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Research Article

Existence of Positive Solutions for Nonlinear Eigenvalue Problems

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We use a fixed point theorem in a cone to obtain the existence of positive solutions of the differential equation, $u'' + \lambda f(t,u) = 0$, $0 < t < 1$, with some suitable boundary conditions, where λ is a parameter.

1. Introduction

We consider the existence of positive solutions of the following two-point boundary value problem:

$$\begin{gathered}(E_\lambda) u'' + \lambda f(t,u) = 0, \quad 0 < t < 1, \\ (BC) u(0) = a, u(1) = b,\end{gathered} \tag{BVP$_\lambda$}$$

where a and b are nonnegative constants, and $f \in C([0,1] \times [0,\infty), [0,\infty))$.

In the last thirty years, there are many mathematician considered the boundary value problem (BVP$_\lambda$) with $a = b = 0$, see, for example, Chu et al. [1], Chu et al. [2], Chu and Zhau [3], Chu and Jiang [4], Coffman and Marcus [5], Cohen and Keller [6], Erbe [7], Erbe et al. [8], Erbe and Wang [9], Guo and Lakshmikantham [10], Iffland [11], Njoku and Zanolin [12], Santanilla [13].

In 1993, Wong [14] showed the following excellent result.

Theorem A (see [14]). *Assume that*

$$f(t,u) := p(t)h(u) \in C([0,1] \times [0,\infty); (0,\infty)) \tag{1.1}$$

is an increasing function with respect to u. If there exists a constant L such that

$$\int_0^c \frac{du}{\sqrt{H(c) - H(u)}} \le L < \infty \quad \forall c > 0, \tag{1.2}$$

where $H(u) := \int_0^u h(y)dy$ for $u \ge 0$, then, there exists $\lambda^* \in (0, 8\, L^2 p_0^{-1})$ such that the boundary value problem (BVP_λ) *with $a = b = 0$ has a positive solution in $C^2(0,1) \bigcap C[0,1]$ for $0 < \lambda \le \lambda^*$, while there is no such solution for $\lambda > \lambda^*$ in which $p_0 := \min\{p(t) \mid t \in [1/4, 3/4]\}$.*

Seeing such facts, we cannot but ask "whether or not we can obtain a similar conclusion for the boundary value problem (BVP_λ)." We give a confirm answer to the question.

First, We observe the following statements.

(1) Let

$$k(t,s) = \begin{cases} s(1-t), & \text{for } 0 \le s \le t \le 1, \\ t(1-s), & \text{for } 0 \le t \le s \le 1, \end{cases} \tag{1.3}$$

on $[0,1] \times [0,1]$, then $k(t,s)$ is the Green's function of the differential equation $u''(t) = 0$ in $(0,1)$ with respect to the boundary value condition $u(0) = u(1) = 0$.

(2) $\mathbb{K} := \{u \in C[0,1] \mid u(t) \ge 0,\ \min_{t\in[1/4,3/4]} u(t) \ge (1/4)\|u\|\}$, is a cone in the Banach space with $\|u\| = \sup_{t\in[0,1]} |u(t)|$.

In order to discuss our main result, we need the follo wing useful lemmas which due to Lian et al. [15] and Guo and Lakshmikantham [10], respectively.

Lemma B (see [10]). Suppose that $k(t,s)$ be defined as in (1). Then, we have the following results.

(R_1) $(k(t,s)/k(s,s) \le 1$, for $t \in [0,1]$ and $s \in [0,1]$,)

(R_2) $(k(t,s)/k(s,s) \ge 1/4$, for $t \in [1/4,3/4]$ and $s \in [0,1]$.)

Lemma C (see [10, Lemmas 2.3.3 and 2.3.1]). Let E be a real Banach space, and let $C \subset E$ be a cone. Assume that $B_\rho := \{u \in C \mid \|u\| < \rho\}$ and $A : \overline{B_\rho} \to C$ is completely continuous. Then

(1) $i(A, B_\rho, C) = 0$ if

$$\begin{gathered} \operatorname{Inf}_{u\in\partial B_\rho} \|Au\| > 0, \\ Au \neq \alpha u \quad \text{for } u \in \partial B_\rho, \alpha \in (0,1], \end{gathered} \tag{1.4}$$

(2) $i(A, B_\rho, C) = 1$ if $Au \neq \alpha u$ for $u \in \partial B_\rho$ and $\alpha \ge 1$,

where $i(A, B_\rho, C)$ is the fixed point index of a compact map $A : \overline{B_\rho} \to C$, such that $Au \neq u$ for $u \in \partial B_\rho$, with respect to B_ρ.

