Hindawi Publishing Corporation
Boundary Value Problems
Volume 2007, Article ID 74517, 10 pages
doi:10.1155/2007/74517

Research Article

Positive Solutions for Nonlinear *n*th-Order Singular Nonlocal Boundary Value Problems

Xin'an Hao, Lishan Liu, and Yonghong Wu

Received 23 June 2006; Revised 16 January 2007; Accepted 26 January 2007

Recommended by Ivan Kiguradze

We study the existence and multiplicity of positive solutions for a class of nth-order singular nonlocal boundary value problems $u^{(n)}(t)+a(t)f(t,u)=0$, $t\in(0,1)$, $u(0)=0$, $u'(0)=0,\dots,u^{(n-2)}(0)=0$, $\alpha u(\eta)=u(1)$, where $0<\eta<1$, $0<\alpha\eta^{n-1}<1$. The singularity may appear at $t=0$ and/or $t=1$. The Krasnosel'skii-Guo theorem on cone expansion and compression is used in this study. The main results improve and generalize the existing results.

1. Introduction

In this paper, we study the existence and multiplicity of positive solutions for the following nth-order nonlinear singular nonlocal boundary value problems (BVPs):

$$\begin{gathered} u^{(n)}(t)+a(t)f(t,u)=0, \quad t\in(0,1), \\ u(0)=0, \qquad u'(0)=0,\dots,u^{(n-2)}(0)=0, \qquad \alpha u(\eta)=u(1), \end{gathered} \tag{1.1}$$

where $0<\eta<1$, $0<\alpha\eta^{n-1}<1$, a may be singular at $t=0$ and/or $t=1$. We call $a(t)$ singular if $\lim_{t\to 0^+}a(t)=\infty$ or $\lim_{t\to 1^-}a(t)=\infty$.

The BVPs for nonlinear differential equations arise in a variety of areas of applied mathematics, physics, and variational problems of control theory. Many authors have discussed the existence of solutions of second-order or higher-order BVPs, for instance, [1–4]. Singular BVPs have also been widely studied because of their importance in both practical and theoretical aspects. In many practical problems, it is frequent that only positive solutions are useful. There have been many papers available in literature concerning the positive solutions of singular BVPs, see [5–9] and references therein. The study of singular nonlocal BVPs for nonlinear differential equations was initiated by Kiguradze and Lomtatidze [10] and Lomtatidze [11, 12]. Since then, more general nonlinear singular nonlocal BVPs have been studied extensively. Recently, Eloe and Ahmad [13] studied the positive solutions for the nth-order differential equation

$$u^{(n)}(t)+a(t)f(u)=0, \quad t\in(0,1), \tag{1.2}$$

subject to the nonlocal boundary conditions

$$u(0)=0, \qquad u'(0)=0,\dots,u^{(n-2)}(0)=0, \qquad \alpha u(\eta)=u(1), \tag{1.3}$$

where $0<\eta<1$, $0<\alpha\eta^{n-1}<1$. For the case in which a is nonsingular, Eloe and Ahmad established the existence of one positive solution for BVPs (1.2) and (1.3) if f is either superlinear (i.e., $\lim_{u\to0^+}(f(u)/u)=0$, $\lim_{u\to\infty}(f(u)/u)=\infty$) or sublinear (i.e., $\lim_{u\to0^+}(f(u)/u)=\infty$, $\lim_{u\to\infty}(f(u)/u)=0$) by applying the fixed point theorem on cones duo to Krasnosel'skii and Guo. However, research for existence of multiple positive solutions for higher-order singular nonlocal BVPs has proceeded very slowly and the related results are very limited.

Motivated by the above works, we consider the nth-order nonlinear singular BVPs (1.1) for the more general equations. In this paper, the results of existence and multiplicity of positive solutions are obtained under certain suitable weak conditions. The theorems and corollaries improve and generalize the results of [13]. The main results extend and include the results obtained by others. The main tool used for the study in this paper is the following Krasnosel'skii and Guo fixed point theorem.

