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Research Article

The Solution of Two-Point Boundary Value Problem of a Class of Duffing-Type Systems with Non-C^1 Perturbation Term

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This paper deals with a two-point boundary value problem of a class of Duffing-type systems with non-C^1 perturbation term. Several existence and uniqueness theorems were presented.

1. Introduction

Minimax theorems are one of powerful tools for investigation on the solution of differential equations and differential systems. The investigation on the solution of differential equations and differential systems with non-C^1 perturbation term using minimax theorems came into being in the paper of Stepan A.Tersian in 1986 [1]. Tersian proved that the equation $Lu(t) = f(t, u(t))$ $(L = -(\mathrm{d}^2/\mathrm{d}t^2))$ exists exactly one generalized solution under the operators $B_j (j = 1, 2)$ related to the perturbation term $f(t, u(t))$ being selfadjoint and commuting with the operator $L(= -(\mathrm{d}^2/\mathrm{d}t^2))$ and some other conditions in [1]. Huang Wenhua extended Tersian's theorems in [1] in 2005 and 2006, respectively, and studied the existence and uniqueness of solutions of some differential equations and differential systems with non-C^1 perturbation term [2–4], the conditions attached to the non-C^1 perturbation term are that the operator $B(u)$ related to the term is self-adjoint and commutes with the operator A (where A is a selfadjoint operator in the equation $Au = f(t, u)$). Recently, by further research, we observe that the conditions imposed upon $B(u)$ can be weakened, the self-adjointness of $B(u)$ can be removed and $B(u)$ is not necessarily commuting with the operator A.

In this note, we consider a two-point boundary value problem of a class of Duffing-type systems with non-C^1 perturbation term and present a result as the operator $B(\mathbf{u})$ related to the perturbation term is not necessarily a selfadjoint and commuting with the operator L. We obtain several valuable results in the present paper under the weaker conditions than those in [2–4].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively, let X and Y be two orthogonal closed subspaces of H such that $H = X \oplus Y$. Let $P : H \to X, Q : H \to Y$ denote the projections from H to X and from H to Y, respectively. The following theorem will be employed to prove our main theorem.

Theorem 2.1 ([2]). *Let H be a real Hilbert space, $f : H \to \mathbb{R}$ an everywhere defined functional with Gâteaux derivative $\nabla f : H \to H$ everywhere defined and hemicontinuous. Suppose that there exist two closed subspaces X and Y such that $H = X \oplus Y$and two nonincreasing functions $\alpha : [0,+\infty) \to (0,+\infty), \beta : [0,+\infty) \to (0,+\infty)$ satisfying*

$$s\cdot\alpha(s) \longrightarrow +\infty, \quad s\cdot\beta(s) \longrightarrow +\infty, \quad \text{as } s \longrightarrow +\infty \tag{2.1}$$

and

$$\langle \nabla f(h_1 + y) - \nabla f(h_2 + y), h_1 - h_2 \rangle \leq -\alpha(\|h_1 - h_2\|)\|h_1 - h_2\|^2, \tag{2.2}$$

for all $h_1, h_2 \in X, y \in Y$, and

$$\langle \nabla f(x + k_1) - \nabla f(x + k_2), k_1 - k_2 \rangle \geq \beta(\|k_1 - k_2\|)\|k_1 - k_2\|^2, \tag{2.3}$$

for all $x \in X, k_1, k_2 \in Y$. Then

(a) *f has a unique critical point $v_0 \in H$ such that $\nabla f(v_0) = 0$;*

(b) *$f(v_0) = \max_{x\in X} \min_{y\in Y} f(x+y) = \min_{y\in Y} \max_{x\in X} f(x+y)$.*

We also need the following lemma in the present work. To the best of our knowledge, the lemma seems to be new.

Lemma 2.2. *Let $\mathbf{A}$ and $\mathbf{B}$ be two diagonalization $n \times n$ matrices, let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$, respectively, where each eigenvalue is repeated according to its multiplicity. If $\mathbf{A}$ commutes with $\mathbf{B}$, that is, $\mathbf{AB} = \mathbf{BA}$, then $\mathbf{A} + \mathbf{B}$ is a diagonalization matrix and $\mu_1 + \lambda_1 \leq \mu_2 + \lambda_2 \leq \cdots \leq \mu_n + \lambda_n$ are the eigenvalues of $\mathbf{A} + \mathbf{B}$.*

Proof. Since $\mathbf{A}$ is a diagonalization $n \times n$ matrix, there exists an inverse matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \text{diag}\,(\overline{\mu}_1\mathbf{E}_1, \overline{\mu}_2\mathbf{E}_2, \ldots, \overline{\mu}_s\mathbf{E}_s)$, where $\overline{\mu}_1 < \overline{\mu}_2 < \cdots < \overline{\mu}_s (1 \leq s \leq n)$ are the distinct eigenvalues of $\mathbf{A}$, $\mathbf{E}_i (i = 1, 2, \ldots, s)$ are the $r_i \times r_i (r_1 + r_2 + \cdots + r_s = n)$ identity matrices. And since $\mathbf{AB} = \mathbf{BA}$, that is,

$$\mathbf{P}\ \text{diag}\,(\overline{\mu}_1\mathbf{E}_1, \overline{\mu}_2\mathbf{E}_2, \ldots, \overline{\mu}_s\mathbf{E}_s)\mathbf{P}^{-1}\mathbf{B} = \mathbf{BP}\ \text{diag}\,(\overline{\mu}_1\mathbf{E}_1, \overline{\mu}_2\mathbf{E}_2, \ldots, \overline{\mu}_s\mathbf{E}_s)\mathbf{P}^{-1}, \tag{2.4}$$

we have

$$\text{diag}\,(\overline{\mu}_1\mathbf{E}_1, \overline{\mu}_2\mathbf{E}_2, \ldots, \overline{\mu}_s\mathbf{E}_s)\mathbf{P}^{-1}\mathbf{BP} = \mathbf{P}^{-1}\mathbf{BP}\ \text{diag}\,(\overline{\mu}_1\mathbf{E}_1, \overline{\mu}_2\mathbf{E}_2, \ldots, \overline{\mu}_s\mathbf{E}_s). \tag{2.5}$$

