_2}\right) < c. \tag{4.18}$$

To minimize the left-hand part of the second condition (4.18) put $\lambda_2 = \sqrt{q_3/q_2}$. Then (4.18) takes the form

$$\tau^2\alpha\lambda_1 q_2 < 1, \quad \frac{\tau^2}{c}\left(2\beta\sqrt{q_2 q_3} + \frac{\alpha q_3}{\lambda_1}\right) < 1. \tag{4.19}$$

Choosing $\lambda_1 > 0$ from the condition

$$\alpha\lambda_1 q_2 = \frac{1}{c}\left(2\beta\sqrt{q_2 q_3} + \frac{\alpha q_3}{\lambda_1}\right), \tag{4.20}$$

we obtain

$$\lambda_1 = \frac{\sqrt{q_3}\left(\beta + \sqrt{\beta^2 + \alpha^2 c}\right)}{\alpha c\sqrt{q_2}}. \tag{4.21}$$

Substituting (4.21) into (4.19) we get stability condition in the form

$$\gamma\left(\beta + \sqrt{\beta^2 + \alpha^2 c}\right) < c, \quad \gamma = \tau^2\sqrt{q_2 q_3}. \tag{4.22}$$

From here it follows

$$\gamma(\gamma\alpha^2 + 2\beta) < c. \tag{4.23}$$

Put

$$\hat{k}_i = \sum_{l=1}^{2} |b_l| h_l^i, \quad i = 0, 1, 2. \tag{4.24}$$

Then

$$\begin{aligned} q_2\tau^2 &= \frac{1}{2}\sum_{l=1}^{2} |b_l| h_l(\tau + h_l) = \frac{1}{2}(\tau\hat{k}_1 + \hat{k}_2), \\ q_3\tau^2 &= \frac{1}{2}\sum_{l=1}^{2} |b_l| \left(\frac{\tau}{2} + h_l\right)^2 = \frac{1}{2}\left(\frac{\tau^2}{4}\hat{k}_0 + \tau\hat{k}_1 + \hat{k}_2\right). \end{aligned} \tag{4.25}$$

Therefore,

$$\gamma = \frac{1}{2}\sqrt{(\tau\widehat{k}_1 + \widehat{k}_2)\left(\frac{\tau^2}{4}\widehat{k}_0 + \tau\widehat{k}_1 + \widehat{k}_2\right)}. \tag{4.26}$$

Using dependence α and β on c put

$$\alpha = B^{-1}(A + 2a_1c), \qquad \beta = B^{-1}(\tau + Gc), \tag{4.27}$$

where

$$A = 2 - \tau(k_1 - a_1\tau), \qquad B = (k_1 - a_1\tau)(4 - G\tau), \qquad G = 2k_1 - a_1\tau. \tag{4.28}$$

Substituting (4.27) into (4.23) we obtain

$$\gamma B^{-2}\left(\frac{A^2\gamma + 2B\tau}{c} + 4\gamma a_1^2 c + 4A\gamma a_1 + 2BG\right) < 1. \tag{4.29}$$

After minimization of the left-hand part of (4.29) with respect to $c > 0$ one can rewrite (4.29) in the form

$$\delta(\tau) = 2\gamma B^{-2}\left(2a_1\sqrt{\gamma(A^2\gamma + 2B\tau)} + 2A\gamma a_1 + BG\right) < 1. \tag{4.30}$$

One has remember that in condition (4.30) a_1 is defined by (1.2), A, B, G are defined by (4.28) and γ is defined by (4.26), (4.24). So, A, B, G and γ depend on τ.

Thus, the following theorem is proven.

