Hindawi Publishing Corporation
Boundary Value Problems
Volume 2010, Article ID 543503, 18 pages
doi:10.1155/2010/543503

Research Article

Two Conservative Difference Schemes for the Generalized Rosenau Equation

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Received 31 October 2009; Accepted 26 January 2010

Academic Editor: Sandro Salsa

Numerical solutions for generalized Rosenau equation are considered and two energy conservative finite difference schemes are proposed. Existence of the solutions for the difference scheme has been shown. Stability, convergence, and priori error estimate of the scheme are proved using energy method. Numerical results demonstrate that two schemes are efficient and reliable.

1. Introduction

Consider the following initial-boundary value problem for generalized Rosenau equation:

$$u_t + u_{xxxxt} + u_x + (u^p)_x = 0, \quad x \in [0, L], \; t \in [0, T], \tag{1.1}$$

with an initial condition

$$u(x, 0) = u_0(x), \quad x \in [0, L], \tag{1.2}$$

and boundary conditions

$$u(0, t) = u(L, t) = 0, \qquad u_{xx}(0, t) = u_{xx}(L, t) = 0, \quad t \in [0, T], \tag{1.3}$$

where $p \geq 2$ is a integer.

When $p = 2$, (1.1) is called as usual Rosenau equation proposed by Rosenau [1] for treating the dynamics of dense discrete systems. Since then, the Cauchy problem of the Rosenau equation was investigated by Park [2]. Many numerical schemes have been proposed, such as C^1-conforming finite element method by Chung and Pani [3], discontinuous Galerkin method by Choo et al. [4], orthogonal cubic spline collocation method by Manickam [5], and finite difference method by Chung [6] and Omrani et al. [7]. As for the generalized case, however, there are few studies on theoretical analysis and numerical methods.

It can be proved easily that the problem (1.1)–(1.3) has the following conservative law:

$$E(t) = \|u\|_{L_2}^2 + \|u_{xx}\|_{L_2}^2 = E(0). \tag{1.4}$$

Hence, we propose two conservative difference schemes which simulate conservative law (1.4). The outline of the paper is as follows. In Section 2, a nonlinear difference scheme is proposed and corresponding convergence and stability of the scheme are proved. In Section 3, a linearized difference scheme is proposed and theoretical results are obtained. In Section 4, some numerical experiments are shown.

2. Nonlinear Finite Difference Scheme

Let h and τ be the uniform step size in the spatial and temporal direction, respectively. Denote $x_j = jh\ (0 \le j \le J), t_n = n\tau\ (0 \le n \le N)$, $u_j^n \approx u(x_j, t_n)$, and $Z_h^0 = \{u = (u_j) \mid u_0 = u_J = 0,\ j = 0, 1, 2, \ldots, J\}$. Define

$$\begin{gathered}
\left(u_j^n\right)_x = \frac{u_{j+1}^n - u_j^n}{h}, \qquad \left(u_j^n\right)_{\bar{x}} = \frac{u_j^n - u_{j-1}^n}{h}, \qquad \left(u_j^n\right)_{\hat{x}} = \frac{u_{j+1}^n - u_{j-1}^n}{2h}, \\
\left(u_j^n\right)_t = \frac{u_j^{n+1} - u_j^n}{\tau}, \qquad \left(u_j^n\right)_{\hat{t}} = \frac{u_j^{n+1} - u_j^{n-1}}{2\tau}, \qquad \left(u_j^n\right)_{x\bar{x}} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}, \\
\overline{u}_j^n = \frac{u_j^{n+1} + u_j^{n-1}}{2}, \qquad u_j^{n+1/2} = \frac{u_j^{n+1} + u_j^n}{2}, \\
(u^n, v^n) = h\sum_{j=0}^{J-1} u_j^n v_j^n, \qquad \|u^n\|^2 = (u^n, u^n), \qquad \|u^n\|_\infty = \max_{0\le j\le J-1}\left|u_j^n\right|,
\end{gathered} \tag{2.1}$$

and in the paper, C denotes a general positive constant which may have different values in different occurrences.

Since $(u^p)_x = (p/(p+1))[u^{p-1}u_x + (u^p)_x]$, then the following finite difference scheme is considered:

$$\left(u_j^n\right)_t + \left(u_j^n\right)_{xx\bar{x}\bar{x}t} + \left(u_j^{n+1/2}\right)_{\hat{x}} + \frac{p}{p+1}\left\{\left(u_j^{n+1/2}\right)^{p-1}\left(u_j^{n+1/2}\right)_{\hat{x}} + \left[\left(u_j^{n+1/2}\right)^p\right]_{\hat{x}}\right\} = 0, \tag{2.2}$$

$$u_j^0 = u_0(x_j), \quad 0 \le j \le J-1, \tag{2.3}$$

$$u_0^n = u_J^n = 0, \qquad (u_0^n)_{x\bar{x}} = \left(u_J^n\right)_{x\bar{x}} = 0. \tag{2.4}$$

Lemma 2.1 (see [8]). *For any two mesh functions, $u, v \in Z_h^0$, one has*

$$\left((u_j)_x, v_j\right) = -\left(u_j, (v_j)_{\bar{x}}\right),$$
$$\left(v_j, (u_j)_{x\bar{x}}\right) = -\left((v_j)_x, (u_j)_x\right), \tag{2.5}$$
$$\left(u_j, (u_j)_{x\bar{x}}\right) = -\left((u_j)_x, (u_j)_x\right) = -\|u_x\|^2.$$

Furthermore, if $(u_0^n)_{x\bar{x}} = (u_J^n)_{x\bar{x}} = 0$, then

$$\left(u_j, (u_j)_{xx\bar{x}\bar{x}}\right) = \|u_{xx}\|^2. \tag{2.6}$$

Theorem 2.2. *Suppose $u_0 \in H_0^2[0, L]$, then the scheme (2.2)–(2.4) is conservative for discrete energy, that is,*

$$E^n = \|u^n\|^2 + \|u_{xx}^n\|^2 = E^{n-1} = \cdots = E^0. \tag{2.7}$$

Proof. Computing the inner product of (2.2) with $2u^{n+1/2}$, according to boundary condition (2.4) and Lemma 2.1, we have

$$\frac{1}{\tau}\left(\left\|u^{n+1}\right\|^2 - \|u^n\|^2\right) + \frac{1}{\tau}\left(\left\|u_{xx}^{n+1}\right\|^2 - \|u_{xx}^n\|^2\right) + \left(\left(u_j^{n+1/2}\right)_{\hat{x}}, 2u_j^{n+1/2}\right) + \left(P_1, 2u_j^{n+1/2}\right) = 0, \tag{2.8}$$

where

$$P_1 = \frac{p}{p+1}\left\{\left(u_j^{n+1/2}\right)^{p-1}\left(u_j^{n+1/2}\right)_{\hat{x}} + \left[\left(u_j^{n+1/2}\right)^p\right]_{\hat{x}}\right\}. \tag{2.9}$$