2. Main Results

Now, we can state and prove our main result.

Theorem 2.1. *Suppose that there exist two distinct positive constants* η, θ *and a function* $g \in C([\xi_2, \theta]; [0, \infty))$ *with* $\theta > \max\{a, b\} := \xi_1$ *and* $\xi_2 = \min\{a, b\}$ *such that*

$$f(t, u) \geq \eta\left(\int_{1/4}^{3/4} k\left(\frac{1}{2}, s\right) ds\right)^{-1} \quad \text{on } \left[\frac{1}{4}, \frac{3}{4}\right] \times \left[\frac{1}{4}\eta, \eta\right], \tag{2.1}$$

$$f(t, u) \leq g(u) \quad \text{on } [0, 1] \times [\xi_2, \theta]. \tag{2.2}$$

Then (BVP_λ) *has a positive solution* u *with* $\|u\|$ *between* η *and* θ *if*

$$\lambda \in \left[1, 2\left(\int_{\xi_1}^{\theta} \frac{ds}{\sqrt{G(\theta) - G(s)}}\right)^2\right], \tag{2.3}$$

where

$$G(u) := \begin{cases} \displaystyle\int_{\xi_1}^{u} g(s) ds, & \text{if } u \in [\xi_1, \theta], \\ 0, & \text{if } u \in [\xi_2, \xi_1]. \end{cases} \tag{2.4}$$

Proof. It is clear that (BVP_λ) has a solution $u = u(t)$ if, and only if, u is the solution of the operator equation

$$u(t) = a(1 - t) + bt + \lambda \int_0^1 k(t, s) f(s, u(s)) ds := Au(t). \tag{2.5}$$

It follows from the definition of $\mathbb{K}$ in our observation (2) and Lemma B that

$$\begin{aligned} \min_{t \in [1/4, 3/4]} (Au)(t) &= \min_{t \in [1/4, 3/4]} \left(a(1 - t) + bt + \lambda \int_0^1 k(t, s) f(s, u(s)) ds\right) \\ &\geq \frac{1}{4}\left(a(1 - t) + bt + \lambda \int_0^1 k(s, s) f(s, u(s))\, ds\right) \quad (\text{using}(R_2)) \\ &\geq \frac{1}{4}\left(a(1 - t) + bt + \lambda \int_0^1 k(t, s) f(s, u(s)) ds\right) \quad (\text{using}(R_1)). \end{aligned} \tag{2.6}$$

Hence, $\min_{t\in[1/4,3/4]}(Au)(t) \geq (1/4)\|Au\|$, which implies $A\mathbb{K} \subset \mathbb{K}$. Furthermore, it is easy to check that $A : \mathbb{K} \to \mathbb{K}$ is completely continuous. If there exists a $u \in \partial B_\eta \cup \partial B_\theta$ such that $Au = u$, then we obtain the desired result. Thus, we may assume that

$$Au \neq u \quad \text{for } u \in \partial B_\eta \cup \partial B_\theta, \tag{2.7}$$

where $B_\eta := \{u \in \mathbb{K} \mid \|u\| < \eta\}$ and $B_\theta := \{u \in \mathbb{K} \mid \|u\| < \theta\}$. We now separate the rest proof into the following three steps.

Step 1. It follows from the definitions of $\|u\|$ and $\mathbb{K}$ that, for $u \in \partial B_\eta$,

$$\begin{aligned} u(t) &\leq \|u\| = \eta \quad \text{for } t \in [0,1], \\ u(t) &\geq \min_{t\in[1/4,3/4]} u(t) \geq \frac{1}{4}\|u\| = \frac{1}{4}\eta \quad \text{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right], \end{aligned} \tag{2.8}$$

which implies

$$\frac{1}{4}\eta \leq u(t) \leq \eta \quad \text{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right]. \tag{2.9}$$