Lemma 1.1 [14]. *Let X be a Banach space, and let P be a cone in X. Assume that Ω_1 and Ω_2 are two bounded open subsets of X with $0\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$. Let $A:P\cap(\overline{\Omega}_2\backslash\Omega_1)\to P$ be a completely continuous operator, satisfying either*

$$\text{(i)} \quad \|Ax\|\le\|x\|, \quad x\in P\cap\partial\Omega_1, \qquad \|Ax\|\ge\|x\|, \quad x\in P\cap\partial\Omega_2, \tag{1.4}$$

or

$$\text{(ii)} \quad \|Ax\|\ge\|x\|, \quad x\in P\cap\partial\Omega_1, \qquad \|Ax\|\le\|x\|, \quad x\in P\cap\partial\Omega_2. \tag{1.5}$$

Then A has at least one fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$.

Let G be Green's function for the $u^{(n)}(t)=0$ subjected to the nonlocal boundary conditions (1.3), then

$$G(t,s)=\begin{cases}\dfrac{\phi(s)t^{n-1}}{(n-1)!}, & 0\le t\le s\le 1,\\[2ex] \dfrac{\phi(s)t^{n-1}+(t-s)^{n-1}}{(n-1)!}, & 0\le s\le t\le 1,\end{cases} \tag{1.6}$$

where

$$\phi(s)=\begin{cases}-\dfrac{(1-s)^{n-1}}{1-\alpha\eta^{n-1}}, & \eta\le s,\\[2ex] -\dfrac{(1-s)^{n-1}-\alpha(\eta-s)^{n-1}}{1-\alpha\eta^{n-1}}, & s\le\eta.\end{cases} \tag{1.7}$$

It is easy to see that

$$G(t,s)<0,\quad t\in(0,1),\ s\in(0,1). \tag{1.8}$$

Lemma 1.2 [13]. *Let* $0<\alpha\eta^{n-1}<1$. *If* u *satisfies* $u^{(n)}(t)\le 0$, $0<t<1$, *with the nonlocal conditions (1.3), then*

$$\min_{t\in[\eta,1]}u(t)\ge\gamma\|u\|, \tag{1.9}$$

where $\gamma=\min\{\alpha\eta^{n-1},\alpha(1-\eta)(1-\alpha\eta)^{-1},\eta^{n-1}\}$.

Define $g(s)=\max_{t\in[0,1]}|G(t,s)|$. From the proof of Lemma 1.2 in [13], we know that

$$|G(t,s)|\ge\gamma g(s),\quad t\in[\eta,1],\ s\in[0,1]. \tag{1.10}$$

We first list some hypotheses for convenience.

(H_1) $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous and does not vanish identically on any subinterval of $[0,1]$.

(H_2) $a:(0,1)\to[0,\infty)$ is continuous and may be singular at $t=0$ and/or $t=1$.

(H_3) There exists $t_0\in[\eta,1)$ such that $a(t_0)>0$ and $\int_0^1 g(s)a(s)ds<+\infty$.

By (H_3) we can choose η_1, $\eta_2:\eta\le\eta_1\le t_0<\eta_2<1$ such that $a(t)>0$ for $t\in(\eta_1,\eta_2)$ and $0<\int_{\eta_1}^{\eta_2}g(s)a(s)ds<+\infty$. Under the conditions of Lemma 1.2, we also have $\min_{t\in[\eta_1,\eta_2]}u(t)\ge\gamma\|u\|$.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries and a lemma which establishes a completely continuous operator. In Section 3, Theorems 3.1 and 3.2, and results for the existence of at least one positive solution are established. Two corollaries on eigenvalue problems are also given. Section 4 deals with the existence of two positive solutions. Finally, in Section 5, we give three examples to illustrate the application of our main results.

2. Preliminaries

In what follows, we will impose the following conditions.

(H_4) $0\le f^0<L$, $l<f_\infty\le\infty$.

(H_5) $l<f_0\le\infty$, $0\le f^\infty<L$.

(H_6) $f_0=f_\infty=\infty$.

(H_7) There exists $\rho>0$ such that $f(t,u)<L\rho$, $0<u\le\rho$, $t\in[0,1]$.

(H_8) $f^0=f^\infty=0$.

(H_9) There exists $\rho>0$ such that $f(t,u)>l\rho$, $\gamma\rho\le u\le\rho$, $t\in[\eta_1,\eta_2]$.

