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Research Article

Multiple Positive Solutions for Singular Elliptic Equations with Concave-Convex Nonlinearities and Sign-Changing Weights

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We study existence and multiplicity of positive solutions for the following Dirichlet equations: $-\Delta u-(\mu/|x|^2)u=\lambda f(x)|u|^{q-2}u+g(x)|u|^{2^*-2}u$ in Ω, $u=0$ on $\partial\Omega$, where $0\in\Omega\subset\mathbb{R}^N(N\geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda>0$, $0\leq\mu<\overline{\mu}=(N-2)^2/4$, $2^*=2N/(N-2)$, $1\leq q<2$, and f,g are continuous functions on $\overline{\Omega}$ which are somewhere positive but which may change sign on Ω.

1. Introduction and Main Results

In this paper, we study the existence and multiplicity of positive solutions for the following singular elliptic equation:

$$\begin{gathered}-\Delta u-\frac{\mu}{|x|^2}u=\lambda f(x)|u|^{q-2}u+g(x)|u|^{p-2}u\quad\text{in }\Omega,\\ u=0\quad\text{on }\partial\Omega,\end{gathered}\tag{$P_{\mu,\lambda,f,g}$}$$

where $0\in\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda>0$, $0\leq\mu<\overline{\mu}=(N-2)^2/4$, $\overline{\mu}$ is the best constant in the Hardy inequality, $1\leq q<2<p$, and $f,g:\overline{\Omega}\to\mathbb{R}$ are continuous functions which are somewhere positive but which may change sign on Ω. We will assume in this paper that p is a critical Sobolev exponent, that is, $p=2^*=2N/(N-2)$.

When $\mu=0$ and weight functions $f(x)\equiv g(x)\equiv 1$ on $\overline{\Omega}$, $(P_{\mu,\lambda,f,g})$ has been studied extensively for $2<p\leq 2^*$ and various $q>1$. See, for example, [1–3] and the references therein. In [4], Wu has proved that there exists $\lambda_0>0$ such that $(P_{\mu,\lambda,f,g})$ admits at least two solutions for all $\lambda \in (0, \lambda_0)$ with $1 \le q < 2$, a subcritical exponent $p \in (2, 2^*)$, $g(x) \equiv 1$ on $\overline{\Omega}$ and f is a continuous function which change sign in Ω. In a recent work [5], Hsu-Lin have showed the existence and multiplicity of positive solutions of $(P_{\mu,\lambda,f,g})$ with a critical exponent $p= 2^*$ and sign-changing weight functions f, g.

To proceed, we make some motivations of the present paper. In [6], Chen studied $(P_{\mu,\lambda,f,g})$ assuming that $0 \le \mu < \overline{\mu} - 1$, $1 \le q < 2$, $p= 2^*$ and $f(x) \equiv g(x) \equiv 1$ on $\overline{\Omega}$. He proved that there exists $\Lambda > 0$ such that $(P_{\mu,\lambda,f,g})$ has at least two positive solutions in $H_0^1(\Omega)$ for any $\lambda \in (0, \Lambda)$. But we do not see any multiplicity results about $(P_{\mu,\lambda,f,g})$ in the case of the critical exponent $p= 2^*$ and the weight functions f, g sign-changing. In the present paper, we continue the study of [5] by considering the general case $\mu \in [0, \overline{\mu})$. We will extend the results of [6] to the more general case with $\mu \in [0, \overline{\mu})$ and the weight functions f, g which may change sign on Ω. Our assumptions are

$(f1)$ $f \in C(\overline{\Omega})$ and $f^+ = \max\{f, 0\} \not\equiv 0$ in Ω,

$(g1)$ $g \in C(\overline{\Omega})$ and $g^+ = \max\{g, 0\} \not\equiv 0$ in Ω.

Set

$$\Lambda_1 = \left(\frac{2-q}{(2^* - q)|g^+|_\infty} \right)^{(2-q)/(2^*-2)} \left(\frac{2^* - 2}{(2^* - q)|f^+|_\infty} \right) |\Omega|^{(q-2^*)/2^*} S_\mu^{(N/2)-(N/4)q+(q/2)} > 0, \quad (1.1)$$

where $|\Omega|$ is the Lebesgue measure of Ω, and S_μ is the best Sobolev constant (see (2.2)). Now, we state the first main result about the existence of positive solution of $(P_{\mu,\lambda,f,g})$.

Theorem 1.1. *Assume $(f1)$ and $(g1)$ hold. If $\lambda \in (0, \Lambda_1)$, then $(P_{\mu,\lambda,f,g})$ (simply written as (P_μ) from now on) has at least one positive solution in $H_0^1(\Omega)$.*

In order to get the second positive solution of (P_μ), we need some additional assumptions about f and g. We assume the following conditions on f and g:

$(f2)$ there exist β_0 and $\rho_0 > 0$ such that $B(0, 2\rho_0) \subset \Omega$ and $f(x) \ge \beta_0$ for all $x \in B(0, 2\rho_0)$;

$(g2)$ $|g^+|_\infty = g(0) = \max_{x \in \overline{\Omega}} g(x)$, $g(x) > 0$ for all $x \in B(0, 2\rho_0)$ and there exists $\beta \in (\sqrt{\overline{\mu} - \mu} N / \sqrt{\overline{\mu}}, \sqrt{\overline{\mu} - \mu}(N+1)/\sqrt{\overline{\mu}})$ such that

$$g(x) = g(0) + o(|x|^\beta) \quad \text{as} \quad x \longrightarrow 0. \quad (1.2)$$

Theorem 1.2. *Assume that $(f1)$-$(f2)$ and $(g1)$-$(g2)$ hold. Then there exists $\Lambda_2 > 0$ such that for $\lambda \in (0, \Lambda_2)$, (P_μ) has at least two positive solutions in $H_0^1(\Omega)$.*

This paper is organized as follows. In Sections 2 and 3, we give some preliminaries and some properties of Nehari manifold. In Sections 4 and 5, we complete proofs of Theorems 1.1 and 1.2.

2. Preliminaries

Throughout this paper, $(f1)$ and $(g1)$ will be assumed. The dual space of a Banach space E will be denoted by E^{-1}. $H_0^1(\Omega)$ denotes the standard Sobolev space, whose norm $\| \cdot \|$ is induced by the standard inner product. We denote the norm in $L^2(\Omega)$ by $|\cdot|_2$ and the norm in $L^2(\mathbb{R}^N)$ by $|\cdot|_{L^2(\mathbb{R}^N)}$. $\mathfrak{D}^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$ with usual norm $\|\cdot\|_{\mathfrak{D}}^2 = \int_{\mathbb{R}^N}|\nabla\cdot|^2dx$. $|\Omega|$ is the Lebesgue measure of Ω. $B(x,r)$ is a ball centered at x with radius r. $O(\varepsilon^t)$ denotes $|O(\varepsilon^t)|/\varepsilon^t \leq C$, $o(\varepsilon^t)$ denotes $|o(\varepsilon^t)|/\varepsilon^t \to 0$ as $\varepsilon \to 0$, and $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$. All integrals are taken over Ω unless stated otherwise. C, C_i will denote various positive constants, the exact values of which are not important. On $H_0^1(\Omega)$, we use the norm

$$\|u\|_\mu^2 = \int\left(|\nabla u|^2 - \frac{\mu}{|x|^2}u^2\right)dx. \tag{2.1}$$