Hence, by (2.5),

$$\begin{aligned} (Au)\left(\frac{1}{2}\right) &= \frac{1}{2}(a+b) + \lambda\int_0^1 k\left(\frac{1}{2}, s\right) f(s, u(s))ds \\ &\geq \int_0^1 k\left(\frac{1}{2}, s\right) f(s, u(s))ds \quad (\text{using } \lambda \geq 1, a, b \geq 0) \\ &\geq \int_{1/4}^{3/4} k\left(\frac{1}{2}, s\right) f(s, u(s))ds \\ &\geq \eta\left(\int_{1/4}^{3/4} k\left(\frac{1}{2}, s\right)ds\right)^{-1}\left(\int_{1/4}^{3/4} k\left(\frac{1}{2}, s\right)ds\right)\frac{\|u\|}{\eta} \\ &= \|u\|, \end{aligned} \tag{2.10}$$

which implies

$$\|Au\| \geq \|u\| \quad \text{for } u \in \partial B_\eta. \tag{2.11}$$

Hence

$$\inf_{u\in\partial B_\eta} \|Au\| \geq \inf_{u\in\partial B_\eta} \|u\| = \eta > 0. \tag{2.12}$$

We now claim that

$$Au \neq \alpha u, \quad \text{for } u \in \partial B_\eta, \alpha \in (0,1). \tag{2.13}$$

In fact, if there exist $u \in \partial B_\eta$ and $\alpha \in (0,1)$ such that $Au = \alpha u$, then, by (2.11),

$$\|u\| \leq \|Au\| = \alpha\|u\| < \|u\|, \tag{2.14}$$

which gives a contradiction. This proves that (2.13) holds. Thus, by Lemma C,

$$i(A, B_\eta, \mathbb{K}) = 0. \tag{2.15}$$

Step 2. First, we claim that

$$Au \neq \alpha u \quad \text{for } u \in \partial \mathrm{B}_\theta, \alpha > 1. \tag{2.16}$$

Suppose to the contrary that there exist $u \in \partial B_\theta$ and $\alpha > 1$ such that

$$Au = \alpha u. \tag{2.17}$$

It is clear that (2.17) is equivalent to

$$u''(t) + \frac{\lambda}{\alpha} f(t,u) = 0. \tag{2.18}$$

Since $u \in C[0,1]$ and $\|u\| = \theta > 0$, it follows that there exists a $t^* \in (0,1)$ such that

$$u(t^*) = \|u\| = \theta. \tag{2.19}$$

Let

$$t_1 = \min\{t \in [0,1] \mid u(t) = \theta\}, \quad t_2 = \max\{t \in [0,1] \mid u(t) = \theta\}. \tag{2.20}$$

Then $0 < t_1 \leq t^* \leq t_2 < 1$. From $u'' < 0$ on $(0,1)$, we see that $u'(t) > 0$ on $(0,t_1)$ $u'(t) < 0$ on $(t_2, 1)$ and $u'(t) = 0$ on $[t_1, t_2]$. It follows from

$$u''(t) = -\frac{\lambda}{\alpha} f(t, u(t)) \geq -\frac{\lambda}{\alpha} g(u(t)) \quad \text{for } t \in [0,1] \tag{2.21}$$

and $u'(t) = 0$ on $[t_1, t_2]$ that

$$0 < u'(t) \le \sqrt{\frac{2\lambda}{\alpha}(G(\theta) - G(u(t)))} \quad \text{for } t \in [0, t_1),$$
$$0 > u'(t) \ge -\sqrt{\frac{2\lambda}{\alpha}(G(\theta) - G(u(t)))} \quad \text{for } t \in (t_2, 1]. \tag{2.22}$$

Hence,

$$\int_a^\theta \frac{ds}{\sqrt{(2\lambda/\alpha)(G(\theta) - G(s))}} \le \int_0^{t_1} dt = t_1,$$
$$\int_b^\theta \frac{ds}{\sqrt{(2\lambda/\alpha)(G(\theta) - G(s))}} \le \int_{t_2}^1 dt = 1 - t_2. \tag{2.23}$$

Thus

$$\begin{aligned} 1 &\ge 1 - t_2 + t_1 \\ &\ge \frac{2}{\sqrt{2\lambda/\alpha}} \int_{\xi_1}^\theta \frac{ds}{\sqrt{G(\theta) - G(s)}} \\ &> \sqrt{\frac{2}{\lambda}} \int_{\xi_1}^\theta \frac{ds}{\sqrt{G(\theta) - G(s)}} \quad (\text{since } \alpha > 1) \\ &\ge 1 \quad \left(\text{because } \lambda \in \left[1, 2\left(\int_{\xi_1}^\theta \frac{ds}{\sqrt{G(\theta) - G(s)}}\right)^2\right]\right). \end{aligned} \tag{2.24}$$