Theorem 4.1. *Let conditions (1.5) hold and the step of quantization $\tau > 0$ satisfies condition (4.30). Then the trivial solution of system (2.3) is asymptotically stable.*

Lemma 4.2. *If condition (1.6) holds then there exists enough small $\tau > 0$ that condition (4.30) holds too.*

Proof. For $\tau = 0$ condition (4.30) takes the form

$$\widehat{k}_2 < 4\left(1 + \sqrt{1 + \frac{4a_1}{k_1^2}}\right)^{-1}. \tag{4.31}$$

It is easy to see that if condition (1.6) holds then condition (4.31) (or condition (4.30) for $\tau = 0$) holds too. Since the function $\delta(\tau)$ is continuous in the point $\tau = 0$ then if condition (4.30) holds for $\tau = 0$ then it holds for enough small $\tau > 0$ also. The proof is completed. □

Corollary 4.3. *Let conditions (1.5), (1.6) hold then there exists enough small $\tau > 0$ that the trivial solution of system (2.3) is asymptotically stable.*

5. Numerical analysis

Here we consider some numerical examples which illustrate the theoretical results obtained above. For illustration of Corollary 4.3 consider the following example.

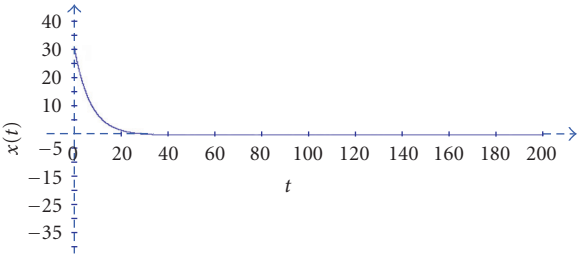

Figure 5.1

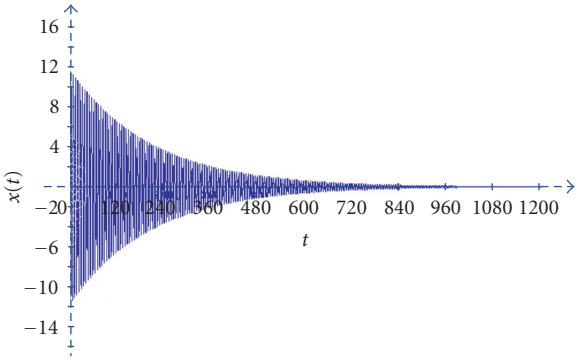

Figure 5.2

Example 5.1. Put in equation (1.7) $a = 9.5$, $b_1 = 10$, $b_2 = -20$, $h_1 = 0.4$, $h_2 = 0.02$. Then $a_1 = 0.5 > 0$, $k_1 = 3.6 > 0$, $\widehat{k}_2 = 1.608 < k_m = 1.92$. Conditions (1.5), (1.6) hold, therefore (Theorem 1.2), the trivial solution of equation (1.7) is asymptotically stable. Besides there exists enough small $\tau > 0$ that condition (4.30) holds. Using $\tau = 0.01$ we obtain $\delta(0.01) = 0.869 < 1$, that is condition (4.30) holds. Therefore, the trivial solution of difference system (2.3) is asymptotically stable. On Figure 5.1 it is shown that the trajectory of solution of system (2.3) with the initial condition $x_j = 33$, $j \le 0$, $y_0 = 0$ goes to zero.

If conditions (1.5) hold but condition (1.6) does not hold then the trivial solution of equation (1.7) can be asymptotically stable or unstable. If in this case for some $\tau > 0$ condition (4.30) does not hold too then the trivial solution of difference system (2.3) can be also asymptotically stable or unstable. In the following two examples one can see the both situations.

Example 5.2. Put in equation (1.7) $a = 3$, $b_1 = 1$, $b_2 = -5$, $h_1 = 0.55$, $h_2 = 0.1$. Then $a_1 = 1 > 0$, $k_1 = 0.05 > 0$, $\widehat{k}_2 = 0.3525 > k_m = 0.0975$, $\delta(0.01) = 20.83 > 1$. So, conditions (1.5) hold but conditions (1.6) and (4.30) do not hold. On Figure 5.2 it is shown that the trajectory of solution of system (2.3) with the initial condition $x_j = 12$, $j \le 0$, $y_0 = 0$ goes to zero.