Denote $\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = (\mathbf{C}_{ij})$, where $\mathbf{C}_{ij}$ are the submatrices such that $\mathbf{E}_i\mathbf{C}_{ij}$ and $\mathbf{C}_{ij}\mathbf{E}_i (i = 1,2,\ldots,s)$ are defined, then, by (2.5),

$$\overline{\mu}_i \mathbf{C}_{ij} = \overline{\mu}_j \mathbf{C}_{ij} \quad (i,j = 1,2,\ldots,s). \tag{2.6}$$

Noticed that $\overline{\mu}_i \neq \overline{\mu}_j$ $(i \neq j)$, we have $\mathbf{C}_{ij} = \mathrm{O}$ $(i \neq j)$, and hence

$$\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \operatorname{diag}\,(\mathbf{C}_{11}, \mathbf{C}_{22}, \ldots, \mathbf{C}_{ss}), \tag{2.7}$$

where $\mathbf{C}_{ii}$ and $\mathbf{E}_i\,(i = 1,2,\ldots,s)$ are the same order square matrices. Since $\mathbf{B}$ is a diagonalization $n \times n$ matrix, there exists an invertible matrix $\mathbf{Q} = \operatorname{diag}\,(\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_s)$ such that

$$\begin{aligned} \mathbf{Q}^{-1}\left(\mathbf{P}^{-1}\mathbf{B}\mathbf{P}\right)\mathbf{Q} &= \operatorname{diag}\left(\mathbf{Q}_1^{-1}, \mathbf{Q}_2^{-1}, \ldots, \mathbf{Q}_s^{-1}\right) \cdot \operatorname{diag}\,(\mathbf{C}_{11}, \mathbf{C}_{22}, \ldots, \mathbf{C}_{ss}) \cdot \operatorname{diag}\,(\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_s) \\ &= \operatorname{diag}\left(\mathbf{Q}_1^{-1}\mathbf{C}_{11}\mathbf{Q}_1, \mathbf{Q}_2^{-1}\mathbf{C}_{22}\mathbf{Q}_2, \ldots, \mathbf{Q}_s^{-1}\mathbf{C}_{ss}\mathbf{Q}_s\right) = \operatorname{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n), \end{aligned} \tag{2.8}$$

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of $\mathbf{B}$.

Let $\mathbf{R} = \mathbf{P}\mathbf{Q}$, then $\mathbf{R}$ is an invertible matrix such that $\mathbf{R}^{-1}\mathbf{B}\mathbf{R} = \operatorname{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and

$$\begin{aligned} \mathbf{R}^{-1}(\mathbf{A}+\mathbf{B})\mathbf{R} &= \mathbf{R}^{-1}\mathbf{A}\mathbf{R} + \mathbf{R}^{-1}\mathbf{B}\mathbf{R} = \mathbf{Q}^{-1}\left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right)\mathbf{Q} + \mathbf{R}^{-1}\mathbf{B}\mathbf{R} \\ &= \operatorname{diag}\left(\mathbf{Q}_1^{-1}, \mathbf{Q}_2^{-1}, \ldots, \mathbf{Q}_s^{-1}\right) \cdot \operatorname{diag}\,(\overline{\mu}_1\mathbf{E}_1, \overline{\mu}_2\mathbf{E}_2, \ldots, \overline{\mu}_s\mathbf{E}_s) \cdot \operatorname{diag}\,(\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_s) \\ &\quad + \operatorname{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n) \\ &= \operatorname{diag}\,(\overline{\mu}_1\mathbf{E}_1, \overline{\mu}_2\mathbf{E}_2, \ldots, \overline{\mu}_s\mathbf{E}_s) + \operatorname{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n) \\ &= \operatorname{diag}\,(\mu_1, \mu_2, \ldots, \mu_n) + \operatorname{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n) \\ &= \operatorname{diag}\,(\mu_1 + \lambda_1, \mu_2 + \lambda_2, \ldots, \mu_n + \lambda_n). \end{aligned} \tag{2.9}$$

$\mathbf{A} + \mathbf{B}$ is a diagonalization matrix and $\mu_1 + \lambda_1 \leq \mu_2 + \lambda_2 \leq \cdots \leq \mu_n + \lambda_n$ are the eigenvalues of $\mathbf{A} + \mathbf{B}$.

The proof of Lemma 2.2 is fulfilled. □

Let $(\cdot,\cdot)$ denote the usual inner product on $\mathbb{R}^n$ and denote the corresponding norm by $|\mathbf{u}| = \{\sum_{i=1}^n u_i^2\}^{1/2}$, where $\mathbf{u} = (u_1, u_2, \ldots, u_n)^{\mathrm{T}}$. Let $[\cdot,\cdot]$ denote the inner product on $L^2([0,\pi], \mathbb{R}^n)$. It is known very well that $L^2([0,\pi], \mathbb{R}^n)$ is a Hilbert space with inner product

$$[\mathbf{u}, \mathbf{v}] = \int_0^{\pi} (\mathbf{u}(t), \mathbf{v}(t))\mathrm{d}t, \quad \left(\mathbf{u}, \mathbf{v} \in L^2([0,\pi], \mathbb{R}^n)\right) \tag{2.10}$$

and norm $\|\mathbf{u}\| = \sqrt{[\mathbf{u}, \mathbf{u}]} = (\int_0^{\pi} (\mathbf{u}(t), \mathbf{u}(t))\mathrm{d}t)^{1/2}$, respectively.

Now, we consider the boundary value problem

$$\begin{cases} \mathbf{u}'' + Au + g(t,\mathbf{u}) = h(t), & t \in (0,\pi), \\ u(0) = a, & u(\pi) = b, \end{cases} \tag{2.11}$$

where $\mathbf{u} : [0,\pi] \to \mathbb{R}^n$, $\mathbf{A}$ is a real constant diagonalization $n \times n$ matrix with real eigenvalues $\mu_1 \le \mu_2 \le \cdots \le \mu_n$ (each eigenvalue is repeated according to its multiplicity), $\mathbf{g} : [0,\pi] \times \mathbb{R}^n \to \mathbb{R}^n$ is a potential Carathéodory vector-valued function , $\mathbf{h} : [0,\pi] \to \mathbb{R}^n$ is continuous, $\mathbf{a} = (a_1, a_2, \ldots, a_n)^{\mathrm{T}}$, $\mathbf{b} = (b_1, b_2, \ldots, b_n)^{\mathrm{T}}$, $a_i, b_i \in \mathbb{R}$, $(i = 1, 2, \ldots, n)$.