This contradiction implies

$$Au \ne \alpha u, \quad \text{for } u \in \partial B_\theta, \alpha > 1. \tag{2.25}$$

Therefore, by Lemma C,

$$i(A, B_\theta, \mathbb{K}) = 1. \tag{2.26}$$

Step 3. It follows from Steps (1) and (2) and the property of the fixed point index (see, for example, [10, Theorem 2.3.2]) that the proof is complete. □

Remark 2.2. It follows from the conclusion of Theorem 2.1 that the positive constant θ and nonnegative function $g(u)$ satisfy

$$\int_{\xi_1}^{\theta} \frac{ds}{\sqrt{G(\theta) - G(s)}} \geq \frac{1}{\sqrt{2}}. \tag{2.27}$$

There are many functions $g(u)$ and positive constants θ satisfying (2.27). For example, Suppose that $M \in (0, 8]$ and $\theta \in (\xi_1, \infty)$. Let $g(u) := M(\theta - \xi_1)$ on $[\xi_2, \theta]$, then $G(u) = M(\theta - \xi_1)(u - \xi_1)$ on $[\xi_1, \theta]$ and

$$\begin{aligned} \int_{\xi_1}^{\theta} \frac{1}{\sqrt{G(\theta) - G(u)}} du &= \frac{1}{\sqrt{M(\theta - \xi_1)}} \int_{\xi_1}^{\theta} \frac{1}{\sqrt{\theta - u}} du \\ &= \frac{1}{\sqrt{M(\theta - \xi_1)}} \left(2\sqrt{\theta - \xi_1} \right) \\ &= \frac{2}{\sqrt{M}} \geq \frac{1}{\sqrt{2}}. \end{aligned} \tag{2.28}$$

Remark 2.3. We now define

$$\begin{aligned} \max f_0 &:= \lim_{u \to 0^+} \max_{t \in [0,1]} \frac{f(t, u)}{u}, \\ \min f_0 &:= \lim_{u \to 0^+} \min_{t \in [0,1]} \frac{f(t, u)}{u}, \\ \max f_\infty &:= \lim_{u \to \infty} \max_{t \in [0,1]} \frac{f(t, u)}{u}, \\ \min f_\infty &:= \lim_{u \to \infty} \min_{t \in [0,1]} \frac{f(t, u)}{u}. \end{aligned} \tag{2.29}$$

A simple calculation shows that

$$\int_{1/4}^{3/4} k\left(\frac{1}{2}, s\right) ds = \frac{3}{32}. \tag{2.30}$$

Then, we have the following results.

(i) Suppose that $\max f_0 := C_1 \in [0,M) \subseteq [0,8)$. Taking $\epsilon = M - C_1 > 0$, there exists $1 > \theta_1 > 0$ (θ_1 can be chosen small arbitrarily) such that

$$\max_{t\in[0,1]} \frac{f(t,u)}{u} \le \epsilon + C_1 = M \quad \text{on } (0,\theta_1]. \tag{2.31}$$

Hence,

$$f(t,u) \le Mu \le M\theta_1 \quad \text{on } [0,1]\times[\xi_2,\theta_1] \subset [0,1]\times[0,\theta_1]. \tag{2.32}$$

It follows from Remark 2.2 that the hypothesis (2.2) of Theorem 2.1 is satisfied if $\lambda \in [1, 8/M]$.

(ii) Suppose that $\min f_\infty := C_2 \in (128/3,\infty]$. Taking $\epsilon = C_2 - 128/3 > 0$, there exists $\eta_1 > 0$ (η_1 can be chosen large arbitrarily) such that

$$\min_{t\in[0,1]} \frac{f(t,u)}{u} \ge -\epsilon + C_2 = \frac{128}{3} \quad \text{on } \left[\frac{1}{4}\eta_1, \infty\right). \tag{2.33}$$

Hence,

$$f(t,u) \ge \frac{128}{3}u \ge \frac{128}{3}\frac{1}{4}\eta_1 \ge \frac{32}{3}\eta_1 \quad \text{on } \left[\frac{1}{4},\frac{3}{4}\right]\times\left[\frac{1}{4}\eta_1,\eta_1\right] \subset [0,1]\times\left[\frac{1}{4}\eta_1,\infty\right), \tag{2.34}$$

which satisfies the hypothesis (2.1) of Theorem 2.1.