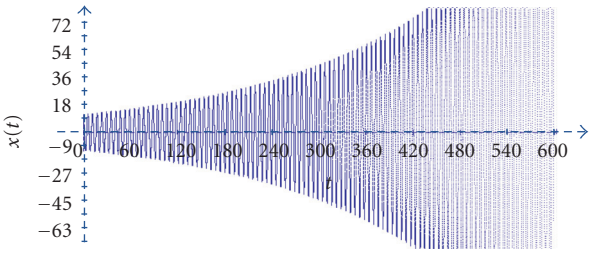

Figure 5.3

Example 5.3. Putting in the previous example $h_1 = 0.53$ (without changing the values of the other parameters) we obtain $a_1 = 1 > 0$, $k_1 = 0.03 > 0$, $\widehat{k}_2 = 0.3309 > k_m = 0.0591$, $\delta(0.01) = 73.06 > 1$. As in the previous example conditions (1.5) hold, conditions (1.6) and (4.30) do not hold but in this case the trajectory of solution of system (2.3) with the initial condition $x_j = 12$, $j \le 0$, $y_0 = 0$ goes to infinity (Figure 5.3).

References

[1] D. J. Acheson, *A pendulum theorem*, Proc. Roy. Soc. London Ser. A **443** (1993), no. 1917, 239–245.

[2] D. J. Acheson and T. Mullin, *Upside-down pendulums*, Nature **366** (1993), no. 6452, 215–216.

[3] J. A. Blackburn, H. J. T. Smith, and N. Grønbech-Jensen, *Stability and Hopf bifurcations in an inverted pendulum*, Amer. J. Phys. **60** (1992), no. 10, 903–908.

[4] P. Borne, V. B. Kolmanovskii, and L. E. Shaikhet, *Stabilization of inverted pendulum by control with delay*, Dynam. Systems Appl. **9** (2000), no. 4, 501–514.

[5] P. L. Kapitza, *Dynamical stability of a pendulum when its point of suspension vibrates, and Pendulum with a vibrating suspension*, In Collected Papers of P. L. Kapitza (D. ter Haar, ed.), vol. 2, Pergamon Press, London, 1965, pp. 714–737.

[6] V. B. Kolmanovskii and L. E. Shaikhet, *New results in stability theory for stochastic functional-differential equations (SFDEs) and their applications*, Proceedings of Dynamic Systems and Applications, Vol. 1 (Atlanta, GA, 1993), Dynamic, Georgia, 1994, pp. 167–171.

[7] ———, *General method of Lyapunov functionals construction for stability investigation of stochastic difference equations*, Dynamical Systems and Applications, World Sci. Ser. Appl. Anal., vol. 4, World Scientific Publishing, New Jersey, 1995, pp. 397–439.

[8] ———, *Method for constructing Lyapunov functionals for stochastic differential equations of neutral type*, Differential Equations **31** (1995), no. 11, 1819–1825.

[9] ———, *Construction of Lyapunov functionals for stochastic hereditary systems: a survey of some recent results*, Math. Comput. Modelling **36** (2002), no. 6, 691–716.

[10] ———, *Some peculiarities of the general method of Lyapunov functionals construction*, Appl. Math. Lett. **15** (2002), no. 3, 355–360.

[11] ———, *About one application of the general method of Lyapunov functionals construction*, Internat. J. Robust Nonlinear Control **13** (2003), no. 9, 805–818, Special Issue on Time Delay Systems.

[12] ———, *About some features of general method of Lyapunov functionals construction*, Stab. Control Theory Appl. **6** (2004), no. 1, 49–76.

[13] M. Levi, *Stability of the inverted pendulum—a topological explanation*, SIAM Rev. **30** (1988), no. 4, 639–644.

[14] M. Levi and W. Weckesser, *Stabilization of the inverted linearized pendulum by high frequency vibrations*, SIAM Rev. **37** (1995), no. 2, 219–223.

[15] L. E. Shaikhet, *Necessary and sufficient conditions of asymptotic mean square stability for stochastic linear difference equations*, Appl. Math. Lett. **10** (1997), no. 3, 111–115.

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