Thanks to the Hardy inequality, the norm $\|\cdot\|_\mu$ is equivalent to the usual norm $\|\cdot\|$ of $H_0^1(\Omega)$. $H_0^1(\Omega)$ with the norm $\|\cdot\|_\mu$ is simply denoted by H. For all $\mu \in [0,\overline{\mu})$, we define the constant

$$S_\mu = \inf_{u\in\mathfrak{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^2 - (\mu/|x|^2)u^2)dx}{\left(\int_{\mathbb{R}^N}|u|^{2^*}dx\right)^{2/2^*}}. \tag{2.2}$$

From [7, 8], S_μ is independent of $\Omega \subset \mathbb{R}^N$ in the sense that if

$$S_\mu(\Omega) = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_{\Omega}(|\nabla u|^2 - (\mu/|x|^2)u^2)dx}{\left(\int_{\Omega}|u|^{2^*}dx\right)^{2/2^*}}, \tag{2.3}$$

then $S_\mu(\Omega) = S_\mu(\mathbb{R}^N) = S_\mu$.

Let $\overline{\mu} = ((N-2)/2)^2$, $\gamma_1 = \sqrt{\overline{\mu}} - \sqrt{\overline{\mu}-\mu}$, $\gamma_2 = \sqrt{\overline{\mu}} + \sqrt{\overline{\mu}-\mu}$; Catrina and Wang [9], Terracini [10] proved that S_μ is attained by the function

$$U(x) = \frac{1}{\left[|x|^{\gamma_1/\sqrt{\overline{\mu}}} + |x|^{\gamma_2/\sqrt{\overline{\mu}}}\right]^{\sqrt{\overline{\mu}}}}. \tag{2.4}$$

Moreover, for $\varepsilon > 0$, $U_\varepsilon(x) = \varepsilon^{-(N-2)/2}[4N(\overline{\mu}-\mu)/(N-2)]^{(N-2)/4}U(x/\varepsilon)$ satisfies

$$\begin{aligned} -\Delta u - \frac{\mu}{|x|^2}u &= |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N \setminus \{0\}, \\ u &\longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty. \end{aligned} \tag{2.5}$$

From [11, Theorem B], all the positive solutions of problem (2.5) must have the form of U_ε. Moreover, U_ε attains S_μ.

We end these preliminaries by the following definition.

Definition 2.1. Let $c \in \mathbb{R}$, E be a Banach space and $I \in C^1(E,\mathbb{R})$.

(i) $\{u_n\}$ is a $(\mathrm{PS})_c$-sequence in E for I if $I(u_n) = c + o_n(1)$ and $I'(u_n) = o_n(1)$ strongly in E^{-1} as $n \to \infty$.

(ii) We say that I satisfies the $(\mathrm{PS})_c$-condition if any $(\mathrm{PS})_c$-sequence $\{u_n\}$ in E for I has a convergent subsequence.

3. Nehari Manifold

Associated with (P_μ), we consider the energy functional J_λ in H, for each $u \in H$ as follows:

$$J_\lambda(u) = \frac{1}{2}\|u\|_\mu^2 - \frac{\lambda}{q}\int f|u|^q dx - \frac{1}{2^*}\int g|u|^{2^*} dx. \tag{3.1}$$

It is well known that J_λ is of C^1 in H, and the solutions of (P_μ) are the critical points of the energy functional J_λ (see Rabinowitz [12]).

As the energy functional J_λ is not bounded below on H, it is useful to consider the functional Nehari manifold

$$\mathcal{N}_\lambda = \left\{u \in H \setminus \{0\} : \langle J_\lambda'(u), u\rangle = 0\right\}. \tag{3.2}$$

Thus, $u \in \mathcal{N}_\lambda$ if and only if

$$\langle J_\lambda'(u), u\rangle = \|u\|_\mu^2 - \lambda\int f|u|^q dx - \int g|u|^{2^*} dx = 0. \tag{3.3}$$

Note that $\mathcal{N}_\lambda$ contains every nonzero solution of (P_μ). Moreover, we have the following results.

Lemma 3.1. *The energy functional J_λ is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof. If $u \in \mathcal{N}_\lambda$, then by $(f1)$, (3.3), the Hölder inequality and the Sobolev embedding theorem

$$J_\lambda(u) = \frac{2^*-2}{2^*2}\|u\|_\mu^2 - \lambda\left(\frac{2^*-q}{2^*q}\right)\int f|u|^q dx \tag{3.4}$$

$$\geq \frac{1}{N}\|u\|_\mu^2 - \lambda\left(\frac{2^*-q}{2^*q}\right)S_\mu^{-(q/2)}|\Omega|^{\left(2^*-q\right)/2^*}\|u\|_\mu^q\left|f^+\right|_\infty. \tag{3.5}$$

Thus, J_λ is coercive and bounded below on $\mathcal{N}_\lambda$. □

Define

$$\psi_\lambda(u) = \langle J_\lambda'(u), u\rangle. \tag{3.6}$$

Then for $u \in \mathcal{N}_\lambda$,

$$\begin{aligned}\langle \psi_\lambda'(u), u\rangle &= 2\|u\|_\mu^2 - \lambda q\int f|u|^q dx - 2^*\int g|u|^{2^*} dx \\ &= (2-q)\|u\|_\mu^2 - \left(2^*-q\right)\int g|u|^{2^*} dx \\ &= \lambda\left(2^*-q\right)\int f|u|^q dx - \left(2^*-2\right)\|u\|_\mu^2.\end{aligned} \tag{3.7}$$

Similar to the method used in Tarantello [13], we split $\mathcal{N}_\lambda$ into three parts:

$$\begin{aligned}
\mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : \langle \psi_\lambda'(u), u\rangle > 0\},\\
\mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : \langle \psi_\lambda'(u), u\rangle = 0\},\\
\mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : \langle \psi_\lambda'(u), u\rangle < 0\}.
\end{aligned} \tag{3.8}$$

Then, we have the following results.

Lemma 3.2. *Assume that u_λ is a local minimizer for J_λ on $\mathcal{N}_\lambda$ and $u_\lambda \notin \mathcal{N}_\lambda^0$. Then $J_\lambda'(u_\lambda) = 0$ in $H^{-1}(\Omega)$.*

Proof. Our proof is almost the same as that in Brown-Zhang [14, Theorem 2.3] (or see Binding-Drábek-Huang [15]) . □

Lemma 3.3. *If $\lambda \in (0, \Lambda_1)$, then $\mathcal{N}_\lambda^0 = \emptyset$, where Λ_1 is the same as in* (1.1).