Let $\mathbf{u}(t) = \mathbf{v}(t) + \boldsymbol{\omega}(t)$, $\boldsymbol{\omega}(t) = (1 - (t/\pi))\mathbf{a} + (t/\pi)\mathbf{b}$, $t \in [0,\pi]$, then (2.11) may be written in the form

$$\begin{cases} \mathbf{v}'' + Av + \mathbf{g}^*(t,\mathbf{v}) = \mathbf{h}^*(t), \\ v(0) = v(\pi) = 0, \end{cases} \tag{2.12}$$

where $\mathbf{g}^*(t,\mathbf{v}) = \mathbf{g}(t,\mathbf{v}+\boldsymbol{\omega})$, $\mathbf{h}^*(t) = \mathbf{h}(t) - \mathbf{A}\boldsymbol{\omega}(\mathbf{t})$. Clearly, $\mathbf{g}^*(t,\mathbf{v})$ is a potential Carathéodory vector-valued function, $\mathbf{h}^* : [0,\pi] \to \mathbb{R}^n$. Clearly, if $\mathbf{v}_0$ is a solution of (2.12), $\mathbf{u}_0 = \mathbf{v}_0 + \boldsymbol{\omega}$ will be a solution of (2.11).

Assume that there exists a real bounded diagonalization $n \times n$ matrix $\mathbf{B}(t,\mathbf{u})$ ($t \in [0,\pi], \mathbf{u} \in \mathbb{R}^n$) such that for a.e. $t \in [0,\pi]$ and $\boldsymbol{\xi}, \boldsymbol{\eta} \in L^2([0,\pi],\mathbb{R}^n)$

$$\mathbf{g}(t,\boldsymbol{\eta}) - \mathbf{g}(t,\boldsymbol{\xi}) = \mathbf{B}(t,\boldsymbol{\xi} + \boldsymbol{\tau}(\boldsymbol{\eta} - \boldsymbol{\xi}))(\boldsymbol{\eta} - \boldsymbol{\xi}), \tag{2.13}$$

where $\boldsymbol{\tau} = \operatorname{diag}(\tau_1, \tau_2, \ldots, \tau_n), \tau_i \in [0,1]$ $(i = 1, 2, \ldots, n)$, $\mathbf{B}(t,\mathbf{u})$ commutes with $\mathbf{A}$ and is possessed of real eigenvalues $\lambda_1(t,\mathbf{u}) \le \lambda_2(t,\mathbf{u}) \le \cdots \le \lambda_n(t,\mathbf{u})$. In the light of Lemma 2.2, $\mathbf{A} + \mathbf{B}(t,\mathbf{u})$ is a diagonalization $n \times n$ matrix with real eigenvalues $\mu_1 + \lambda_1(t,\mathbf{u}) \le \mu_2 + \lambda_2(t,\mathbf{u}) \le \cdots \le \mu_n + \lambda_n(t,\mathbf{u})$ (each eigenvalue is repeated according to its multiplicity). Assume that there exist positive integers N_i $(i = 1, 2, \ldots, n)$ such that for $\mathbf{u} \in L^2([0,\pi],\mathbb{R}^n)$

$$N_i^2 - \mu_i < \lambda_i(t,\mathbf{u}) < (N_i + 1)^2 - \mu_i \quad (i = 1, 2, \ldots, n). \tag{2.14}$$

Let $\boldsymbol{\xi}_i$ $(i = 1, 2, \ldots, n)$ be n linearly independent eigenvectors associated with the eigenvalues $\mu_i + \lambda_i(t,\mathbf{u})$ $(i = 1, 2, \ldots, n)$ and let $\boldsymbol{\gamma}_i$ $(i = 1, 2, \ldots, n)$ be the orthonormal vectors obtained by orthonormalizing to the eigenvectors $\boldsymbol{\xi}_i$ $(i = 1, 2, \ldots, n)$ of $\mu_i + \lambda_i(t,\mathbf{u})$ $(i = 1, 2, \ldots, n)$. Then for every $\mathbf{u} \in \mathbb{R}^n$

$$(\mathbf{A} + \mathbf{B}(t,\mathbf{u}))\boldsymbol{\gamma}_i = \left(\mu_i + \lambda_i(t,\mathbf{u})\right)\boldsymbol{\gamma}_i \quad (i = 1, 2, \ldots, n). \tag{2.15}$$

And let the set $\{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \ldots, \boldsymbol{\gamma}_n\}$ be a basis for the space $\mathbb{R}^n$, then for every $\mathbf{u} \in \mathbb{R}^n$,

$$\mathbf{u} = u_1\boldsymbol{\gamma}_1 + u_2\boldsymbol{\gamma}_2 + \cdots + u_n\boldsymbol{\gamma}_n. \tag{2.16}$$

It is well known that each $\mathbf{v} \in L^2([0,\pi],\mathbb{R}^n)$ can be represented by the absolutely convergent Fourier series

$$\mathbf{v} = \sqrt{\frac{2}{\pi}}\sum_{i=1}^{n}\sum_{k=1}^{\infty}(C_{ki}\sin kt)\boldsymbol{\gamma}_i, \quad C_{ki} = \sqrt{\frac{2}{\pi}}\int_0^{\pi} v_i(t)\sin kt\mathrm{d}t \quad (i = 1,2,\ldots,n; k = 1,2,\ldots). \tag{2.17}$$

Define the linear operator $L = -(\mathrm{d}^2/\mathrm{d}t^2) : \mathfrak{D}(L) \subset L^2([0,\pi],\mathbb{R}^n) \to L^2([0,\pi],\mathbb{R}^n)$,

$$\mathfrak{D}(L) = \left\{ \mathbf{v} \in L^2([0,\pi],\mathbb{R}^n) \mid \mathbf{v}(0) = \mathbf{v}(\pi) = \mathbf{0},\ \mathbf{v}(t) = \sqrt{\frac{2}{\pi}}\sum_{i=1}^{n}\sum_{k=1}^{\infty}(C_{ki}\sin kt)\boldsymbol{\gamma}_{\mathbf{i}}, \right.$$
$$\left. C_{ki} = \sqrt{\frac{2}{\pi}}\int_0^{\pi} v_i(t)\sin kt\mathrm{d}t,\ (i = 1,2,\ldots,n), \sum_{i=1}^{n}\sum_{k=1}^{\infty} C_{ki}^2 k^4 < +\infty \right\}, \tag{2.18}$$
$$L\mathbf{v} = \sqrt{\frac{2}{\pi}}\sum_{i=1}^{n}\sum_{k=1}^{\infty}k^2(C_{ki}\sin kt)\boldsymbol{\gamma}_{\mathbf{i}}, \quad \sigma(L) = \left\{ n^2 \mid n \in \mathbb{N} \right\}.$$