According to

$$\left(\left(u_j^{n+1/2}\right)_{\hat{x}}, 2u_j^{n+1/2}\right) = 0, \tag{2.10}$$

$$\begin{aligned}
&\left(P_1, 2u_j^{n+1/2}\right) \\
&= \frac{2p}{p+1} h \sum_{j=0}^{J-1} \left\{ \left(u_j^{n+1/2}\right)^{p-1} \left(u_j^{n+1/2}\right)_{\hat{x}} + \left[\left(u_j^{n+1/2}\right)^p\right]_{\hat{x}} \right\} u_j^{n+1/2} \\
&= \frac{p}{p+1} \sum_{j=0}^{J-1} \left\{ \left(u_j^{n+1/2}\right)^p \left(u_{j+1}^{n+1/2} - u_{j-1}^{n+1/2}\right) + \left[\left(u_{j+1}^{n+1/2}\right)^p - \left(u_{j-1}^{n+1/2}\right)^p\right] u_j^{n+1/2} \right\} \\
&= \frac{p}{p+1} \sum_{j=0}^{J-1} \left[\left(u_{j+1}^{n+1/2}\right)^{p-1} u_j^{n+1/2} + \left(u_j^{n+1/2}\right)^p\right] u_{j+1}^{n+1/2} \\
&\quad - \frac{p}{p+1} \sum_{j=0}^{J-1} \left[\left(u_j^{n+1/2}\right)^{p-1} u_{j-1}^{n+1/2} + \left(u_{j-1}^{n+1/2}\right)^p\right] u_j^{n+1/2} \\
&= 0,
\end{aligned} \tag{2.11}$$

we obtain

$$\left(\left\|u^{n+1}\right\|^2 - \|u^n\|^2\right) + \left(\left\|u_{xx}^{n+1}\right\|^2 - \|u_{xx}^n\|^2\right) = 0. \tag{2.12}$$

By the definition of E^n, (2.7) holds. □

To prove the existence of solution for scheme (2.2)–(2.4), the following Browder fixed point Theorem should be introduced. For the proof, see [9].

Lemma 2.3 (Browder fixed point Theorem). *Let H be a finite dimensional inner product space. Suppose that $g : H \to H$ is continuous and there exists an $\alpha > 0$ such that $(g(x), x) > 0$ for all $x \in H$ with $\|x\| = \alpha$. Then there exists $x^* \in H$ such that $g(x^*) = 0$ and $\|x^*\| \le \alpha$.*

Theorem 2.4. *There exists $u^n \in Z_h^0$ satisfying the difference scheme* (2.2)–(2.4).

Proof. By the mathematical induction, for $n \le N-1$, assume that $u^0, u^1, \ldots, u^n$ satisfy (2.2)–(2.4). Next we prove that there exists u^{n+1} satisfying (2.2)–(2.4).

Define a operator g on Z_h^0 as follows:

$$g(v) = 2v - 2u^n + 2v_{xx\bar{x}\bar{x}} - 2u_{xx\bar{x}\bar{x}}^n + \tau v_{\hat{x}} + \frac{\tau p}{p+1}\left\{ (v_j)^{p-1} (v_j)_{\hat{x}} + \left[(v_j)^p\right]_{\hat{x}} \right\}. \tag{2.13}$$

Taking the inner product of (2.13) with v, we get

$$(v_{\hat{x}}, v) = 0, \qquad \left((v_j)^{p-1}(v_j)_{\hat{x}} + \left[(v_j)^p\right]_{\hat{x}}, v\right) = 0,$$

$$\begin{aligned}
(g(v),v) &= 2\|v\|^2 - 2(u^n,v) + 2\|v_{xx}\|^2 - 2(u^n_{xx},v_{xx}) \\
&\geq 2\|v\|^2 - 2\|u^n\|\cdot\|v\| + 2\|v_{xx}\|^2 - 2\|u^n_{xx}\|\cdot\|v_{xx}\| \\
&\geq 2\|v\|^2 - \left(\|u\|^2 + \|v\|^2\right) + 2\|v_{xx}\|^2 - \left(\|u_{xx}\|^2 + \|v_{xx}\|^2\right) \\
&\geq \|v\|^2 - \left(\|u^n\|^2 + \|u_{xx}\|^2\right) + \|v_{xx}\|^2 \\
&\geq \|v\|^2 - \left(\|u^n\|^2 + \|u^n_{xx}\|^2\right).
\end{aligned} \tag{2.14}$$

Obviously, for all $v \in Z^0_h$, $(g(v),v) \geq 0$ with $\|v\|^2 = \|u^n\|^2 + \|u^n_{xx}\|^2 + 1$. It follows from Lemma 2.3 that there exists $v^* \in Z^0_h$ which satisfies $g(v^*) = 0$. Let $u^{n+1} = 2v^* - u^n$, it can be proved that u^{n+1} is the solution of the scheme (2.2)–(2.4). □

Next, we discuss the convergence and stability of the scheme (2.2)–(2.4). Let $v(x,t)$ be the solution of problem (1.1)–(1.3), $v^n_j = v(x_j,t_n)$, then the truncation of the scheme (2.2)–(2.4) is

$$r^n_j = \left(v^n_j\right)_t + \left(v^n_j\right)_{xx\overline{x}\overline{x}t} + \left(v^{n+1/2}_j\right)_{\hat{x}} + \frac{p}{p+1}\left\{\left(v^{n+1/2}_j\right)^{p-1}\left(v^{n+1/2}_j\right)_{\hat{x}} + \left[\left(v^{n+1/2}_j\right)^p\right]_{\hat{x}}\right\}. \tag{2.15}$$

Using Taylor expansion, we know that $r^n_j = O(\tau^2 + h^2)$ holds if $\tau, h \to 0$.