(iii) Suppose that $\min f_0 := C_3 \in (128/3,\infty]$. Taking $\epsilon = C_3 - 128/3 > 0$, there exists $1 > \eta_2 > 0$ (η_2 can be chosen small arbitrarily) such that

$$\min_{t\in[0,1]} \frac{f(t,u)}{u} \ge -\epsilon + C_3 = \frac{128}{3} \quad \text{on } (0,\eta_2]. \tag{2.35}$$

Hence,

$$f(t,u) \ge \frac{128}{3}u \ge \frac{128}{3}\frac{1}{4}\eta_2 = \frac{32}{3}\eta_2 \quad \text{on} \left[\frac{1}{4},\frac{3}{4}\right]\times\left[\frac{1}{4}\eta_2,\eta_2\right] \subset [0,1]\times[0,\eta_2], \tag{2.36}$$

which satisfies the hypothesis (2.1) of Theorem 2.1.

(iv) Suppose that $\max f_\infty := C_4 \in [0,M) \subseteq [0,8)$. Taking $\epsilon = M - C_4 > 0$, there exists a $\delta > 0$ (δ can be chosen large arbitrarily) such that

$$\max_{t\in[0,1]} \frac{f(t,u)}{u} \le \epsilon + C_4 = M \quad \text{on } [\delta,\infty). \tag{2.37}$$

Hence, we have the following two cases.

Case (i). Assume that $\max_{t\in[0,1]} f(t,u)$ is bounded, say

$$f(t,u) \le L \quad \text{on } [0,1]\times[0,\infty), \tag{2.38}$$

for some constant L. Taking $\theta_2 = L/M > 1$ (since L can be chosen large arbitrarily, θ_2 can be chosen large arbitrarily, too),

$$f(t,u) \le L = M\theta_2 \quad \text{on } [0,1]\times[0,\theta_2] \subset [0,1]\times[0,\infty). \tag{2.39}$$

Case (ii). Assume that $\max_{t\in[0,1]} f(t,u)$ is unbounded, then there exist a $\theta_2 \ge \max\{\delta,\xi_2\}$ (θ_2 can be chosen large arbitrarily) and $t_0 \in [0,1]$ such that

$$f(t,u) \le f(t_0,\theta_2) \quad \text{on } [0,1]\times[0,\theta_2]. \tag{2.40}$$

It follows from $\theta_2 \ge \delta$ and (2.37) that

$$f(t,u) \le f(t_0,\theta_2) \le M\theta_2 \quad \text{on } [0,1]\times[\xi_2,\theta_2] \subset [0,1]\times[0,\theta_2]. \tag{2.41}$$

By Cases (i), (ii) and Remark 2.2, we see that the hypothesis (2.2) of Theorem 2.1 is satisfied if $\lambda \in [1, 8/M]$.

We immediately conclude the following corollaries.

Corollary 2.4. (BVP$_\lambda$) *has at least one positive solution for* $\lambda \in [1, 8M]$ *if one of the following conditions holds:*

(H_1) $\max f_0 = C_1 \in [0,M) \subseteq [0,8), \min f_\infty = C_2 \in (128/3,\infty]$,

(H_2) $\min f_0 = C_3 \in (128/3,\infty], \max f_\infty = C_4 \in [0,M) \subseteq [0,8)$.

Proof. It follows from Remark 2.3 and Theorem 2.1 that the desired result holds, immediately. □

Corollary 2.5. *Let*

(H_3) $\min f_\infty = C_2, \min f_0 = C_3 \in (128/3,\infty]$,

(H_4) $f(t,u) \le M\theta^*$ *on* $[0,1]\times[\xi_2,\theta^*]$ *for some* $M \in (0,8]$ *and* $\theta^* > 0$.