Clearly, $L = -(\mathrm{d}^2/\mathrm{d}t^2)$ is a selfadjoint operator and $\mathfrak{D}(L)$ is a Hilbert space for the inner product

$$\langle \mathbf{u},\mathbf{v} \rangle = \int_0^{\pi} \left[\left(\mathbf{u}'(t),\mathbf{v}'(t) \right) + (\mathbf{u}(t),\mathbf{v}(t)) \right]\mathrm{d}t, \quad (\mathbf{u},\mathbf{v} \in \mathfrak{D}(L)), \tag{2.19}$$

and the norm induced by the inner product is

$$\|\mathbf{v}\|^2 = \int_0^{\pi} \left[\left(\mathbf{v}'(t),\mathbf{v}'(t) \right) + (\mathbf{v}(t),\mathbf{v}(t)) \right]\mathrm{d}t, \quad (\mathbf{v} \in \mathfrak{D}(L)). \tag{2.20}$$

Define

$$X = \left\{ \mathbf{x} \in L^2([0,\pi],\mathbb{R}^n) \mid \mathbf{x}(t) = \sqrt{\frac{2}{\pi}}\sum_{i=1}^{n}\sum_{k=1}^{N_i}(C_{ki}\sin kt)\boldsymbol{\gamma}_i, t \in [0,\pi], \right.$$
$$\left. C_{ki} = \sqrt{\frac{2}{\pi}}\int_0^{\pi} x_i(t)\sin kt\mathrm{d}t \right\}, \tag{2.21}$$

$$Y = \left\{ \mathbf{y} \in L^2([0,\pi],\mathbb{R}^n) \mid \mathbf{y}(t) = \sqrt{\frac{2}{\pi}}\sum_{i=1}^{n}\sum_{k=N_i+1}^{\infty}(C_{ki}\sin kt)\boldsymbol{\gamma}_i,\ t \in [0,\pi], \right.$$
$$\left. C_{ki} = \sqrt{\frac{2}{\pi}}\int_0^{\pi} y_i(t)\sin kt\mathrm{d}t, \sum_{i=1}^{n}\sum_{k=N_i+1}^{\infty} C_{ki}^2 k^4 < +\infty \right\}. \tag{2.22}$$

Clearly, X and Y are orthogonal closed subspaces of $\mathfrak{D}(L)$ and $\mathfrak{D}(L) = X \oplus Y$.

Define two projective mappings $P:\mathfrak{D}(L)\to X$ and $Q:\mathfrak{D}(L)\to Y$ by $P\mathbf{v}=\mathbf{x}\in X$ and $Q\mathbf{v}=\mathbf{y}\in Y$, $\mathbf{v}=\mathbf{x}+\mathbf{y}\in\mathfrak{D}(L)$, then $S=P-Q$ is a selfadjoint operator.

Using the Riesz representation theorem , we can define a mapping $T: L^2([0,\pi],\mathbb{R}^n)\to L^2([0,\pi],\mathbb{R}^n)$ by

$$\langle T(\mathbf{u}),\mathbf{v}\rangle=\int_0^{\pi}\left[(\mathbf{u}',\mathbf{v}')-(\mathbf{A}\mathbf{u},\mathbf{v})-(\mathbf{g}(t,\mathbf{u}),\mathbf{v})+(\mathbf{h}(t),\mathbf{v})\right]dt,\quad \forall\mathbf{v}\in L^2([0,\pi],\mathbb{R}^n). \tag{2.23}$$

We observe that T in (2.23) is defined implicity. Let $T(\mathbf{u})=\nabla F(\mathbf{u})$ in (2.23), we have

$$\langle \nabla F(\mathbf{u}),\mathbf{v}\rangle=\int_0^{\pi}\left[(\mathbf{u}',\mathbf{v}')-(\mathbf{A}\mathbf{u},\mathbf{v})-(\mathbf{g}(t,\mathbf{u}),\mathbf{v})+(\mathbf{h}(t),\mathbf{v})\right]dt,\quad \forall\mathbf{v}\in \mathfrak{D}(L)\subset L^2([0,\pi],\mathbb{R}^n). \tag{2.24}$$

Clearly, ∇F and hence F is defined implicity by (2.24). It can be proved that $\mathbf{u}$ is a solution of (2.11) if and only if $\mathbf{u}$ satisfies the operator equation

$$\nabla F(\mathbf{u})=\mathbf{0}. \tag{2.25}$$

3. The Main Theorems

Now, we state and prove the following theorem concerning the solution of problem (2.11).

Theorem 3.1. *Assume that there exists a real diagonalization $n\times n$ matrix $\mathbf{B}(t,\mathbf{u})$ ($\mathbf{u}\in L^2([0,\pi],\mathbb{R}^n)$) with real eigenvalues $\lambda_1(t,\mathbf{u})\le\lambda_2(t,\mathbf{u})\le\cdots\le\lambda_n(t,\mathbf{u})$ satisfying* (2.14) *and commuting with* $\mathbf{A}$. *Denote*

$$\alpha(\|\mathbf{u}\|)=\min_{\|\tilde{\mathbf{u}}\|\le\|\mathbf{u}\|}\min_{1\le i\le n}\min_{0\le t\le\pi}\left\{\lambda_i(t,\tilde{\mathbf{u}})+\mu_i-N_i^2>0\right\}, \tag{3.1}$$

$$\beta(\|\mathbf{u}\|)=\min_{\|\tilde{\mathbf{u}}\|\le\|\mathbf{u}\|}\min_{1\le i\le n}\min_{0\le t\le\pi}\left\{(N_i+1)^2-\mu_i-\lambda_i(t,\tilde{\mathbf{u}})>0\right\}. \tag{3.2}$$

If

$$\begin{gathered}\alpha:[0,+\infty)\longrightarrow(0,+\infty),\quad \beta:[0,+\infty)\longrightarrow(0,+\infty),\\ s\cdot\alpha(s)\longrightarrow+\infty,\quad s\cdot\beta(s)\longrightarrow+\infty,\quad \text{as } s\longrightarrow+\infty,\end{gathered} \tag{3.3}$$

problem (2.11) *has a unique solution* $\mathbf{u}_0$, *and* $\mathbf{u}_0$ *satisfies* $\nabla F(\mathbf{u}_0)=\mathbf{0}$, *and*

$$F(\mathbf{u}_0)=\max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}F(\mathbf{x}+\mathbf{y}+\boldsymbol{\omega})=\min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}F(\mathbf{x}+\mathbf{y}+\boldsymbol{\omega}), \tag{3.4}$$

where F is a functional defined in (2.24) *and* $\boldsymbol{\omega}=(1-(t/\pi))\mathbf{a}+(t/\pi)\mathbf{b},\ t\in[0,\pi]$.