Lemma 2.5. *Suppose that $u \in H^2_0[0,L]$, then the solution of the initial-boundary value problem* (1.1)–(1.3) *satisfies*

$$\|u\|_{L_2} \leq C, \qquad \|u_x\|_{L_2} \leq C, \qquad \|u\|_\infty \leq C. \tag{2.16}$$

Proof. It follows from (1.4) that

$$\|u\|_{L_2} \leq C, \qquad \|u_{xx}\|_{L_2} \leq C. \tag{2.17}$$

Using Hölder inequality and Schwartz inequality, we get

$$\begin{aligned}
\|u_x\|^2_{L_2} &= \int_0^L u_x u_x dx = u u_x|_0^L - \int_0^L u u_{xx} dx = -\int_0^L u u_{xx} dx \\
&\leq \|u\|_{L_2}\cdot\|u_{xx}\|_{L_2} \leq \frac{1}{2}\left(\|u\|^2_{L_2} + \|u_{xx}\|^2_{L_2}\right).
\end{aligned} \tag{2.18}$$

Hence, $\|u_x\|_{L_2} \leq C$. According to Sobolev inequality, we have $\|u\|_\infty \leq C$. □

Lemma 2.6 (Discrete Sobolev's inequality [10]). *There exist two constant C_1 and C_2 such that*

$$\|u^n\|_\infty \leq C_1\|u^n\| + C_2\|u_x^n\|. \tag{2.19}$$

Lemma 2.7 (Discrete Gronwall inequality [10]). *Suppose $w(k)$, $\rho(k)$ are nonnegative mesh functions and $\rho(k)$ is nondecreasing. If $C > 0$ and*

$$w(k) \leq \rho(k) + C\tau\sum_{l=0}^{k-1} w(l), \quad \forall k, \tag{2.20}$$

then

$$w(k) \leq \rho(k)e^{C\tau k}, \quad \forall k. \tag{2.21}$$

Theorem 2.8. *Suppose $u_0 \in H_0^2[0,L]$, then the solution u^n of* (2.2) *satisfies $\|u^n\| \leq C$, $\|u_x^n\| \leq C$, which yield $\|u^n\|_\infty \leq C$ $(n = 1, 2, \ldots, N)$.*

Proof. It follows from (2.7) that

$$\|u^n\| \leq C, \qquad \|u_{xx}^n\| \leq C. \tag{2.22}$$

Using Lemma 2.1 and Schwartz inequality, we get

$$\|u_x^n\|^2 \leq \|u^n\|\|u_{xx}^n\| \leq \frac{1}{2}\left(\|u^n\|^2 + \|u_{xx}^n\|^2\right) \leq C. \tag{2.23}$$

According to Lemma 2.6, we have $\|u^n\|_\infty \leq C$. □

Theorem 2.9. *Suppose $u_0 \in H_0^2[0,L]$, then the solution u^n of the scheme* (2.2)–(2.4) *converges to the solution of problem* (1.1)–(1.3) *and the rate of convergence is $O(\tau^2 + h^2)$.*

Proof. Subtracting (2.15) from (2.2) and letting $e_j^n = v_j^n - u_j^n$, we have

$$\begin{aligned} r_j^n = {}& \left(e_j^n\right)_t + \left(e_j^n\right)_{xx\bar{x}\bar{x}t} + \left(e_j^{n+1/2}\right)_{\hat{x}} + \frac{p}{p+1}\left\{\left(v_j^{n+1/2}\right)^{p-1}\left(v_j^{n+1/2}\right)_{\hat{x}} + \left[\left(v_j^{n+1/2}\right)^p\right]_{\hat{x}}\right\} \\ & - \frac{p}{p+1}\left\{\left(u_j^{n+1/2}\right)^{p-1}\left(u_j^{n+1/2}\right)_{\hat{x}} + \left[\left(u_j^{n+1/2}\right)^p\right]_{\hat{x}}\right\}. \end{aligned} \tag{2.24}$$

Computing the inner product of (2.24) with $2e^{n+1/2}$, and using $((e_j^{n+1/2})_{\hat{x}},\ 2e_j^{n+1/2}) = 0$, we get

$$\left(r_j^n, 2e^{n+1/2}\right) = \frac{1}{\tau}\left(\left\|e^{n+1}\right\|^2 - \|e^n\|^2\right) + \frac{1}{\tau}\left(\left\|e_{xx}^{n+1}\right\|^2 - \|e_{xx}^n\|^2\right) + \left(Q_1 + Q_2, 2e^{n+1/2}\right), \quad (2.25)$$

where

$$Q_1 = \frac{p}{p+1}\left[\left(v_j^{n+1/2}\right)^{p-1}\left(v_j^{n+1/2}\right)_{\hat{x}} - \left(u_j^{n+1/2}\right)^{p-1}\left(u_j^{n+1/2}\right)_{\hat{x}}\right],$$
$$Q_2 = \frac{p}{p+1}\left\{\left[\left(v_j^{n+1/2}\right)^p\right]_{\hat{x}} - \left[\left(u_j^{n+1/2}\right)^p\right]_{\hat{x}}\right\}. \quad (2.26)$$

According to Lemma 2.5, Theorem 2.8, and Schwartz inequality, we have

$$\begin{aligned}
\left(Q_1, 2e^{n+1/2}\right) &= \frac{2p}{p+1}h\sum_{j=0}^{J-1}\left[\left(v_j^{n+1/2}\right)^{p-1}\left(v_j^{n+1/2}\right)_{\hat{x}} - \left(u_j^{n+1/2}\right)^{p-1}\left(u_j^{n+1/2}\right)_{\hat{x}}\right]e_j^{n+1/2} \\
&= \frac{2p}{p+1}h\sum_{j=0}^{J-1}\left(v_j^{n+1/2}\right)^{p-1}\left(e_j^{n+1/2}\right)_{\hat{x}}e_j^{n+1/2} \\
&\quad + \frac{2p}{p+1}h\sum_{j=0}^{J-1}\left[\left(v_j^{n+1/2}\right)^{p-1} - \left(u_j^{n+1/2}\right)^{p-1}\right]\left(u_j^{n+1/2}\right)_{\hat{x}}e_j^{n+1/2} \\
&= \frac{2p}{p+1}h\sum_{j=0}^{J-1}\left(v_j^{n+1/2}\right)^{p-1}\left(e_j^{n+1/2}\right)_{\hat{x}}e_j^{n+1/2} \\
&\quad + \frac{2p}{p+1}h\sum_{j=0}^{J-1}\left[e_j^{n+1/2}\sum_{k=0}^{p-2}\left(v_j^{n+1/2}\right)^{p-2-k}\left(u_j^{n+1/2}\right)^k\right]\left(u_j^{n+1/2}\right)_{\hat{x}}e_j^{n+1/2} \\
&\le Ch\sum_{j=0}^{J-1}\left|\left(e_j^{n+1/2}\right)_{\hat{x}}\right| \cdot \left|e_j^{n+1/2}\right| + Ch\sum_{j=0}^{J-1}\left|\left(e_j^{n+1/2}\right)^2\right| \\
&\le C\left[\left\|e_x^{n+1/2}\right\|^2 + \left\|e^{n+1/2}\right\|^2\right] \\
&\le C\left[\left\|e_x^{n+1}\right\|^2 + \|e_x^n\|^2 + \left\|e^{n+1}\right\|^2 + \|e^n\|^2\right],
\end{aligned} \quad (2.27)$$