Proof. Suppose otherwise, that is there exists $\lambda \in (0, \Lambda_1)$ such that $\mathcal{N}_\lambda^0 \neq \emptyset$. Then by (3.7), for $u \in \mathcal{N}_\lambda^0$, we have

$$\begin{aligned}
\|u\|_\mu^2 &= \frac{2^* - q}{2 - q}\int g|u|^{2^*}dx,\\
\|u\|_\mu^2 &= \lambda\frac{2^* - q}{2^* - 2}\int f|u|^q dx.
\end{aligned} \tag{3.9}$$

Moreover, by $(f1)$, $(g1)$, the Hölder inequality, and the Sobolev embedding theorem, we have

$$\begin{aligned}
\|u\|_\mu &\geq \left(\frac{2 - q}{(2^* - q)|g^+|_\infty} S_\mu^{2^*/2}\right)^{1/(2^*-2)},\\
\|u\|_\mu &\leq \left[\lambda\frac{2^* - q}{2^* - 2} S_\mu^{-(q/2)}|\Omega|^{(2^*-q)/2^*}\left|f^+\right|_\infty\right]^{1/(2-q)}.
\end{aligned} \tag{3.10}$$

This implies

$$\lambda \geq \left(\frac{2 - q}{(2^* - q)\left|g^+\right|_\infty}\right)^{(2-q)/(2^*-2)}\left(\frac{2^* - 2}{(2^* - q)\left|f^+\right|_\infty}\right)|\Omega|^{(q-2^*)/2^*} S_\mu^{(N/2)-(N/4)q+(q/2)} = \Lambda_1, \tag{3.11}$$

which is a contradiction. Thus, we can conclude that if $\lambda \in (0, \Lambda_1)$, we have $\mathcal{N}_\lambda^0 = \emptyset$. □

By Lemma 3.3, we write $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ and define

$$\alpha_\lambda = \inf_{u\in\mathcal{N}_\lambda} J_\lambda(u), \qquad \alpha_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u), \qquad \alpha_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u). \tag{3.12}$$

Then we get the following result.

Lemma 3.4. (i) *If* $\lambda \in (0, \Lambda_1)$, *then one has* $\alpha_\lambda \leq \alpha_\lambda^+ < 0$.

(ii) *If* $\lambda \in (0, (q/2)\Lambda_1)$, *then* $\alpha_\lambda^- > d_0$ *for some positive constant* d_0 *depending on* λ, μ, q, N, S_μ, $|f^+|_\infty$, $|g^+|_\infty$ *and* $|\Omega|$.

Proof. (i) Let $u \in \mathcal{N}_\lambda^+$. By (3.7)

$$\frac{2-q}{2^*-q}\|u\|_\mu^2 > \int g|u|^{2^*}dx, \tag{3.13}$$

and so

$$\begin{aligned} J_\lambda(u) &= \left(\frac{1}{2}-\frac{1}{q}\right)\|u\|_\mu^2 + \left(\frac{1}{q}-\frac{1}{2^*}\right)\int g|u|^{2^*}dx \\ &< \left[\left(\frac{1}{2}-\frac{1}{q}\right) + \left(\frac{1}{q}-\frac{1}{2^*}\right)\frac{2-q}{2^*-q}\right]\|u\|_\mu^2 \\ &= -\frac{2-q}{qN}\|u\|_\mu^2 < 0. \end{aligned} \tag{3.14}$$

Therefore, from the definitions of α_λ, α_λ^+, we can deduce that $\alpha_\lambda \leq \alpha_\lambda^+ < 0$.

(ii) Let $u \in \mathcal{N}_\lambda^-$. By (3.7)

$$\frac{2-q}{2^*-q}\|u\|_\mu^2 < \int g|u|^{2^*}dx. \tag{3.15}$$

Moreover, by $(g1)$ and the Sobolev embedding theorem,

$$\int g|u|^{2^*}dx \leq S_\mu^{-(2^*/2)}\|u\|_\mu^{2^*}\left|g^+\right|_\infty. \tag{3.16}$$

This implies

$$\|u\|_\mu > \left(\frac{2-q}{(2^*-q)|g^+|_\infty}\right)^{1/(2^*-2)} S_\mu^{N/4} \quad \forall u \in \mathcal{N}_\lambda^-. \tag{3.17}$$

By (3.5) in the proof of Lemma 3.1

$$\begin{aligned} J_\lambda(u) &\geq \|u\|_\mu^q\left[\frac{1}{N}\|u\|_\mu^{2-q} - \lambda S_\mu^{-(q/2)}\frac{2^*-q}{2^*q}|\Omega|^{(2^*-q)/2^*}|f^+|_\infty\right] \\ &> \left(\frac{2-q}{(2^*-q)\left|g^+\right|_\infty}\right)^{q/\left(2^*-2\right)} S_\mu^{qN/4}\left[\frac{1}{N}S_\mu^{(2-q)N/4}\left(\frac{2-q}{(2^*-q)\left|g^+\right|_\infty}\right)^{(2-q)/(2^*-2)} \right. \\ &\qquad \left. - \lambda S_\mu^{-(q/2)}\frac{2^*-q}{2^*q}|\Omega|^{(2^*-q)/2^*}\left|f^+\right|_\infty\right]. \end{aligned} \tag{3.18}$$

Thus, if $\lambda \in (0, (q/2)\Lambda_1)$, then

$$J_\lambda(u) > d_0 \quad \forall u \in \mathcal{N}_\lambda^-, \tag{3.19}$$

for some positive constant $d_0 = d_0(\lambda, q, N, S_\mu, |f^+|_\infty, |g^+|_\infty, |\Omega|)$. This completes the proof. □

For each $u \in H$ with $\int g|u|^{2^*}dx > 0$, we write

$$t_{\max} = \left(\frac{(2-q)\|u\|_\mu^2}{(2^* - q)\int g|u|^{2^*}dx} \right)^{1/(2^*-2)} > 0. \tag{3.20}$$

Then the following lemma holds.

Lemma 3.5. *Let $\lambda \in (0, \Lambda_1)$. For each $u \in H$ with $\int g|u|^{2^*}dx > 0$, one has the following:*

(i) *if $\int f|u|^q dx \le 0$, then there exists a unique $t^- > t_{\max}$ such that $t^- u \in \mathcal{N}_\lambda^-$ and*

$$J_\lambda(t^- u) = \sup_{t\ge 0} J_\lambda(tu), \tag{3.21}$$

(ii) *if $\int f|u|^q dx > 0$, then there exist unique $0 < t^+ < t_{\max} < t^-$ such that $t^+ u \in \mathcal{N}_\lambda^+$, $t^- u \in \mathcal{N}_\lambda^-$ and*

$$J_\lambda(t^+ u) = \inf_{0\le t\le t_{\max}} J_\lambda(tu), \qquad J_\lambda(t^- u) = \sup_{t\ge 0} J_\lambda(tu). \tag{3.22}$$

Proof. The proof is almost the same as that in Brown-Wu [16, Lemma 2.6], and is omitted here. □

4. Proof of Theorem 1.1

First, we will use the idea of Tarantello [13] to get the following results.

Proposition 4.1. (i) *If $\lambda \in (0, \Lambda_1)$, then there exists a $(PS)_{\alpha_\lambda}$-sequence $\{u_n\} \subset \mathcal{N}_\lambda$ in H for J_λ.*

(ii) *If $\lambda \in (0, (q/2)\Lambda_1)$, then there exists a $(PS)_{\alpha_\lambda^-}$-sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ in H for J_λ.*

Proof. The proof is almost the same as that in Wu [4, Proposition 9] (or see Hsu-Lin [5, Proposition 3.3]). □

Now, we establish the existence of a local minimum for J_λ on $\mathcal{N}_\lambda^+$.