Proof. First, by virtue of (2.21) and (2.22), we have

$$\begin{aligned}
\int_0^{\pi} (\mathbf{x}', \mathbf{x}')\mathrm{d}t &= \int_0^{\pi} (-\mathbf{x}'', \mathbf{x})\mathrm{d}t \\
&\leq \int_0^{\pi} \left(\sqrt{\frac{2}{\pi}} \sum_{i=1}^{n} N_i^2 \sum_{k=1}^{N_i} (C_{ki} \sin kt)\boldsymbol{\gamma}_i, \sqrt{\frac{2}{\pi}} \sum_{i=1}^{n} \sum_{k=1}^{N_i} (C_{ki} \sin kt)\boldsymbol{\gamma}_i \right) \mathrm{d}t \\
&\leq \left(\max_{1\leq i \leq n} N_i \right)^2 \int_0^{\pi} (\mathbf{x}, \mathbf{x})\mathrm{d}t,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\int_0^{\pi} (\mathbf{y}', \mathbf{y}')\mathrm{d}t &= \int_0^{\pi} (-\mathbf{y}'', \mathbf{y})\mathrm{d}t \\
&= \int_0^{\pi} \left(\sqrt{\frac{2}{\pi}} \sum_{i=1}^{n} \sum_{k=N_i+1}^{\infty} k^2 (C_{ki} \sin kt)\boldsymbol{\gamma}_i, \sqrt{\frac{2}{\pi}} \sum_{i=1}^{n} \sum_{k=N_i+1}^{\infty} (C_{ki} \sin kt)\boldsymbol{\gamma}_i \right) \mathrm{d}t,
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\frac{1}{\max_{1\leq i\leq n} (N_i+1)^2} \int_0^{\pi} (\mathbf{y}', \mathbf{y}')\mathrm{d}t \\
&= \int_0^{\pi} \left(\sqrt{\frac{2}{\pi}} \sum_{i=1}^{n} \sum_{k=N_i+1}^{\infty} \frac{k^2}{\max_{1\leq i\leq n} (N_i+1)^2} (C_{ki} \sin kt)\boldsymbol{\gamma}_i, \mathbf{y} \right) \mathrm{d}t \\
&\geq \int_0^{\pi} (\mathbf{y}, \mathbf{y})\mathrm{d}t.
\end{aligned} \tag{3.7}$$

Denote $\nabla F(\mathbf{u}) = \nabla F(\mathbf{v} + \boldsymbol{\omega}) = \nabla F^*(\mathbf{v})$.

By (2.24), (2.13), (3.5), (3.6), (3.7), (3.1), and (3.2), for all $\mathbf{x_1}, \mathbf{x_2} \in X, y \in Y$, let $\mathbf{v_1} = \mathbf{x_1} + \mathbf{y} \in \mathfrak{D}(L)$, $\mathbf{v_2} = \mathbf{x_2} + \mathbf{y} \in \mathfrak{D}(L)$, $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 = \mathbf{x}_1 - \mathbf{x}_2 = \mathbf{x} \in X$, $\mathbf{x}_1 = P\mathbf{v}_1 \in X$, $\mathbf{x}_2 = P\mathbf{v}_2 \in X$, $\mathbf{y} = Q\mathbf{v}_1 = Q\mathbf{v}_2 \in Y$, we have

$$\begin{aligned}
\langle \nabla F^*(\mathbf{v}_1) - \nabla F^*(\mathbf{v}_2), \mathbf{x}_1 - \mathbf{x}_2 \rangle &= \langle \nabla F(\mathbf{u}_1) - \nabla F(\mathbf{u}_2), \mathbf{x}_1 - \mathbf{x}_2 \rangle = \langle \nabla F(\mathbf{u}_1), \mathbf{x} \rangle - \langle \nabla F(\mathbf{u}_2), \mathbf{x} \rangle \\
&= \int_0^{\pi} \left[\left(\mathbf{u}_1^{'}, \mathbf{x}' \right) - (\mathbf{A}\mathbf{u}_1, \mathbf{x}) - (\mathbf{g}(t, \mathbf{u}_1), \mathbf{x}) + (\mathbf{h}(t), \mathbf{x}) \right] \mathrm{d}t \\
&\quad - \int_0^{\pi} \left[\left(\mathbf{u}_2^{'}, \mathbf{x}' \right) - (\mathbf{A}\mathbf{u}_2, \mathbf{x}) - (\mathbf{g}(t, \mathbf{u}_2), \mathbf{x}) + (\mathbf{h}(t), \mathbf{x}) \right] \mathrm{d}t \\
&= \int_0^{\pi} \left[((\mathbf{u}_1 - \mathbf{u}_2)', \mathbf{x}') - (\mathbf{A}(\mathbf{u}_1 - \mathbf{u}_2), \mathbf{x}) - (\mathbf{g}(t, \mathbf{u}_1) - \mathbf{g}(t, \mathbf{u}_2), \mathbf{x}) \right] \mathrm{d}t \\
&= \int_0^{\pi} \left[(-\mathbf{v}'', \mathbf{x}) - (\mathbf{A}\mathbf{v}, \mathbf{x}) - (\mathbf{B}(t, \mathbf{v}_2 + \boldsymbol{\omega} + \boldsymbol{\tau}\mathbf{v})\mathbf{v}, \mathbf{x}) \right] \mathrm{d}t \\
&= \int_0^{\pi} \left[(-\mathbf{x}'', \mathbf{x}) - (\mathbf{A}\mathbf{x}, \mathbf{x}) - (\mathbf{B}(t, \mathbf{v}_2 + \boldsymbol{\omega} + \boldsymbol{\tau}\mathbf{v})\mathbf{x}, \mathbf{x}) \right] \mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^{\pi}\left[\left(\sum_{i=1}^{n} N_i^2 \cdot \sqrt{\frac{2}{\pi}}\sum_{k=1}^{N_i}(C_{ki}\sin kt)\boldsymbol{\gamma}_i, \mathbf{x}\right)\right.\\
&\qquad\left.-\left(\sum_{i=1}^{n}\sqrt{\frac{2}{\pi}}\sum_{k=1}^{N_i}(C_{ki}\sin kt)(\mathbf{A}+\mathbf{B}(t,\tilde{\mathbf{v}}))\boldsymbol{\gamma}_i, \mathbf{x}\right)\right]\mathrm{d}t\\
&\leq \int_0^{\pi}\left(\sum_{i=1}^{n}\left(N_i^2-\mu_i-\lambda_i(t,\tilde{\mathbf{v}})\right)\sqrt{\frac{2}{\pi}}\sum_{k=1}^{N_i}(C_{ki}\sin kt)\boldsymbol{\gamma}_i, \mathbf{x}\right)\mathrm{d}t\\
&\leq -\alpha(\|\mathbf{v}\|)\int_0^{\pi}(\mathbf{x},\mathbf{x})\mathrm{d}t\\
&= -\alpha(\|\mathbf{v}_1-\mathbf{v}_2\|)\frac{1}{(\max_{1\leq i\leq n} N_i)^2+1}\\
&\quad\times\int_0^{\pi}\left[\left((\max_{1\leq i\leq n} N_i)^2\right)(\mathbf{x},\mathbf{x})+(\mathbf{x},\mathbf{x})\right]\mathrm{d}t\\
&\leq -\alpha^*(\|\mathbf{v}_1-\mathbf{v}_2\|)\|\mathbf{x}_1-\mathbf{x}_2\|^2,\\
&\qquad\left(\alpha^*(\|\mathbf{v}_1-\mathbf{v}_2\|)=\frac{\alpha(\|\mathbf{v}_1-\mathbf{v}_2\|)}{(\max_{1\leq i\leq n} N_i)^2+1}\right),
\end{aligned}
\tag{3.8}
$$