$$\begin{aligned}\left(Q_2, 2e^{n+1/2}\right) &= \frac{2p}{p+1} h \sum_{j=0}^{J-1} \left\{ \left[\left(v_j^{n+1/2} \right)^p \right]_{\hat{x}} - \left[\left(u_j^{n+1/2} \right)^p \right]_{\hat{x}} \right\} e_j^{n+1/2} \\ &= -\frac{2p}{p+1} h \sum_{j=0}^{J-1} \left[\left(v_j^{n+1/2} \right)^p - \left(u_j^{n+1/2} \right)^p \right] \left(e_j^{n+1/2} \right)_{\hat{x}} \\ &= -\frac{2p}{p+1} h \sum_{j=0}^{J-1} e_j^{n+1/2} \left[\sum_{k=0}^{p-1} \left(v_j^{n+1/2} \right)^{p-1-k} \left(u_j^{n+1/2} \right)^k \right] \left(e_j^{n+1/2} \right)_{\hat{x}} \\ &\leq C \left[\left\| e_x^{n+1/2} \right\|^2 + \left\| e^{n+1/2} \right\|^2 \right] \\ &\leq C \left[\left\| e_x^{n+1} \right\|^2 + \| e_x^n \|^2 + \left\| e^{n+1} \right\|^2 + \| e^n \|^2 \right]. \end{aligned} \tag{2.28}$$

Furthermore,

$$\left(r_j^n, 2e^{n+1/2} \right) = \left(r_j^n, e^{n+1} + e^n \right) \leq \| r^n \|^2 + \frac{1}{2} \left[\left\| e^{n+1} \right\|^2 + \| e^n \|^2 \right]. \tag{2.29}$$

Substituting (2.27)–(2.29) into (2.25), we get

$$\left(\left\| e^{n+1} \right\|^2 - \| e^n \|^2 \right) + \left(\left\| e_{xx}^{n+1} \right\|^2 - \| e_{xx}^n \|^2 \right) \leq C\tau \left[\left\| e^{n+1} \right\|^2 + \| e^n \|^2 + \left\| e_x^{n+1} \right\|^2 + \| e_x^n \|^2 \right] + \tau \| r^n \|^2. \tag{2.30}$$

Similarly to the proof of (2.23), we have

$$\left\| e_x^{n+1} \right\|^2 \leq \frac{1}{2} \left(\left\| e^{n+1} \right\|^2 + \left\| e_{xx}^{n+1} \right\|^2 \right), \qquad \| e_x^n \|^2 \leq \frac{1}{2} \left(\| e^n \|^2 + \| e_{xx}^n \|^2 \right), \tag{2.31}$$

and (2.30) can be rewritten as

$$\left(\left\| e^{n+1} \right\|^2 - \| e^n \|^2 \right) + \left(\left\| e_{xx}^{n+1} \right\|^2 - \| e_{xx}^n \|^2 \right) \leq C\tau \left[\left\| e^{n+1} \right\|^2 + \| e^n \|^2 + \left\| e_{xx}^{n+1} \right\|^2 + \| e_{xx}^n \|^2 \right] + \tau \| r^n \|^2. \tag{2.32}$$

Let $B^n = \| e^n \|^2 + \| e_{xx}^n \|^2$, then (2.32) is written as follows:

$$(1 - C\tau) \left[B^{n+1} - B^n \right] \leq 2C\tau B^n + \tau \| r^n \|^2. \tag{2.33}$$

If τ is sufficiently small which satisfies $1 - C\tau > 0$, then

$$\left[B^{n+1} - B^n \right] \leq C\tau B^n + C\tau \| r^n \|^2. \tag{2.34}$$

Summing up (2.34) from 0 to $n-1$, we have

$$B^n \le B^0 + C\tau\sum_{l=0}^{n-1}\left\|r^l\right\|^2 + C\tau\sum_{l=0}^{n-1}B^l. \tag{2.35}$$

Noticing

$$\tau\sum_{l=0}^{n-1}\left\|r^l\right\|^2 \le n\tau \max_{0\le l\le n-1}\left\|r^l\right\|^2 \le T\cdot O\left(\tau^2+h^2\right)^2, \tag{2.36}$$

and $e^0=0$, we have $B^0 = O(\tau^2+h^2)^2$. Hence

$$B^n \le O\left(\tau^2+h^2\right)^2 + C\tau\sum_{l=0}^{n-1}B^l. \tag{2.37}$$

According to Lemma 2.7, we get $B^n \le O(\tau^2+h^2)^2$, that is,

$$\|e^n\| \le O\left(\tau^2+h^2\right), \qquad \|e^n_{xx}\| \le O\left(\tau^2+h^2\right). \tag{2.38}$$

It follows from (2.31) that

$$\|e^n_x\| \le O\left(\tau^2+h^2\right). \tag{2.39}$$

By using Lemma 2.6, we have

$$\|e^n\|_\infty \le O\left(\tau^2+h^2\right). \tag{2.40}$$

This completes the proof of Theorem 2.9. □

Similarly, the following theorem can be proved.

Theorem 2.10. *Under the conditions of Theorem 2.9, the solution of the scheme* (2.2)–(2.4) *is stable by* $\|\cdot\|_\infty$.

3. Linearized Finite Difference Scheme

In this section, we propose a linear-implicit finite difference scheme as follows:

$$\left(u_j^n\right)_{\hat{t}} + \left(u_j^n\right)_{xx\bar{x}\bar{x}\hat{t}} + \left(u_j^n\right)_{\hat{x}} + \frac{p}{p+1}\left\{\left(u_j^n\right)^{p-1}\left(\overline{u}_j^n\right)_{\hat{x}} + \left[\left(u_j^n\right)^{p-1}\overline{u}_j^n\right]_{\hat{x}}\right\} = 0. \tag{3.1}$$

Theorem 3.1. *Suppose $u_0 \in H_0^2[0,L]$, then the scheme* (3.1), (2.3), *and* (2.4) *are conservative for discrete energy, that is,*

$$\widetilde{E}^n = \frac{1}{2}\left(\left\|u^{n+1}\right\|^2 + \|u^n\|^2\right) + \frac{1}{2}\left(\left\|u_{xx}^{n+1}\right\|^2 + \|u_{xx}^n\|^2\right) + \tau h\sum_{j=0}^{J-1}\left(u_j^n\right)_{\hat{x}} u_j^{n+1} = \widetilde{E}^{n-1} = \cdots = \widetilde{E}^0. \tag{3.2}$$