Theorem 4.2. *If $\lambda \in (0, \Lambda_1)$, then J_λ has a minimizer u_λ in $\mathcal{N}_\lambda^+$ and it satisfies*

(i) $J_\lambda(u_\lambda) = \alpha_\lambda = \alpha_\lambda^+$,

(ii) *u_λ is a positive solution of (P_μ),*

(iii) $J_\lambda(u_\lambda) \to 0$ *as* $\lambda \to 0^+$.

Proof. By Proposition 4.1(i), there exists a minimizing sequence $\{u_n\}$ for J_λ on $\mathcal{N}_\lambda$ such that

$$J_\lambda(u_n) = \alpha_\lambda + o_n(1), \quad J'_\lambda(u_n) = o_n(1) \quad \text{in } H^{-1}. \tag{4.1}$$

Since J_λ is coercive on $\mathcal{N}_\lambda$ (see Lemma 3.1), we get that $\{u_n\}$ is bounded in H. Going if necessary to a subsequence, we can assume that there exists $u_\lambda \in H$ such that

$$\begin{gathered} u_n \rightharpoonup u_\lambda \quad \text{weakly in } H, \\ u_n \longrightarrow u_\lambda \quad \text{almost every where in } \Omega, \\ u_n \longrightarrow u_\lambda \quad \text{strongly in } L^s(\Omega) \quad \forall 1 \leq s < 2^*. \end{gathered} \tag{4.2}$$

First, we claim that u_λ is a nontrivial solution of (P_μ). By (4.1) and (4.2), it is easy to see that u_λ is a solution of (P_μ). From $u_n \in \mathcal{N}_\lambda$ and (3.4), we deduce that

$$\lambda \int f|u_n|^q dx = \frac{q(2^* - 2)}{2(2^* - q)} \|u_n\|_\mu^2 - \frac{2^* q}{2^* - q} J_\lambda(u_n). \tag{4.3}$$

Let $n \to \infty$ in (4.3), by (4.1), (4.2), and $\alpha_\lambda < 0$, we get

$$\lambda \int f|u_\lambda|^q dx \geq -\frac{2^* q}{2^* - q} \alpha_\lambda > 0. \tag{4.4}$$

Thus, $u_\lambda \in \mathcal{N}_\lambda$ is a nontrivial solution of (P_μ). Now we prove that $u_n \to u_\lambda$ strongly in H and $J_\lambda(u_\lambda) = \alpha_\lambda$. By (4.3), if $u \in \mathcal{N}_\lambda$, then

$$J_\lambda(u) = \frac{1}{N} \|u\|_\mu^2 - \frac{2^* - q}{2^* q} \lambda \int f|u|^q dx. \tag{4.5}$$

In order to prove that $J_\lambda(u_\lambda) = \alpha_\lambda$, it suffices to recall that $u_\lambda \in \mathcal{N}_\lambda$, by (4.5) and applying Fatou's lemma to get

$$\begin{aligned} \alpha_\lambda \leq J_\lambda(u_\lambda) &= \frac{1}{N} \|u_\lambda\|_\mu^2 - \frac{2^* - q}{2^* q} \lambda \int f|u_\lambda|^q dx \\ &\leq \liminf_{n \to \infty} \left(\frac{1}{N} \|u_n\|_\mu^2 - \frac{2^* - q}{2^* q} \lambda \int f|u_n|^q dx \right) \\ &\leq \liminf_{n \to \infty} J_\lambda(u_n) = \alpha_\lambda. \end{aligned} \tag{4.6}$$

This implies that $J_\lambda(u_\lambda) = \alpha_\lambda$ and $\lim_{n \to \infty} \|u_n\|_\mu^2 = \|u_\lambda\|_\mu^2$. Let $v_n = u_n - u_\lambda$, then by Brézis-Lieb lemma [17] implies that

$$\|v_n\|_\mu^2 = \|u_n\|_\mu^2 - \|u_\lambda\|_\mu^2 + o_n(1). \tag{4.7}$$

Therefore, $u_n \to u_\lambda$ strongly in H. Moreover, we have $u_\lambda \in \mathcal{N}_\lambda^+$. On the contrary, if $u_\lambda \in \mathcal{N}_\lambda^-$, then by Lemma 3.5, there are unique t_0^+ and t_0^- such that $t_0^+ u_\lambda \in \mathcal{N}_\lambda^+$ and $t_0^- u_\lambda \in \mathcal{N}_\lambda^-$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt} J_\lambda(t_0^+ u_\lambda) = 0, \qquad \frac{d^2}{dt^2} J_\lambda(t_0^+ u_\lambda) > 0, \tag{4.8}$$

there exists $t_0^+ < \bar{t} \le t_0^-$ such that $J_\lambda(t_0^+ u_\lambda) < J_\lambda(\bar{t} u_\lambda)$. By Lemma 3.5,

$$J_\lambda(t_0^+ u_\lambda) < J_\lambda(\bar{t} u_\lambda) \le J_\lambda(t_0^- u_\lambda) = J_\lambda(u_\lambda), \tag{4.9}$$

which is a contradiction. Since $J_\lambda(u_\lambda) = J_\lambda(|u_\lambda|)$ and $|u_\lambda| \in \mathcal{N}_\lambda^+$, by Lemma 3.2 we may assume that u_λ is a nontrivial nonnegative solution of (P_μ). Standard arguments implies that u_λ is a positive solution of (P_μ). Moreover, by Lemma 3.4 (i) and (3.5), we have

$$0 > \alpha_\lambda > -\lambda\left(\frac{2^* - q}{2^* q}\right) S_\mu^{-(q/2)} |\Omega|^{(2^*-q)/2^*} \|u_\lambda\|_\mu^q |f^+|_\infty. \tag{4.10}$$

This implies that $J_\lambda(u_\lambda) \to 0$ as $\lambda \to 0^+$. □

Now, we begin the proof of Theorem 1.1: By Theorem 4.2, we obtain (P_μ) has a positive solution u_λ.

5. Proof of Theorem 1.2

Next, we will establish the existence of the second positive solution of (P_μ) by proving that J'_λ satisfies the $(\mathrm{PS})_{\alpha_\lambda^-}$-condition.

Lemma 5.1. *Assume that* $(f1)$ *and* $(g1)$ *hold. If* $\{u_n\}$ *is a* $(PS)_c$*-sequence for* J_λ *with* $u_n \rightharpoonup u$ *in* H*, then* $J'_\lambda(u) = 0$*, and there exists a constant* C_0 *depending on* $q, N, S_\mu, |f^+|_\infty$ *and* $|\Omega|$*, such that* $J_\lambda(u) \ge -C_0 \lambda^{2/(2-q)}$.