for all $\mathbf{x} \in X, \mathbf{y_1}, \mathbf{y_2} \in Y$, let $\mathbf{v_1} = \mathbf{x}+\mathbf{y_1} \in \mathfrak{D}(L)$, $\mathbf{v_2} = \mathbf{x}+\mathbf{y_2} \in \mathfrak{D}(L)$, $\mathbf{v} = \mathbf{v}_1-\mathbf{v}_2 = \mathbf{y}_1-\mathbf{y}_2 = \mathbf{y} \in Y$, $\mathbf{y}_1 = Q\mathbf{v}_1 \in Y$, $\mathbf{y}_2 = Q\mathbf{v}_2 \in Y$, $\mathbf{x} = P\mathbf{v}_1 = P\mathbf{v}_2 \in X$, we have

$$
\begin{aligned}
&\langle \nabla F^*(\mathbf{v}_1)-\nabla F^*(\mathbf{v}_2), \mathbf{y}_1-\mathbf{y}_2\rangle\\
&\quad= \langle \nabla F(\mathbf{u}_1)-\nabla F(\mathbf{u}_2), \mathbf{y}_1-\mathbf{y}_2\rangle\\
&\quad= \int_0^{\pi}\left[((\mathbf{u}_1-\mathbf{u}_2)', \mathbf{y}')-(\mathbf{A}(\mathbf{u}_1-\mathbf{u}_2), \mathbf{y})-(\mathbf{g}(t,\mathbf{u}_1)-\mathbf{g}(t,\mathbf{u}_2), \mathbf{y})\right]\mathrm{d}t\\
&\quad= \int_0^{\pi}\left[(\mathbf{v}', \mathbf{y}')-(\mathbf{A}\mathbf{v}, \mathbf{y})-(\mathbf{B}(t,\tilde{\mathbf{v}})\mathbf{v}, \mathbf{y})\right]\mathrm{d}t\\
&\quad= \int_0^{\pi}\left[(\mathbf{y}', \mathbf{y}')-((\mathbf{A}+\mathbf{B}(t,\tilde{\mathbf{v}}))\mathbf{y}, \mathbf{y})\right]\mathrm{d}t\\
&\quad\geq \int_0^{\pi}\left[(\mathbf{y}', \mathbf{y}')-\left(\sqrt{\frac{2}{\pi}}\sum_{i=1}^{n}\sum_{k=N_i+1}^{\infty}\frac{k^2}{\max_{1\leq i\leq n}(N_i+1)^2}(C_{ki}\sin kt)\left(\mu_i+\boldsymbol{\lambda}_i(t,\tilde{\mathbf{v}})\right)\boldsymbol{\gamma}_i, \mathbf{y}\right)\right]\mathrm{d}t\\
&\quad= \int_0^{\pi}\left[(-\mathbf{y}'', \mathbf{y})-\left(\frac{1}{\max\limits_{1\leq i\leq n}(N_i+1)^2}\sqrt{\frac{2}{\pi}}\sum_{i=1}^{n}\sum_{k=N_i+1}^{\infty}k^2(C_{ki}\sin kt)\left(\mu_i+\lambda_i(t,\tilde{\mathbf{v}})\right)\boldsymbol{\gamma}_i, \mathbf{y}\right)\right]\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&\geq \int_0^{\pi}\left[\left(\sqrt{\frac{2}{\pi}}\sum_{i=1}^{n}\sum_{k=N_i+1}^{\infty}k^2(C_{ki}\sin kt)\gamma_i,\mathbf{y}\right)\right.\\
&\qquad\left.-\left(\frac{1}{\max_{1\leq i\leq n}(N_i+1)^2}\sqrt{\frac{2}{\pi}}\sum_{i=1}^{n}\sum_{k=N_i+1}^{\infty}k^2(C_{ki}\sin kt)(\mu_i+\lambda_i(t,\tilde{\mathbf{v}}))\gamma_i,\mathbf{y}\right)\right]\mathrm{d}t\\
&\geq \frac{1}{\max_{1\leq i\leq n}(N_i+1)^2}\int_0^{\pi}\left(\sqrt{\frac{2}{\pi}}\sum_{i=1}^{n}\sum_{k=N_i+1}^{\infty}k^2(C_{ki}\sin kt)\Big((N_i+1)^2-(\mu_i+\lambda_i(t,\tilde{\mathbf{v}}))\Big)\gamma_i,\mathbf{y}\right)\mathrm{d}t\\
&\geq \frac{\min_{\|\tilde{\mathbf{v}}\|\leq\|\mathbf{v}\|}\min_{1\leq i\leq n}\min_{t\in[0,\pi]}\left\{(N_i+1)^2-\mu_i-\lambda_i(t,\tilde{\mathbf{v}})>0\right\}}{\max_{1\leq i\leq n}(N_i+1)^2+1}\cdot\\
&\left(1+\frac{1}{\max_{1\leq i\leq n}(N_i+1)^2}\right)\int_0^{\pi}(\mathbf{y}',\mathbf{y}')\mathrm{d}t\\
&\geq \frac{\beta(\|\mathbf{v}\|)}{\max_{1\leq i\leq n}(N_i+1)^2+1}\int_0^{\pi}\left[(\mathbf{y}',\mathbf{y}')+(\mathbf{y},\mathbf{y})\right]\mathrm{d}t\\
&=\beta^*(\|\mathbf{v}\|)\|\mathbf{y}\|^2=\beta^*(\|\mathbf{v}_1-\mathbf{v}_2\|)\|\mathbf{y}_1-\mathbf{y}_2\|^2,\quad\left(\beta^*(\|\mathbf{v}_1-\mathbf{v}_2\|)=\frac{\beta(\|\mathbf{v}_1-\mathbf{v}_2\|)}{\max_{1\leq i\leq n}(N_i+1)^2+1}\right).
\end{aligned}
\tag{3.9}
$$