Proof. Computing the inner product of (3.1) with $2\overline{u}^n$, we have

$$\frac{1}{2\tau}\left(\left\|u^{n+1}\right\|^2 - \left\|u^{n-1}\right\|^2\right) + \frac{1}{2\tau}\left(\left\|u_{xx}^{n+1}\right\|^2 - \left\|u_{xx}^{n-1}\right\|^2\right) + \left(\left(u_j^n\right)_{\hat{x}}, 2\overline{u}_j^n\right) + \left(P_2, 2\overline{u}_j^n\right) = 0, \tag{3.3}$$

where

$$P_2 = \frac{p}{p+1}\left\{\left(u_j^n\right)^{p-1}\left(\overline{u}_j^n\right)_{\hat{x}} + \left[\left(u_j^n\right)^{p-1}\overline{u}_j^n\right]_{\hat{x}}\right\}. \tag{3.4}$$

According to Lemma 2.1, we get

$$\begin{aligned}
\left(\left(u_j^n\right)_{\hat{x}}, 2\overline{u}_j^n\right) &= \left(\left(u_j^n\right)_{\hat{x}}, u_j^{n+1}\right) + \left(\left(u_j^n\right)_{\hat{x}}, u_j^{n-1}\right) \\
&= \left(\left(u_j^n\right)_{\hat{x}}, u_j^{n+1}\right) - \left(u_j^n, \left(u_j^{n-1}\right)_{\hat{x}}\right) \\
&= h\sum_{j=0}^{J-1}\left(u_j^n\right)_{\hat{x}} u_j^{n+1} - h\sum_{j=0}^{J-1}\left(u_j^{n-1}\right)_{\hat{x}} u_j^n,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\left(P_2, 2\overline{u}_j^n\right) &= \frac{2p}{p+1} h\sum_{j=0}^{J-1}\left\{\left(u_j^n\right)^{p-1}\left(\overline{u}_j^n\right)_{\hat{x}} + \left[\left(u_j^n\right)^{p-1}\overline{u}_j^n\right]_{\hat{x}}\right\}\overline{u}_j^n \\
&= \frac{p}{p+1}\sum_{j=0}^{J-1}\left[\left(u_j^n\right)^{p-1}\left(\overline{u}_{j+1}^n - \overline{u}_{j-1}^n\right) + \left(u_{j+1}^n\right)^{p-1}\overline{u}_{j+1}^n - \left(u_{j-1}^n\right)^{p-1}\overline{u}_{j-1}^n\right]\overline{u}_j^n \\
&= \frac{p}{p+1}\sum_{j=0}^{J-1}\left[\left(u_j^n\right)^{p-1}\overline{u}_{j+1}^n\overline{u}_j^n - \left(u_{j+1}^n\right)^{p-1}\overline{u}_{j+1}^n\overline{u}_j^n\right] \\
&\quad - \frac{p}{p+1}\sum_{j=0}^{J-1}\left[\left(u_{j-1}^n\right)^{p-1}\overline{u}_j^n\overline{u}_{j-1}^n - \left(u_j^n\right)^{p-1}\overline{u}_j^n\overline{u}_{j-1}^n\right] = 0.
\end{aligned} \tag{3.6}$$

Adding (3.3) and (3.5) to (3.6), we obtain

$$\frac{1}{2\tau}\left(\left\|u^{n+1}\right\|^2 - \|u^n\|^2\right) + \frac{1}{2\tau}\left(\left\|u_{xx}^{n+1}\right\|^2 - \|u_{xx}^n\|^2\right) + h\sum_{j=0}^{J-1}\left(u_j^n\right)_{\hat{x}} u_j^{n+1} - h\sum_{j=0}^{J-1}\left(u_j^{n-1}\right)_{\hat{x}} u_j^n = 0. \tag{3.7}$$

By the definition of $\widetilde{E}^n$, (3.2) holds. □

Theorem 3.2. *The difference scheme* (3.1) *is uniquely solvable.*

Proof. we use the mathematical induction. Obviously, u^0 is determined by (2.3) and we can choose a two-order method to compute u^1 (e.g., by scheme (2.2)). Assuming that $u^0, u^1, \ldots, u^n$ are uniquely solvable, consider u^{n+1} in (3.1) which satisfies

$$\frac{1}{2\tau}u_j^n + \frac{1}{2\tau}\left(u_j^n\right)_{xx\overline{x}\overline{x}} + \frac{p}{2(p+1)}\left\{\left(u_j^n\right)^{p-1}\left(u_j^{n+1}\right)_{\hat{x}} + \left[\left(u_j^n\right)^{p-1}u_j^{n+1}\right]_{\hat{x}}\right\} = 0. \tag{3.8}$$

Taking the inner product of (3.8) with u^{n+1}, we get

$$\frac{1}{2\tau}\left\|u^{n+1}\right\|^2 + \frac{1}{2\tau}\left\|u_{xx}^{n+1}\right\|^2 + \frac{ph}{2(p+1)}\sum_{j=0}^{J-1}\left\{\left(u_j^n\right)^{p-1}\left(u_j^{n+1}\right)_{\hat{x}} + \left[\left(u_j^n\right)^{p-1}u_j^{n+1}\right]_{\hat{x}}\right\}u_j^{n+1}. \tag{3.9}$$

Notice that

$$\begin{aligned}
&\frac{ph}{2(p+1)}\sum_{j=0}^{J-1}\left\{\left(u_j^n\right)^{p-1}\left(u_j^{n+1}\right)_{\hat{x}} + \left[\left(u_j^n\right)^{p-1}u_j^{n+1}\right]_{\hat{x}}\right\}u_j^{n+1}. \\
&\quad = \frac{p}{4(p+1)}\sum_{j=0}^{J-1}\left[\left(u_j^n\right)^{p-1}\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right) + \left(u_{j+1}^n\right)^{p-1}u_{j+1}^{n+1} - \left(u_{j-1}^n\right)^{p-1}u_{j-1}^{n+1}\right]u_j^{n+1} = 0.
\end{aligned} \tag{3.10}$$

It follows from (3.8) that

$$\frac{1}{2\tau}\left\|u^{n+1}\right\|^2 + \frac{1}{2\tau}\left\|u_{xx}^{n+1}\right\|^2 = 0. \tag{3.11}$$

That is, there uniquely exists trivial solution satisfying (3.8). Hence, u_j^{n+1} in (3.1) is uniquely solvable. □

To discuss the convergence and stability of the scheme (3.1), we denote the truncation of the scheme (3.1):

$$\widetilde{r}_j^n = \left(v_j^n\right)_{\hat{t}} + \left(v_j^n\right)_{xx\overline{x}\overline{x}\hat{t}} + \left(v_j^n\right)_{\hat{x}} + \frac{p}{p+1}\left\{\left(v_j^n\right)^{p-1}\left(\overline{v}_j^n\right)_{\hat{x}} + \left[\left(v_j^n\right)^{p-1}\overline{v}_j^n\right]_{\hat{x}}\right\}. \tag{3.12}$$

Using Taylor expansion, we know that $\widetilde{r}_j^n = O(\tau^2 + h^2)$ holds if $\tau, h \to 0$.