Proof. If $\{u_n\}$ is a $(\mathrm{PS})_c$-sequence for J'_λ with $u_n \rightharpoonup u$ in H, it is easy to see that $J'_\lambda(u) = 0$. This implies that $\langle J'_\lambda(u), u\rangle = 0$, and

$$\int g(x)|u|^{2^*} dx = \|u\|_\mu^2 - \lambda \int f(x)|u|^q dx. \tag{5.1}$$

Consequently,

$$J_\lambda(u) = \left(\frac{1}{2} - \frac{1}{2^*}\right)\|u\|_\mu^2 - \left(\frac{1}{q} - \frac{1}{2^*}\right)\lambda \int f(x)|u|^q dx. \tag{5.2}$$

Using the Hölder inequality, the Young inequality, and the Sobolev embedding theorem, we have

$$\begin{aligned}
J_\lambda(u) &= \left(\frac{1}{2}-\frac{1}{2^*}\right)\|u\|_\mu^2 - \left(\frac{1}{q}-\frac{1}{2^*}\right)\lambda\int f(x)|u|^q dx \\
&\geq \frac{1}{N}\|u\|_\mu^2 - \frac{2^*-q}{2^*q}\left|f^+\right|_\infty |u|_{2^*}^q |\Omega|^{(2^*-q)/2^*}\lambda \\
&\geq \frac{1}{N}\|u\|_\mu^2 - \frac{2^*-q}{2^*q}\left|f^+\right|_\infty S_\mu^{-(q/2)}\|u\|_\mu^q |\Omega|^{(2^*-q)/2^*}\lambda \\
&\geq \frac{1}{N}\|u\|_\mu^2 - \frac{1}{N}\|u\|_\mu^2 - C_0\lambda^{2/(2-q)} = -C_0\lambda^{2/(2-q)},
\end{aligned} \tag{5.3}$$

where C_0 is a positive constant depending on $q, N, S_\mu, |f^+|_\infty$, and $|\Omega|$. □

Lemma 5.2. *Assume that* $(f1)$ *and* $(g1)$ *hold. Then the functional* J_λ *satisfies the* $(PS)_c$*-condition for all* $c \in (-\infty, (1/N)|g^+|_\infty^{-(N-2)/2} S_\mu^{N/2} - C_0\lambda^{2/(2-q)})$ *where* C_0 *is the positive constant given in Lemma 5.1.*

Proof. Let $\{u_n\} \subset H$ be a $(\mathrm{PS})_c$-sequence which satisfies $J_\lambda(u_n) = c + o_n(1)$ and $J'_\lambda(u_n) = o_n(1)$. Using standard arguments it follows that $\{u_n\}$ is bounded in H. Thus, there exists a subsequence still denoted by $\{u_n\}$ and a function $u \in H$ such that

$$\begin{gathered}
u_n \rightharpoonup u \quad \text{weakly in } H, \\
u_n \longrightarrow u \quad \text{strongly in } L^s(\Omega) \;\; \forall 1 \leq s < 2^*, \\
u_n \longrightarrow u \quad \text{a.e. on } \Omega.
\end{gathered} \tag{5.4}$$

By $(f1)$, $(g1)$, and Lemma 5.1, we have that $J'_\lambda(u) = 0$ and

$$\lambda\int f(x)\left|u_n\right|^q dx = \lambda\int f(x)|u|^q dx + o_n(1), \tag{5.5}$$

Let $v_n = u_n - u$. Then by g is continuous on $\overline{\Omega}$, Brézis-Lieb lemma (see [17]), and Vitali's theorem, we obtain

$$\left\|v_n\right\|_\mu^2 = \left\|u_n\right\|_\mu^2 - \|u\|_\mu^2 + o_n(1), \tag{5.6}$$

$$\int g(x)\left|v_n\right|^{2^*} dx = \int g(x)\left|u_n\right|^{2^*} dx - \int g(x)|u|^{2^*} dx + o_n(1). \tag{5.7}$$

Since $J_\lambda(u_n) = c + o_n(1)$, $J'_\lambda(u_n) = o_n(1)$ and (5.5)–(5.7), we can deduce that

$$\frac{1}{2}\|v_n\|_\mu^2 - \frac{1}{2^*}\int g(x)|v_n|^{2^*}dx = c - J_\lambda(u) + o_n(1), \tag{5.8}$$

$$\|v_n\|_\mu^2 - \int g(x)|v_n|^{2^*}dx = o_n(1). \tag{5.9}$$

Hence, we may assume that

$$\|v_n\|_\mu^2 \longrightarrow l, \qquad \int g(x)|v_n|^{2^*}dx \longrightarrow l. \tag{5.10}$$

By the Sobolev inequality, we have $\|v_n\|_\mu^2 \geq S_\mu|v_n|_{2^*}^2$, combining with (5.10), we get that $l \geq |g^+|_\infty^{-(N-2)/N} S_\mu l^{(N-2)/N}$. Either $l = 0$ or $l \geq |g^+|_\infty^{-(N-2)/2} S_\mu^{N/2}$. If $l = 0$, this completes the proof. Assume that $l \geq |g^+|_\infty^{-(N-2)/2} S_\mu^{N/2}$, from Lemmas 5.1, (5.8), and (5.10), we get

$$c \geq \left(\frac{1}{2} - \frac{1}{2^*}\right)l + J_\lambda(u) \geq \frac{1}{N}|g^+|_\infty^{-(N-2)/2} S_\mu^{N/2} - C_0\lambda^{2/(2-q)}, \tag{5.11}$$

which is a contradiction. Therefore, $l = 0$ and we conclude that $u_n \to u$ in H. □

Lemma 5.3. *Assume that* $(f1)$*-*$(f2)$ *and* $(g1)$*-*$(g2)$ *hold. Then there exist* $v \in H$ *and* $\Lambda^* > 0$ *such that for* $\lambda \in (0, \Lambda^*)$*, one has*

$$\sup_{t\geq 0} J_\lambda(tv) < \frac{1}{N}|g^+|_\infty^{-(N-2)/2} S_\mu^{N/2} - C_0\lambda^{2/(2-q)}, \tag{5.12}$$

where C_0 *is the positive constant given in Lemma 5.1.*

In particular, $\alpha_\lambda^- < 1/N|g^+|_\infty^{-(N-2)/2} S_\mu^{N/2} - C_0\lambda^{2/(2-q)}$ *for all* $\lambda \in (0, \Lambda^*)$.

Proof. Without loss of generality, we can assume that $|g^+|_\infty = 1$. In fact, if $|g^+|_\infty \neq 1$, we may consider new coefficients $g^*(x) = g(x)/|g^+|_\infty$ whose maximum equals to 1.