By (3.3), $s\cdot\alpha^*(s)\to+\infty, s\cdot\beta^*(s)\to+\infty$, as $s\to+\infty$. Clearly, α^* and β^* are nonincreasing. Now, all the conditions in the Theorem 2.1 are satisfied. By virtue of Theorem 2.1, there exists a unique $\mathbf{v}_0\in\mathfrak{D}(L)$ such that $\nabla F^*(\mathbf{v}_0)=\nabla F(\mathbf{v}_0+\boldsymbol{\omega})=\nabla F(\mathbf{u}_0)=\mathbf{0}$ and $F^*(\mathbf{v}_0)=F(\mathbf{v}_0+\boldsymbol{\omega})=F(\mathbf{u}_0)=\max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y}F(\mathbf{x}+\mathbf{y}+\boldsymbol{\omega})=\min_{\mathbf{y}\in Y}\max_{\mathbf{x}\in X}F(\mathbf{x}+\mathbf{y}+\boldsymbol{\omega})$, where F is a functional defined implicity in (2.24) and $\boldsymbol{\omega}(t)=(1-(t/\pi))\mathbf{a}+(t/\pi)\mathbf{b}, t\in[0,\pi]$. $\mathbf{v}_0(t)$ is just a unique solution of (2.12) and $\mathbf{u}_0(t)=\mathbf{v}_0(t)+\boldsymbol{\omega}(t)$ is exactly a unique solution of (2.11). The proof of Theorem 3.1 is completed. □

Now, we assume that there exists a positive integer N such that

$$
N^2-\mu_i<\lambda_i(t,\mathbf{u})<(N+1)^2-\mu_i\quad(i=1,2,\ldots,n)
\tag{3.10}
$$

for $\mathbf{u}\in L^2([0,\pi],\mathbb{R}^n),\ t\in[0,\pi]$. Define

$$
\begin{aligned}
X=\Bigg\{&\mathbf{x}\in L^2([0,\pi],\mathbb{R}^n)\mid\mathbf{x}(t)=\sqrt{\frac{2}{\pi}}\sum_{i=1}^{n}\sum_{k=1}^{N}(C_{ki}\sin kt)\gamma_i,t\in[0,\pi],\\
&C_{ki}=\sqrt{\frac{2}{\pi}}\int_0^{\pi}x_i(t)\sin kt\mathrm{d}t\Bigg\},
\end{aligned}
\tag{3.11}
$$

$$Y = \left\{ \mathbf{y} \in L^2([0,\pi],\mathbb{R}^n) \mid \mathbf{y}(t) = \sqrt{\frac{2}{\pi}} \sum_{i=1}^{n} \sum_{k=N+1}^{\infty} (C_{ki} \sin kt) \gamma_i, t \in [0,\pi], \right.$$
$$\left. C_{ki} = \sqrt{\frac{2}{\pi}} \int_0^{\pi} y_i(t) \sin kt dt, \sum_{i=1}^{n} \sum_{k=N+1}^{\infty} C_{ki}^2 k^4 < +\infty \right\}, \tag{3.12}$$

$$\alpha(\|\mathbf{u}\|) = \min_{\|\tilde{\mathbf{u}}\| \le \|\mathbf{u}\|} \min_{1 \le i \le n} \min_{0 \le t \le \pi} \left\{ \lambda_i(t, \tilde{\mathbf{u}}) + \mu_i - N^2 > 0 \right\}, \tag{3.13}$$

$$\beta(\|\mathbf{u}\|) = \min_{\|\tilde{\mathbf{u}}\| \le \|\mathbf{u}\|} \min_{1 \le i \le n} \min_{0 \le t \le \pi} \left\{ (N+1)^2 - \mu_i - \lambda_i(t, \tilde{\mathbf{u}}) > 0 \right\}. \tag{3.14}$$

Replace the condition (2.14) by (3.10) and replace (2.21), (2.22), (3.1), and (3.2) by (3.11), (3.12), (3.11), and (3.14), respectively. Using the similar proving techniques in the Theorem 3.1, we can prove the following theorem.

Theorem 3.2. *Assume that there exists a real diagonalization $n \times n$ matrix $\mathbf{B}(t,\mathbf{u})$ ($t \in [0,\pi], \mathbf{u} \in \mathbb{R}^n$) with real eigenvalues $\lambda_1(t,\mathbf{u}) \le \lambda_2(t,\mathbf{u}) \le \cdots \le \lambda_n(t,\mathbf{u})$ satisfying* (2.13) *and* (3.10) *and commuting with $\mathbf{A}$. If the functions α and β defined in (3.11) and* (3.14) *satisfy* (3.3)*, problem* (2.11) *has a unique solution $\mathbf{u}_0$, and $\mathbf{u}_0$ satisfies $\nabla F(\mathbf{u}_0) = \mathbf{0}$ and* (3.4).

It is also of interest to the case of $\mathbf{A} = \mathrm{O}$.