Theorem 3.3. *Suppose* $u_0 \in H_0^2[0, L]$, *then the solution of* (3.1) *satisfies* $\|u^n\| \leq C$, $\|u_x^n\| \leq C$, *which yield* $\|u^n\|_\infty \leq C$ $(n = 1, 2, \ldots, N)$.

Proof. It follows from (3.2) that

$$\left(\left\|u^{n+1}\right\|^2+\|u^n\|^2\right)+\left(\left\|u_{xx}^{n+1}\right\|^2+\|u_{xx}^n\|^2\right)=C-2\tau h\sum_{j=0}^{J-1}\left(u_j^n\right)_{\hat{x}}u_j^{n+1}\leq C+\tau\left(\left\|u^{n+1}\right\|^2+\|u_x^n\|^2\right). \tag{3.13}$$

According to (2.23), we have

$$\left(\left\|u^{n+1}\right\|^2+\|u^n\|^2\right)+\left(\left\|u_{xx}^{n+1}\right\|^2+\|u_{xx}^n\|^2\right)\leq C+\tau\left(\left\|u^{n+1}\right\|^2+\frac{1}{2}\|u^n\|^2+\frac{1}{2}\|u_{xx}^n\|^2\right), \tag{3.14}$$

that is,

$$(1-\tau)\left\|u^{n+1}\right\|^2+\left(1-\frac{\tau}{2}\right)\|u^n\|^2+\left\|u_{xx}^{n+1}\right\|^2+\left(1-\frac{\tau}{2}\right)\|u_{xx}^n\|^2\leq C. \tag{3.15}$$

If τ is sufficiently small which satisfies $1-\tau>0$, we get

$$\left(\left\|u^{n+1}\right\|^2+\|u^n\|^2\right)+\left(\left\|u_{xx}^{n+1}\right\|^2+\|u_{xx}^n\|^2\right)\leq C, \tag{3.16}$$

which yields $\|u^n\|\leq C,\ \|u_{xx}^n\|\leq C$. According to (2.23), we get

$$\|u_x^n\|\leq C. \tag{3.17}$$

Using Lemma 2.6, we obtain

$$\|u^n\|_\infty\leq C. \tag{3.18}$$

□

Theorem 3.4. *Suppose $u_0\in H_0^2[0,L]$, then the solution u^n of the schemes* (3.1), (2.3), *and* (2.4) *converges to the solution of problem* (1.1)–(1.3) *and the rate of convergence is $O(\tau^2+h^2)$.*

Proof. Subtracting (3.12) from (3.1) and letting $e_j^n=v_j^n-u_j^n$, we have

$$\begin{aligned}\widetilde{r}_j^n=&\left(e_j^n\right)_{\hat{t}}+\left(e_j^n\right)_{xx\bar{x}\bar{x}\hat{t}}+\left(e_j^n\right)_{\hat{x}}+\frac{p}{p+1}\left\{\left(v_j^n\right)^{p-1}\left(\overline{v}_j^n\right)_{\hat{x}}+\left[\left(v_j^n\right)^{p-1}\overline{v}_j^n\right]_{\hat{x}}\right\}\\&-\frac{p}{p+1}\left\{\left(u_j^n\right)^{p-1}\left(\overline{u}_j^n\right)_{\hat{x}}+\left[\left(u_j^n\right)^{p-1}\overline{u}_j^n\right]_{\hat{x}}\right\}.\end{aligned} \tag{3.19}$$

Computing the inner product of (3.19) with $2\overline{e}^n$, we get

$$\left(\widetilde{r}_j^n, 2\overline{e}^n\right) = \frac{1}{2\tau}\left(\left\|e^{n+1}\right\|^2 - \left\|e^{n-1}\right\|^2\right) + \frac{1}{2\tau}\left(\left\|e_{xx}^{n+1}\right\|^2 - \left\|e_{xx}^{n-1}\right\|^2\right) + \left(\left(e_j^n\right)_{\hat{x}}, 2\overline{e}^n\right) + \left(Q_3 + Q_4, 2\overline{e}^n\right), \tag{3.20}$$

where

$$Q_3 = \frac{p}{p+1}\left[\left(v_j^n\right)^{p-1}\left(\overline{v}_j^n\right)_{\hat{x}} - \left(u_j^n\right)^{p-1}\left(\overline{u}_j^n\right)_{\hat{x}}\right],$$
$$Q_4 = \frac{p}{p+1}\left\{\left[\left(v_j^n\right)^{p-1}\left(\overline{v}_j^n\right)\right]_{\hat{x}} - \left[\left(u_j^n\right)^{p-1}\overline{u}_j^n\right]_{\hat{x}}\right\}. \tag{3.21}$$

Notice that

$$\begin{aligned}
\left(Q_3, 2\overline{e}^n\right) &= \frac{2ph}{p+1}\sum_{j=0}^{J-1}\left[\left(v_j^n\right)^{p-1}\left(\overline{v}_j^n\right)_{\hat{x}} - \left(u_j^n\right)^{p-1}\left(\overline{u}_j^n\right)_{\hat{x}}\right]\overline{e}^n \\
&= \frac{2ph}{p+1}\sum_{j=0}^{J-1}\left(v_j^n\right)^{p-1}\left(\overline{e}_j^n\right)_{\hat{x}}\overline{e}^n + \frac{2ph}{p+1}\sum_{j=0}^{J-1}\left[\left(v_j^n\right)^{p-1} - \left(u_j^n\right)^{p-1}\right]\left(\overline{u}_j^n\right)_{\hat{x}}\overline{e}^n \\
&= \frac{2ph}{p+1}\sum_{j=0}^{J-1}\left(v_j^n\right)^{p-1}\left(\overline{e}_j^n\right)_{\hat{x}}\overline{e}^n + \frac{2ph}{p+1}\sum_{j=0}^{J-1}\left[e_j^n\sum_{k=0}^{p-2}\left(v_j^n\right)^{p-2-k}\left(u_j^n\right)^k\right]\left(\overline{u}_j^n\right)_{\hat{x}}\overline{e}^n \\
&\le C\left[\left\|\overline{e}_x^n\right\|^2 + \left\|e^n\right\|^2 + \left\|\overline{e}^n\right\|^2\right] \\
&\le C\left[\left\|e_x^{n+1}\right\|^2 + \left\|e_x^{n-1}\right\|^2 + \left\|e^{n+1}\right\|^2 + \left\|e^n\right\|^2 + \left\|e^{n-1}\right\|^2\right],
\end{aligned} \tag{3.22}$$

and similarly

$$\left(Q_4, 2\overline{e}^n\right) \le C\left[\left\|e_x^{n+1}\right\|^2 + \left\|e_x^{n-1}\right\|^2 + \left\|e^{n+1}\right\|^2 + \left\|e^n\right\|^2 + \left\|e^{n-1}\right\|^2\right]. \tag{3.23}$$