For convenience, we introduce the following notations:

$$I(u) = \frac{1}{2}\|u\|_\mu^2 - \frac{1}{2^*}\int g|u|^{2^*}dx,$$

$$\chi_{B(0,2\rho_0)} = \begin{cases} 1 & \text{if } x \in B(0, 2\rho_0), \\ 0 & \text{if } x \notin B(0, 2\rho_0), \end{cases} \tag{5.13}$$

$$Q(u) = \frac{\|u\|_\mu^2}{\left|\left(g\chi_{B(0,2\rho_0)}\right)^{1/2^*}u\right|_{2^*}^2}.$$

From $(g2)$, we know that there exists $0 < \delta_0 \le \rho_0$ such that for all $x \in B(0, 2\delta_0)$,

$$g(x) = g(0) + o\left(|x|^{\beta}\right) \quad \text{for some } \beta \in \left(\frac{\sqrt{\overline{\mu} - \mu}N}{\sqrt{\overline{\mu}}}, \frac{\sqrt{\overline{\mu} - \mu}(N+1)}{\sqrt{\overline{\mu}}}\right). \tag{5.14}$$

Motivated by some ideas of selecting cut-off functions in [18], we take such cut-off function $\eta(x)$ that satisfies $\eta(x) \in C_0^{\infty}(B(0, 2\delta_0))$, $\eta(x) = 1$ for $|x| < \delta_0$, $\eta(x) = 0$ for $|x| > 2\delta_0$, $0 \le \eta \le 1$ and $|\nabla \eta| \le C$. For $\varepsilon > 0$, let

$$u_{\varepsilon}(x) = \frac{\eta(x)}{\left[\varepsilon |x|^{\gamma_1/\sqrt{\overline{\mu}}} + |x|^{\gamma_2/\sqrt{\overline{\mu}}}\right]^{\sqrt{\overline{\mu}}}}, \tag{5.15}$$

where $\mu \in [0, \overline{\mu})$, $\overline{\mu} = ((N-2)/2)^2$, $\gamma_1 = \sqrt{\overline{\mu}} - \sqrt{\overline{\mu} - \mu}$, and $\gamma_2 = \sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu}$.

Step 1. Show that $\sup_{t \ge 0} I(t u_{\varepsilon}) \le (1/N) S_{\mu}^{N/2} + O(\varepsilon^{(N-2)/2})$.

On that purpose, we need to establish the following estimates (as $\varepsilon \to 0$):

$$\left|\left(g \chi_{B(0,2\rho_0)}\right)^{1/2^*} u_{\varepsilon}\right|_{2^*}^{2} = \varepsilon^{-(N-2)/2} |U|_{L^{2^*}(\mathbb{R}^N)}^{2} + O(\varepsilon), \tag{5.16}$$

$$\|u_{\varepsilon}\|_{\mu}^{2} = \varepsilon^{-(N-2)/2} \int_{\mathbb{R}^N} \left(|\nabla U|^2 - \frac{\mu}{|x|^2} U^2\right) dx + O(1), \tag{5.17}$$

where U is defined as in (2.4), and $\omega_N = 2\pi^{N/2} / N\Gamma(N/2)$ is the volume of the unit ball $B(0,1)$ in $\mathbb{R}^N$. We only show that equality (5.16) is valid, proofs of (5.17) are very similar to [18]. By $(g2)$ and the definition of u_{ε}, we get that

$$\begin{aligned} \left|\left(g \chi_{B(0,2\rho_0)}\right)^{1/2^*} u_{\varepsilon}\right|_{2^*}^{2^*} &= \int_{B(0,2\delta_0)} g(x) \left|u_{\varepsilon}\right|^{2^*} dx \\ &= \int_{\mathbb{R}^N} \frac{\eta^{2^*}(x) g(x)}{\left[\varepsilon |x|^{\gamma_1/\sqrt{\overline{\mu}}} + |x|^{\gamma_2/\sqrt{\overline{\mu}}}\right]^{N}} dx. \end{aligned} \tag{5.18}$$

On the other hand, it is clear that

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{1}{\left(\varepsilon |x|^{\gamma_1/\sqrt{\overline{\mu}}} + |x|^{\gamma_2/\sqrt{\overline{\mu}}}\right)^{N}} dx &= \varepsilon^{-(N/2)} \int_{\mathbb{R}^N} \frac{1}{\left[|y|^{\gamma_1/\sqrt{\overline{\mu}}} + |y|^{\gamma_2/\sqrt{\overline{\mu}}}\right]^{N}} dy \\ &= \varepsilon^{-(N/2)} |U|_{L^{2^*}(\mathbb{R}^N)}^{2^*}. \end{aligned} \tag{5.19}$$

Combining the equalities above, we have

$$\varepsilon^{-(N/2)}|U|^{2^*}_{L^{2^*}(\mathbb{R}^N)} - |(g\chi_{B(0,2\rho_0)})^{1/2^*}u_\varepsilon|^{2^*}_{2^*} = \int_{\mathbb{R}^N\setminus B(0,\delta_0)} \frac{1-\eta^{2^*}(x)g(x)}{\left[\varepsilon|x|^{\gamma_1/\sqrt{\bar{\mu}}}+|x|^{\gamma_2/\sqrt{\bar{\mu}}}\right]^N}dx + \int_{B(0,\delta_0)} \frac{1-g(x)}{\left[\varepsilon|x|^{\gamma_1/\sqrt{\bar{\mu}}}+|x|^{\gamma_2/\sqrt{\bar{\mu}}}\right]^N}dx, \tag{5.20}$$

hence

$$\begin{aligned}
0 &\le \varepsilon^{-(N/2)}|U|^{2^*}_{L^{2^*}(\mathbb{R}^N)} - \left|(g\chi_{B(0,2\rho_0)})^{1/2^*}u_\varepsilon\right|^{2^*}_{2^*} \\
&\le \int_{\mathbb{R}^N\setminus B(0,\delta_0)} \frac{1}{\left[\varepsilon|x|^{\gamma_1/\sqrt{\bar{\mu}}}+|x|^{\gamma_2/\sqrt{\bar{\mu}}}\right]^N}dx + \int_{B(0,\delta_0)} \frac{o(|x|^\beta)}{\left[\varepsilon|x|^{\gamma_1/\sqrt{\bar{\mu}}}+|x|^{\gamma_2/\sqrt{\bar{\mu}}}\right]^N}dx, \\
&\le \int_{\mathbb{R}^N\setminus B(0,\delta_0)} \frac{1}{|x|^{\gamma_2 N/\sqrt{\bar{\mu}}}}dx + \int_{B(0,\delta_0)} \frac{o(|x|^\beta)}{|x|^{\gamma_2 N/\sqrt{\bar{\mu}}}}dx, \\
&= N\omega_N\int_{\delta_0}^{\infty} \frac{r^{N-1}}{r^{\gamma_2 N/\sqrt{\bar{\mu}}}}dr + \int_0^{\delta_0} \frac{o(r^\beta)r^{N-1}}{r^{\gamma_2 N/\sqrt{\bar{\mu}}}}dr, \\
&= \frac{\omega_N\sqrt{\bar{\mu}}}{\sqrt{\bar{\mu}-\mu}}\delta_0^{-(\sqrt{\bar{\mu}-\mu}/\sqrt{\bar{\mu}})N} + \frac{o(1)\delta_0^{\beta-(\sqrt{\bar{\mu}-\mu}/\sqrt{\bar{\mu}})N}}{\beta-\left(\sqrt{\bar{\mu}-\mu}/\sqrt{\bar{\mu}}\right)N} \le C_1 = \text{Const.},
\end{aligned} \tag{5.21}$$

which leads to

$$0 \le 1 - \left|(g\chi_{B(0,2\rho_0)})^{1/2^*}u_\varepsilon\right|^{2^*}_{2^*}|U|^{-2^*}_{L^{2^*}(\mathbb{R}^N)}\varepsilon^{N/2} \le C_1|U|^{-2^*}_{L^{2^*}(\mathbb{R}^N)}\varepsilon^{N/2}, \tag{5.22}$$

that is,

$$1 - C_1|U|^{-2^*}_{L^{2^*}(\mathbb{R}^N)}\varepsilon^{N/2} \le \left|(g\chi_{B(0,2\rho_0)})^{1/2^*}u_\varepsilon\right|^{2^*}_{2^*}|U|^{-2^*}_{L^{2^*}(\mathbb{R}^N)}\varepsilon^{N/2} \le 1. \tag{5.23}$$