Then, for $\lambda \in [1, 8/M]$, (BVP$_\lambda$) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$0 < \|u_1\| < \theta^* < \|u_2\|. \tag{2.42}$$

Proof. It follows from Remark 2.3 that there exist two real numbers $\eta_2 < \theta^* < \eta_1$ satisfying

$$\begin{aligned} f(t,u) &\ge \frac{32}{3}\eta_1 \quad \text{on } \left[\frac{1}{4},\frac{3}{4}\right]\times\left[\frac{1}{4}\eta_1,\eta_1\right], \\ f(t,u) &\ge \frac{32}{3}\eta_2 \quad \text{on } \left[\frac{1}{4},\frac{3}{4}\right]\times\left[\frac{1}{4}\eta_2,\eta_2\right]. \end{aligned} \tag{2.43}$$

Hence, by Theorem 2.1 and Remark 2.2, we see that for each $\lambda \in [1, 8/M]$, there exist two positive solutions u_1 and u_2 of (BVP_λ) such that

$$\eta_2 < \|u_1\| < \theta^* < \|u_2\| < \eta_1. \tag{2.44}$$

Thus, we complete the proof. □

Corollary 2.6. *Let*

(H_5) $\max f_0 = C_1$, $\max f_\infty = C_4 \in [0, M) \subseteq [0, 8)$,

(H_6) $f(t, u) \geq (32/3)\ \eta^*$ *on* $[1/4, 3/4] \times [(1/4)\eta^*, \eta^*]$, *for some* $\eta^* > 0$.

Then, for $\lambda \in [1, 8/M]$, (BVP_λ) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$0 < \|u_1\| < \eta^* < \|u_2\|. \tag{2.45}$$

Proof. It follows from Remark 2.3 that there exist two real numbers $\theta_1 < \eta^* < \theta_2$ satisfying

$$\begin{aligned} f(t, u) &\leq M\theta_1 \quad \text{on } [0, 1] \times [\xi_2, \theta_1], \\ f(t, u) &\leq M\theta_2 \quad \text{on } [0, 1] \times [\xi_2, \theta_2]. \end{aligned} \tag{2.46}$$

Hence, by Theorem 2.1 and Remark 2.2, we see that, for each $\lambda \in [1, 8/M]$, (BVP_λ) has two positive solutions u_1 and u_2 such that

$$\theta_1 < \|u_1\| < \eta^* < \|u_2\| < \theta_2. \tag{2.47}$$

Thus, we completed the proof. □

3. Examples

To illustrate the usage of our results, we present the following examples.

Example 3.1. Consider the following boundary value problem:

$$\begin{gathered} u''(t) + \lambda \frac{ue^u}{1 + t^2} = 0 \quad \text{in}(0, 1), \\ (BC_1) \begin{cases} u(0) = a = 1, \\ u(1) = b = 1. \end{cases} \end{gathered} \tag{BVP.1}$$

Clearly,

$$\begin{aligned} \max f_0 &= 1 \in [0, M) \subseteq [0, 8), \\ \min\ f_\infty &= \infty \in \left(\frac{128}{3}, \infty\right]. \end{aligned} \tag{3.1}$$

If we take $M = 2$, then it follows from (H_1) of Corollary 2.4 that (BVP.1) has a solution if $\lambda \in [1,4]$.

Example 3.2. Consider the following boundary value problem:

$$u''(t) + \lambda[u(1-t) + K(1-e^{-u})] = 0 \quad \text{in } (0,1),\ K + \frac{1}{4} > \frac{128}{3},$$
$$(BC_2)\begin{cases} u(0) = a = 1, \\ u(1) = b = 2. \end{cases} \tag{BVP.2}$$

Clearly,

$$\min f_0 = K + \frac{1}{4} \in \left(\frac{128}{3}, \infty\right],$$
$$\max f_\infty = 1 \in [0, M) \subseteq [0, 8). \tag{3.2}$$

If we take $M = 2$, then it follows from (H_2) of Corollary 2.4 that (BVP.2) has a solution if $\lambda \in [1,4]$.

Example 3.3. Consider the following boundary value problem:

$$u''(t) + \left(\lambda u^{3/2} + u^{1/2}\right)/(1+t) = 0 \text{ in } (0,1),$$
$$(BC_3)\begin{cases} u(0) = a = 0, \\ u(1) = b = 1. \end{cases} \tag{BVP.3}$$

Clearly, if we take $M = 2$ and $\theta^* = 1$,

$$\min f_\infty = \infty \in (128/3, \infty],$$
$$\min f_0 = \infty \in (128/3, \infty],$$
$$f(t,u) \leq 2 \quad \text{on}[0,1] \times [0,1]. \tag{3.3}$$

Hence, it follows from Corollary 2.5 that (BVP.3) has two solutions if $\lambda \in [1,4]$.

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