Furthermore, we get

$$\left(\widetilde{r}_j^n, 2\overline{e}^n\right) = \left(\widetilde{r}_j^n, e^{n+1} + e^{n-1}\right) \le \left\|\widetilde{r}^n\right\|^2 + \frac{1}{2}\left[\left\|e^{n+1}\right\|^2 + \left\|e^{n-1}\right\|^2\right],$$
$$\left(\left(e_j^n\right)_{\hat{x}}, 2\overline{e}^n\right) = \left(\left(e_j^n\right)_{\hat{x}}, e^{n+1} + e^{n-1}\right) \le \left\|e_x^n\right\|^2 + \frac{1}{2}\left[\left\|e^{n+1}\right\|^2 + \left\|e^{n-1}\right\|^2\right]. \tag{3.24}$$

Substituting (3.22)–(3.24) into (3.20), we get

$$\begin{aligned}&\left(\left\|e^{n+1}\right\|^2-\left\|e^{n-1}\right\|^2\right)+\left(\left\|e_{xx}^{n+1}\right\|^2-\left\|e_{xx}^{n-1}\right\|^2\right)\\&\quad\leq C\tau\left[\left\|e^{n+1}\right\|^2+\left\|e^{n-1}\right\|^2+\|e^n\|^2+\left\|e_x^{n+1}\right\|^2+\|e_x^n\|^2+\left\|e_x^{n-1}\right\|^2\right]+2\tau\|\widetilde{r}^n\|^2.\end{aligned}\tag{3.25}$$

Similarly to the proof of (2.31), (3.25) can be written as

$$\begin{aligned}&\left[\left(\left\|e^{n+1}\right\|^2+\|e^n\|^2\right)+\left(\left\|e_{xx}^{n+1}\right\|^2+\|e_{xx}^n\|^2\right)\right]-\left[\left(\|e^n\|^2+\left\|e^{n-1}\right\|^2\right)+\left(\|e_{xx}^n\|^2+\left\|e_{xx}^{n-1}\right\|^2\right)\right]\\&\quad\leq C\tau\left[\left\|e^{n+1}\right\|^2+\|e^n\|^2+\left\|e^{n-1}\right\|^2+\left\|e_{xx}^{n+1}\right\|^2+\|e_{xx}^n\|^2+\left\|e_{xx}^{n-1}\right\|^2\right]+2\tau\|\widetilde{r}^n\|^2.\end{aligned}\tag{3.26}$$

Let $D^n=(\|e^{n+1}\|^2+\|e^n\|^2)+(\|e_{xx}^{n+1}\|^2+\|e_{xx}^n\|^2)$, then (3.26) is written as follows:

$$\begin{aligned}D^n-D^{n-1}&\leq C\tau\left[\left\|e^{n+1}\right\|^2+\|e^n\|^2+\left\|e^{n-1}\right\|^2+\left\|e_{xx}^{n+1}\right\|^2+\|e_{xx}^n\|^2+\left\|e_{xx}^{n-1}\right\|^2\right]+2\tau\|\widetilde{r}^n\|^2\\&\leq C\tau\left(D^n+D^{n-1}\right)+2\tau\|\widetilde{r}^n\|^2,\end{aligned}\tag{3.27}$$

that is,

$$(1-C\tau)\ \left[D^n-D^{n-1}\right]\leq 2C\tau D^{n-1}+2\tau\|\widetilde{r}^n\|^2.\tag{3.28}$$

If τ is sufficiently small which satisfies $1-C\tau>0$, then

$$\left[D^n-D^{n-1}\right]\leq C\tau D^{n-1}+C\tau\|\widetilde{r}^n\|^2.\tag{3.29}$$

Summing up (3.29) from 1 to n, we have

$$D^n\leq D^0+C\tau\sum_{l=1}^{n}\left\|\widetilde{r}^l\right\|^2+C\tau\sum_{l=1}^{n}D^l.\tag{3.30}$$

Choosing a two-order method to compute u^1 (e.g., by scheme (2.2)) and noticing

$$\tau\sum_{l=1}^{n}\left\|\widetilde{r}^l\right\|^2\leq n\tau\max_{1\leq l\leq n}\left\|\widetilde{r}^l\right\|^2\leq T\cdot O\left(\tau^2+h^2\right)^2,\tag{3.31}$$

Table 1: The errors estimates in the sense of $\|\cdot\|_\infty$, when $p = 2$ and $\tau = 0.1$.

	$h = 1/8$		$h = 1/16$		$h = 1/32$	
	Scheme (2.2)	Scheme (3.1)	Scheme (2.2)	Scheme (3.1)	Scheme (2.2)	Scheme (3.1)
$t = 2$	$4.7028e-8$	$4.7035e-8$	$5.8077e-9$	$5.8145e-8$	$1.0617e-9$	$1.0692e-9$
$t = 4$	$1.2527e-7$	$1.2528e-7$	$1.8815e-8$	$1.8823e-8$	$3.9301e-9$	$3.9387e-9$
$t = 6$	$2.3471e-7$	$2.3472e-7$	$3.9308e-8$	$3.9318e-8$	$8.6770e-9$	$8.6778e-9$
$t = 8$	$3.7529e-7$	$3.7531e-7$	$6.7171e-8$	$6.7191e-8$	$1.5273e-8$	$1.5287e-8$
$t = 10$	$5.4699e-7$	$5.4701e-7$	$1.0272e-7$	$1.0273e-7$	$2.3778e-8$	$2.3794e-8$

Table 2: The errors estimates in the sense of $\|\cdot\|_\infty$, when $p = 4$ and $\tau = 0.1$.