Corollary 3.3. *Let $\mathbf{h}(t)$, $\mathbf{g}(t,\mathbf{u})$, $\mathbf{a}$ and $\mathbf{b}$ be as in* (2.11)*. Assume that there exists a real diagonalization $n \times n$ matrix $\mathbf{B}(t,\mathbf{u})$ ($t \in [0,\pi], \mathbf{u} \in \mathbb{R}^n$) with real eigenvalues $\lambda_1(t,\mathbf{u}) \le \lambda_2(t,\mathbf{u}) \le \cdots \le \lambda_n(t,\mathbf{u})$ satisfying* (2.13) *and $N_i^2 < \lambda_i(t,\mathbf{u}) < (N_i+1)^2$ ($N_i \in \mathbb{Z}^+, i = 1,2,\ldots,n$). Denote*

$$\alpha(\|\mathbf{u}\|) = \min_{\|\tilde{\mathbf{u}}\| \le \|\mathbf{u}\|} \min_{1 \le i \le n} \min_{0 \le t \le \pi} \left\{ \lambda_i(t, \tilde{\mathbf{u}}) - N_i^2 > 0 \right\},$$
$$\beta(\|\mathbf{u}\|) = \min_{\|\tilde{\mathbf{u}}\| \le \|\mathbf{u}\|} \min_{1 \le i \le n} \min_{0 \le t \le \pi} \left\{ (N_i+1)^2 - \lambda_i(t, \tilde{\mathbf{u}}) > 0 \right\}. \tag{3.15}$$

If α and β satisfy (3.3)*, the problem*

$$\begin{cases} \mathbf{u}'' + g(t,\mathbf{u}) = h(t), & t \in (0,\pi), \\ u(0) = a, & u(\pi) = b \end{cases} \tag{3.16}$$

has a unique solution $\mathbf{u}_0$, and $\mathbf{u}_0$ satisfies $\nabla F(\mathbf{u}_0) = \mathbf{0}$ and (3.4)*, where F is a functional defined in*

$$\langle \nabla F(\mathbf{u}), \mathbf{v} \rangle = \int_0^{\pi} \left[(\mathbf{u}', \mathbf{v}') - (\mathbf{g}(t,\mathbf{u}), \mathbf{v}) + (\mathbf{h}(t), \mathbf{v}) \right] dt, \quad \mathbf{v} \in \mathfrak{D}(L). \tag{3.17}$$

Corollary 3.4. *Let* $\mathbf{h}(t)$*,* $\mathbf{g}(t,\mathbf{u})$*,* $\mathbf{a}$*,* $\mathbf{b}$*, and* $\mathbf{B}(t,\mathbf{u})$ *be as in Corollary 3.3. The eigenvalues of* $\mathbf{B}(t,\mathbf{u})\lambda_1(t,\mathbf{u}) \le \lambda_2(t,\mathbf{u}) \le \cdots \le \lambda_n(t,\mathbf{u})$ *satisfy* $N^2 < \lambda_i(t,\mathbf{u}) < (N+1)^2 (N \in \mathbb{Z}^+)$*. Denote*

$$\alpha(\|\mathbf{u}\|) = \min_{\|\tilde{\mathbf{u}}\|\le\|\mathbf{u}\|} \min_{1\le i\le n} \min_{0\le t\le\pi} \left\{\lambda_i(t,\tilde{\mathbf{u}}) - N^2 > 0\right\},$$
$$\beta(\|\mathbf{u}\|) = \min_{\|\tilde{\mathbf{u}}\|\le\|\mathbf{u}\|} \min_{1\le i\le n} \min_{0\le t\le\pi} \left\{(N+1)^2 - \lambda_i(t,\tilde{\mathbf{u}}) > 0\right\}. \tag{3.18}$$

If α *and* β *satisfy* (3.3)*, problem* (3.16) *has a unique solution* $\mathbf{u}_0$*, and* $\mathbf{u}_0$ *satisfies* $\nabla F(\mathbf{u}_0) = \mathbf{0}$ *and* (3.4)*, where F is a functional defined in* (3.17)*.*

If there exists a C^2 functional $G : [0,\pi] \times \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{g}(t,\mathbf{u}) = \nabla G(t,\mathbf{u})$, then (2.13) should be

$$\mathbf{g}(t,\boldsymbol{\eta}) - \mathbf{g}(t,\boldsymbol{\xi}) = \nabla G(t,\boldsymbol{\eta}) - \nabla G(t,\boldsymbol{\xi}) = \int_0^1 D^2 G(t,\boldsymbol{\xi} + \tau(\boldsymbol{\eta} - \boldsymbol{\xi}))(\boldsymbol{\eta} - \boldsymbol{\xi})\mathrm{d}\tau, \tag{3.19}$$

where D^2G is just a Hessian of G. In this case, the following corollary follows from Theorem 3.1.

Corollary 3.5. *Let the eigenvalues of* $\int_0^1 D^2G(t,\boldsymbol{\xi} + \tau(\boldsymbol{\eta} - \boldsymbol{\xi}))d\tau\lambda_1(t,\mathbf{u}) \le \lambda_2(t,\mathbf{u}) \le \cdots \le \lambda_n(t,\mathbf{u})$ *satisfy* (2.14)*. If* α *and* β *defined in* (3.1) *and* (3.2) *satisfy* (3.3)*, problem* (2.11)*(where* $\mathbf{g}(t,\mathbf{u}) = \nabla G(t,\mathbf{u})$*) has a unique solution* $\mathbf{u}_0$*, and* $\mathbf{u}_0$ *satisfies* $\nabla F(\mathbf{u}_0) = \mathbf{0}$ *and* (3.4)*.*

Using the similar techniques of the present paper, we can also investigate the two-point boundary value problem

$$\begin{cases} \mathbf{u}'' + \mathbf{A}\mathbf{u} + \mathbf{g}(t,\mathbf{u}) = \mathbf{h}(t), & t \in (0,2\pi), \\ \mathbf{u}(0) = \mathbf{a}, \qquad \mathbf{u}(2\pi) = \mathbf{b}, \end{cases} \tag{3.20}$$

where $\mathbf{u}$, $\mathbf{A}$, $\mathbf{h}(t)$, $\mathbf{g}(t,\mathbf{u})$, $\mathbf{a}$ and $\mathbf{b}$ are as in problem (2.11). The corresponding results are similar to the results in the present paper.

The special case of $\mathbf{A} = \mathrm{O}$ and $n = 1$ in problem (3.20) has been studied by Zhou Ting and Huang Wenhua [5]. Zhou and Huang adopted the techniques different from this paper to achieve their research.

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