Now, let ε be small enough such that $C_1|U|^{-2^*}_{2^*}\varepsilon^{N/2} < 1$, then from (5.23) we can deduce that

$$\begin{aligned}
1 - C_1|U|^{-2^*}_{L^{2^*}(\mathbb{R}^N)}\varepsilon^{N/2} &\le \left(1 - C_1|U|^{-2^*}_{L^{2^*}(\mathbb{R}^N)}\varepsilon^{N/2}\right)^{2/2^*} \\
&\le \left|(g\chi_{B(0,2\rho_0)})^{1/2^*}u_\varepsilon\right|^2_{2^*}|U|^{-2}_{L^{2^*}(\mathbb{R}^N)}\varepsilon^{(N-2)/2} \le 1,
\end{aligned} \tag{5.24}$$

which yields that

$$|U|^2_{L^{2^*}(\mathbb{R}^N)}\varepsilon^{-(N-2)/2} - C_1|U|^{2-2^*}_{L^{2^*}(\mathbb{R}^N)}\varepsilon \le \left|(g\chi_{B(0,2\rho_0)})^{1/2^*}u_\varepsilon\right|^2_{2^*} \le |U|^2_{L^{2^*}(\mathbb{R}^N)}\varepsilon^{-(N-2)/2}, \tag{5.25}$$

equivalently, equality (5.16) is valid.

Set $|U|_\mu^2 = \int_{\mathbb{R}^N}(|\nabla U|^2 - (\mu/|x|^2)U^2)dx$. Combining with (5.16) and (5.17), we obtain that

$$Q(u_\varepsilon) = \frac{\varepsilon^{-(N-2)/2}|U|_\mu^2 + O(1)}{\varepsilon^{-(N-2)/2}|U|_{L^{2^*}(\mathbb{R}^N)}^2 + O(\varepsilon)} = \frac{|U|_\mu^2 + O\left(\varepsilon^{(N-2)/2}\right)}{|U|_{L^{2^*}(\mathbb{R}^N)}^2 + O\left(\varepsilon^{N/2}\right)}. \tag{5.26}$$

Hence

$$\begin{aligned} Q(u_\varepsilon) - S_\mu &= \frac{|U|_\mu^2 + O\left(\varepsilon^{(N-2)/2}\right)}{|U|_{L^{2^*}(\mathbb{R}^N)}^2 + O\left(\varepsilon^{N/2}\right)} - \frac{|U|_\mu^2}{|U|_{L^{2^*}(\mathbb{R}^N)}^2} \\ &= \frac{|U|_{L^{2^*}(\mathbb{R}^N)}^2 O\left(\varepsilon^{(N-2)/2}\right) - |U|_\mu^2 O\left(\varepsilon^{N/2}\right)}{\left(|U|_{L^{2^*}(\mathbb{R}^N)}^2 + O\left(\varepsilon^{N/2}\right)\right)|U|_{L^{2^*}(\mathbb{R}^N)}^2} \\ &= O\left(\varepsilon^{(N-2)/2}\right). \end{aligned} \tag{5.27}$$

Using the fact

$$\max_{t\ge 0}\left(\frac{t^2}{2}a - \frac{t^{2^*}}{2^*}b\right) = 1/N\left(\frac{a}{b^{2/2^*}}\right)^{N/2} \quad \text{for any} \quad a, b > 0, \tag{5.28}$$

we can deduce that

$$\sup_{t\ge 0} I(tu_\varepsilon) = \frac{1}{N}(Q(u_\varepsilon))^{N/2}. \tag{5.29}$$

From (5.27), we conclude that $\sup_{t\ge 0} I(tu_\varepsilon) \le (1/N)S_\mu^{N/2} + O(\varepsilon^{(N-2)/2})$.

Step 2. Let $\varepsilon = \lambda^{4/(2-q)(N-2)}$. We claim that there exists $\Lambda^* > 0$ such that $\sup_{t\ge 0} J_\lambda(tu_\varepsilon) < (1/N)S_\mu^{N/2} - C_0\lambda^{2/(2-q)}$ for all $\lambda \in (0, \Lambda^*)$.

Let $\delta_1 > 0$ be such that

$$\frac{1}{N}S_\mu^{N/2} - C_0\lambda^{2/(2-q)} > 0, \quad \forall \lambda \in (0, \delta_1). \tag{5.30}$$

Using the definitions of J_λ, u_ε and by $(f2)$, $(g2)$, we get

$$J_\lambda(tu_\varepsilon) \le \frac{t^2}{2}\|u_\varepsilon\|_\mu^2, \qquad \forall t \ge 0, \quad \lambda > 0, \tag{5.31}$$

which implies that there exists $t_0 \in (0,1)$ satisfying

$$\sup_{0\le t\le t_0} J_\lambda(tu_\varepsilon) < \frac{1}{N} S_\mu^{N/2} - C_0\lambda^{2/(2-q)}, \quad \forall \lambda \in (0,\delta_1). \tag{5.32}$$

Using the definitions of J_λ, u_ε, and by the results in Step 1 and $(f2)$, we have

$$\begin{aligned} \sup_{t\ge t_0} J_\lambda(tu_\varepsilon) &= \sup_{t\ge t_0}\left(I(tu_\varepsilon) - \frac{t^q}{q}\lambda \int f(x)|u_\varepsilon|^q dx \right) \\ &\le \frac{1}{N} S_\mu^{N/2} + O\left(\varepsilon^{(N-2)/2}\right) - \frac{t_0^q}{q}\beta_0 \lambda \int_{B(0,\delta_0)} |u_\varepsilon|^q dx. \end{aligned} \tag{5.33}$$

Let $0 < \varepsilon \le \delta_0^{(\gamma_2-\gamma_1)/\sqrt{\bar\mu}}$, we have

$$\begin{aligned} \int_{B(0,\delta_0)} |u_\varepsilon|^q dx &= \int_{B(0,\delta_0)} \frac{1}{\left[\varepsilon |x|^{\gamma_1/\sqrt{\bar\mu}} + |x|^{\gamma_2/\sqrt{\bar\mu}}\right]^{\sqrt{\bar\mu}q}} dx \\ &\ge \int_{B(0,\delta_0)} \frac{1}{\left(2\delta_0^{\gamma_2/\sqrt{\bar\mu}}\right)^{\sqrt{\bar\mu}q}} dx \\ &= C_1(N,q,\mu,\delta_0). \end{aligned} \tag{5.34}$$

Combining with (5.33) and (5.34), for all $\varepsilon = \lambda^{4/(2-q)(N-2)} \in (0, \delta_0^{(\gamma_2-\gamma_1)/\sqrt{\bar\mu}})$, we get

$$\sup_{t\ge t_0} J_\lambda(tu_\varepsilon) \le \frac{1}{N} S_\mu^{N/2} + O\left(\lambda^{2/(2-q)}\right) - \frac{t_0^q}{q}\beta_0 C_1 \lambda. \tag{5.35}$$

Hence, we can choose $\delta_2 > 0$ such that

$$O\left(\lambda^{2/(2-q)}\right) - \frac{t_0^q}{q}\beta_0 C_1\lambda < -C_0\lambda^{2/(2-q)} \quad \lambda \in (0,\delta_2). \tag{5.36}$$

If we set $\Lambda^* = \min\{\delta_1, \delta_0^{(2-q)\sqrt{\bar\mu-\mu}}, \delta_2\} > 0$, then for $\lambda \in (0,\Lambda^*)$ and $\varepsilon = \lambda^{4/(2-q)(N-2)}$, we have

$$\sup_{t\ge 0} J_\lambda(tu_\varepsilon) < \frac{1}{N} S_\mu^{N/2} - C_0\lambda^{2/(2-q)}. \tag{5.37}$$

Step 3. Prove that $\alpha_\lambda^- < (1/N)S_\mu^{N/2} - C_0\lambda^{2/(2-q)}$ for all $\lambda \in (0,\Lambda^*)$.