	$h = 1/8$		$h = 1/16$		$h = 1/32$	
	Scheme (2.2)	Scheme (3.1)	Scheme (2.2)	Scheme (3.1)	Scheme (2.2)	Scheme (3.1)
$t = 2$	$4.6363e-8$	$4.6358e-8$	$5.5803e-9$	$5.5749e-8$	$1.0400e-9$	$1.0340e-9$
$t = 4$	$1.2377e-7$	$1.2376e-7$	$1.8485e-8$	$1.8479e-8$	$3.8859e-9$	$3.8798e-9$
$t = 6$	$2.3221e-7$	$2.3220e-7$	$3.8575e-8$	$3.8743e-8$	$8.6015e-9$	$8.5936e-9$
$t = 8$	$3.7165e-7$	$3.7164e-7$	$6.6364e-8$	$6.6352e-8$	$1.5181e-8$	$1.5176e-8$
$t = 10$	$5.4204e-7$	$5.4203e-7$	$1.0197e-7$	$1.0197e-7$	$2.3657e-8$	$2.3649e-8$

we have

$$D^n \le O\left(\tau^2 + h^2\right)^2 + C\tau \sum_{l=0}^{n-1} D^l. \tag{3.32}$$

According to Lemma 2.7, we get $D^n \le O(\tau^2 + h^2)^2$, that is,

$$\|e^n\| \le O\left(\tau^2 + h^2\right), \qquad \|e^n_{xx}\| \le O\left(\tau^2 + h^2\right). \tag{3.33}$$

According to (2.31), we get

$$\|e^n_x\| \le O\left(\tau^2 + h^2\right). \tag{3.34}$$

By using Lemma 2.6, we have

$$\|e^n\|_\infty \le O\left(\tau^2 + h^2\right). \tag{3.35}$$

This completes the proof of Theorem 3.4. □

Similarly, the following theorem can be proved that.

Theorem 3.5. *Under the conditions of Theorem 3.4, the solution of the schemes* (3.1), (2.3), *and* (2.4) *are stable by* $\|\cdot\|_\infty$.

Table 3: The errors estimates in the sense of $\|\cdot\|_\infty$, when $p = 8$ and $\tau = 0.1$.

	$h = 1/8$		$h = 1/16$		$h = 1/32$	
	Scheme (2.2)	Scheme (3.1)	Scheme (2.2)	Scheme (3.1)	Scheme (2.2)	Scheme (3.1)
$t = 2$	$4.6349e-8$	$4.6353e-8$	$5.5673e-9$	$5.5571e-8$	$1.0254e-9$	$1.0301e-9$
$t = 4$	$1.2375e-7$	$1.2377e-7$	$1.8468e-8$	$1.8476e-8$	$3.8683e-9$	$3.8764e-9$
$t = 6$	$2.3219e-7$	$2.3220e-7$	$3.8733e-8$	$3.8741e-8$	$8.5827e-9$	$8.5914e-9$
$t = 8$	$3.7163e-7$	$3.7164e-7$	$6.6344e-8$	$6.6352e-8$	$1.5165e-8$	$1.5173e-8$
$t = 10$	$5.4202e-7$	$5.4203e-7$	$1.0195e-7$	$1.0196e-7$	$2.3631e-8$	$2.3645e-8$

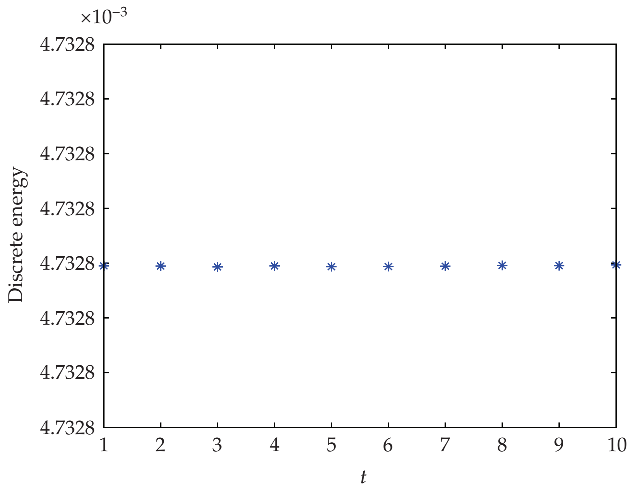

Figure 1: Energy of scheme (2.2) when $h = 1/32$ and $\tau = 0.1$.

4. Numerical Experiments

Consider the generalized Rosenau equation:

$$u_t + u_{xxxxt} + u_x + (u^p)_x = 0, \quad (x,t) \in [0,1] \times [0,T], \tag{4.1}$$

with an initial condition

$$u(x,0) = x^4(1-x)^4, \quad x \in [0,1], \tag{4.2}$$

and boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad t \in [0,T]. \tag{4.3}$$

We construct two schemes to (4.1)–(4.3) as nonlinear scheme (2.2) and linearized scheme (3.1). Since we do not know the exact solution of (4.1)–(4.3), we consider the solution on mesh $h = 1/160$ as reference solution and obtain the error estimates on mesh $h = 1/8, 1/16, 1/32$, respectively, for different choices of p, where we take $p = 2, 4, 8$. To verify

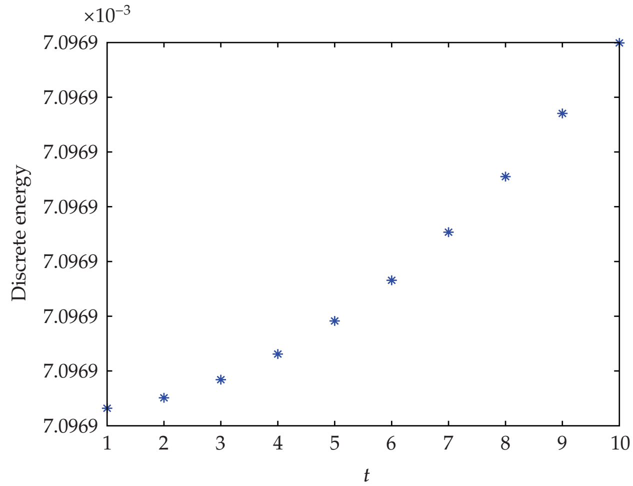

Figure 2: Energy of scheme (3.1) when $h = 1/32$ and $\tau = 0.1$.

the stability of schemes, we take $T = 10$. The maximal errors e^n are listed on Tables 1, 2, and 3.

We have shown in Theorems 2.2 and 3.1 that the numerical solutions u^n of Scheme (2.2) and Scheme (3.1) satisfy the conservation of energy, respectively. In Figure 1, we give the values of $\|u^n\|^2 + \|u^n_{xx}\|^2$ for $h = 1/32$ with fixed $\tau = 0.1$ for Scheme (2.2). In Figure 2, the values of $(1/2)(\|u^{n+1}\|^2 + \|u^{n+1}_{xx}\|^2) + (1/2)(\|u^n\|^2 + \|u^n_{xx}\|^2) + \tau h \sum_{j=0}^{J-1}(u^n_j)_{\hat{x}} u^{n+1}_j$ for Scheme (3.1) are presented. We can see that scheme (2.2) preserves the discrete energy better than scheme (3.1).

From the numerical results, two finite difference schemes of this paper are efficient.

Acknowledgment

This work was supported by the Youth Research Foundation of SWUST (no. 08zx3125).

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