By $(f2)$, $(g2)$, and the definition of u_ε, we have

$$\int f(x)|u_\varepsilon|^q dx > 0, \qquad \int g(x)|u_\varepsilon|^{2^*} dx > 0. \tag{5.38}$$

Combining this with Lemma 3.5, from the definition of α_λ^- and the results in Step 2, we obtain that there exists $t_\varepsilon > 0$ such that $t_\varepsilon u_\varepsilon \in \mathcal{N}_\lambda^-$ and

$$\alpha_\lambda^- \leq J_\lambda(t_\varepsilon u_\varepsilon) \leq \sup_{t\geq 0} J_\lambda(tu_\varepsilon) < \frac{1}{N} S_\mu^{N/2} - C_0 \lambda^{2/(2-q)} \tag{5.39}$$

for all $\lambda \in (0, \Lambda^*)$. □

Now, we establish the existence of a local minimum of J_λ on $\mathcal{N}_\lambda^-$.

Theorem 5.4. *There exists $\Lambda_2 > 0$ such that for $\lambda \in (0, \Lambda_2)$ the functional J_λ has a minimizer U_λ in $\mathcal{N}_\lambda^-$ and satisfies*

(i) $J_\lambda(U_\lambda) = \alpha_\lambda^-$,

(ii) *U_λ is a positive solution of (P_μ) in H,*

where $\Lambda_2 = \min\{\Lambda^*, (q/2)\Lambda_1\}$, Λ^* is defined as in Lemma 5.3, and Λ_1 is defined as in (1.1).

Proof. By Proposition 4.1(ii), there exists a $(PS)_{\alpha_\lambda^-}$-sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ in H for J_λ for all $\lambda \in (0, (q/2)\Lambda_1)$. From Lemmas 5.2, 5.3 and 3.4(ii), for $\lambda \in (0, \Lambda^*)$, J_λ satisfies $(PS)_{\alpha_\lambda^-}$-condition and $\alpha_\lambda^- > 0$. Since J_λ is coercive on $\mathcal{N}_\lambda$ (see Lemma 3.1), we get that $\{u_n\}$ is bounded in H. Therefore, there exist a subsequence still denoted by $\{u_n\}$ and $U_\lambda \in \mathcal{N}_\lambda^-$ such that $u_n \to U_\lambda$ strongly in H and $J_\lambda(U_\lambda) = \alpha_\lambda^- > 0$ for all $\lambda \in (0, \Lambda_2)$. Finally, by using the same arguments as in the proof of Theorem 4.2, for all $\lambda \in (0, \Lambda_2)$, we have that U_λ is a positive solution of (P_μ). □

Now, we complete the proof of Theorem 1.2: By Theorems 4.2 and 5.4, we obtain (P_μ) has two positive solutions u_λ and U_λ such that $u_\lambda \in \mathcal{N}_\lambda^+$, $U_\lambda \in \mathcal{N}_\lambda^-$. Since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, this implies that u_λ and U_λ are distinct.

References

[1] A. Ambrosetti, J. Garcia Azorero, and I. Peral, "Multiplicity results for some nonlinear elliptic equations," *Journal of Functional Analysis*, vol. 137, no. 1, pp. 219–242, 1996.

[2] T. Bartsch and M. Willem, "On an elliptic equation with concave and convex nonlinearities," *Proceedings of the American Mathematical Society*, vol. 123, no. 11, pp. 3555–3561, 1995.

[3] A. Capozzi, D. Fortunato, and G. Palmieri, "An existence result for nonlinear elliptic problems involving critical Sobolev exponent," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 2, no. 6, pp. 463–470, 1985.

[4] T.-F. Wu, "On semilinear elliptic equations involving concave-convex nonlinearities and sign-changing weight function," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 253–270, 2006.

[5] T.-S. Hsu and H.-L. Lin, "On critical semilinear elliptic equations involving concave-convex nonlinearities and sign-changing weight functions," submitted.

[6] J. Chen, "Multiple positive solutions for a class of nonlinear elliptic equations," *Journal of Mathematical Analysis and Applications*, vol. 295, no. 2, pp. 341–354, 2004.

[7] H. Egnell, "Elliptic boundary value problems with singular coefficients and critical nonlinearities," *Indiana University Mathematics Journal*, vol. 38, no. 2, pp. 235–251, 1989.

[8] A. Ferrero and F. Gazzola, "Existence of solutions for singular critical growth semilinear elliptic equations," *Journal of Differential Equations*, vol. 177, no. 2, pp. 494–522, 2001.

[9] F. Catrina and Z.-Q. Wang, "On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions," *Communications on Pure and Applied Mathematics*, vol. 54, no. 2, pp. 229–258, 2001.

[10] S. Terracini, "On positive entire solutions to a class of equations with a singular coefficient and critical exponent," *Advances in Differential Equations*, vol. 1, no. 2, pp. 241–264, 1996.

[11] K. S. Chou and C. W. Chu, "On the best constant for a weighted Sobolev-Hardy inequality," *Journal of the London Mathematical Society*, vol. 48, no. 1, pp. 137–151, 1993.

[12] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, vol. 65 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Washington, DC, USA, 1986.

[13] G. Tarantello, "On nonhomogeneous elliptic equations involving critical Sobolev exponent," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 9, no. 3, pp. 281–304, 1992.

[14] K. J. Brown and Y. Zhang, "The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function," *Journal of Differential Equations*, vol. 193, no. 2, pp. 481–499, 2003.

[15] P. A. Binding, P. Drábek, and Y. X. Huang, "On Neumann boundary value problems for some quasilinear elliptic equations," *Electronic Journal of Differential Equations*, no. 5, pp. 1–11, 1997.

[16] K. J. Brown and T.-F. Wu, "A semilinear elliptic system involving nonlinear boundary condition and sign-changing weight function," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 2, pp. 1326–1336, 2008.

[17] H. Brézis and E. Lieb, "A relation between pointwise convergence of functions and convergence of functionals," *Proceedings of the American Mathematical Society*, vol. 88, no. 3, pp. 486–490, 1983.

[18] J. Chen, "Existence of solutions for a nonlinear PDE with an inverse square potential," *Journal of Differential Equations*, vol. 195, no. 2, pp. 497–519, 2003.