In the above assumptions, we write

$$L := \left(\int_0^1 g(s)a(s)ds\right)^{-1}, \qquad l := \left(\gamma^2 \int_{\eta_1}^{\eta_2} g(s)a(s)ds\right)^{-1},$$
$$f^{\alpha} := \limsup_{u\to\alpha} \max_{t\in[0,1]} \frac{f(t,u)}{u}, \quad f_{\beta} := \liminf_{u\to\beta} \min_{t\in[\eta_1,\eta_2]} \frac{f(t,u)}{u}, \quad \alpha,\beta = 0^+, +\infty. \tag{2.1}$$

Set $E = C[0,1] = \{u : [0,1] \to R \mid u \text{ is continues on } [0,1]\}$. It is easy to testify that E is a Banach space with the norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$. We define a cone P as follows:

$$P = \left\{u \in E : u(t) \geq 0,\ t \in [0,1],\ \min_{t\in[\eta_1,\eta_2]} u(t) \geq \gamma\|u\|\right\}, \tag{2.2}$$

where γ is given in Lemma 1.2. Define an operator $A : P \to E$ by

$$Au(t) = -\int_0^1 G(t,s)a(s)f(s,u(s))\,ds. \tag{2.3}$$

By (H_1)–(H_3) and the properties of the function $G(t,s)$, we see that operator A is well defined. It is clear that the positive solution of singular BVP (1.1) is equivalent to the fixed point of A in P.

Before presenting the main results, we first give the following lemma establishing the conditions for A to be a completely continuous operator.

Lemma 2.1. *Assume that conditions (H_1)–(H_3) hold. Then $A : P \to P$ is a completely continuous operator.*

Proof. By (H_1)–(H_3), (1.8) and (2.3), we know that $Au(t) \geq 0$, $t \in [0,1]$. For any $u \in P$ and $t \in [0,1]$, we have

$$Au(t) = \int_0^1 |G(t,s)|\,a(s)f(s,u(s))\,ds \leq \int_0^1 g(s)a(s)f(s,u(s))\,ds. \tag{2.4}$$

Hence,

$$\|Au\| \leq \int_0^1 g(s)a(s)f(s,u(s))\,ds. \tag{2.5}$$

On the other hand, by (1.10) and (2.5), we have

$$\begin{aligned}\min_{t\in[\eta_1,\eta_2]} Au(t) &= \min_{t\in[\eta_1,\eta_2]} \int_0^1 |G(t,s)|\,a(s)f(s,u(s))\,ds \\ &\geq \gamma \int_0^1 g(s)a(s)f(s,u(s))\,ds \geq \gamma\|Au\|.\end{aligned} \tag{2.6}$$

Therefore, $A(P) \subset P$.

Now let us prove that A is completely continuous. Define $a_n : (0,1) \to [0,+\infty)$ by

$$a_n(t) = \begin{cases} \inf\left\{a(t), a\left(\frac{1}{n}\right)\right\}, & 0 < t \le \frac{1}{n}, \\ a(t), & \frac{1}{n} \le t \le \frac{n-1}{n}, \\ \inf\left\{a(t), a\left(\frac{n-1}{n}\right)\right\}, & \frac{n-1}{n} \le t < 1. \end{cases} \tag{2.7}$$

It is easy to see that $a_n \in C(0,1)$ is bounded and

$$0 \le a_n(t) \le a(t), \quad t \in (0,1). \tag{2.8}$$

Furthermore, we define an operator $A_n : P \to P$ as follows:

$$A_n u(t) = -\int_0^1 G(t,s)a_n(s)f(s,u(s))\,ds, \quad n \ge 2. \tag{2.9}$$

Obviously, A_n is a completely continuous operator on P for each $n \ge 2$. For any $R > 0$, set $B_R = \{u \in P : \|u\| \le R\}$, then A_n converges uniformly to A on B_R as $n \to \infty$. In fact, for $R > 0$ and $u \in B_R$, by (2.3) and (2.9), we get

$$\begin{aligned} |A_n u(t) - Au(t)| &= \left| \int_0^1 G(t,s)[a(s) - a_n(s)]f(s,u(s))\,ds \right| \\ &\le \left| \int_0^{1/n} G(t,s)[a(s) - a_n(s)]f(s,u(s))\,ds \right| \\ &\quad + \left| \int_{(n-1)/n}^1 G(t,s)[a(s) - a_n(s)]f(s,u(s))\,ds \right| \\ &\le M\left[\int_0^{1/n} g(s)[a(s) - a_n(s)]\,ds \right. \\ &\quad \left. + \int_{(n-1)/n}^1 g(s)[a(s) - a_n(s)]\,ds \right] \longrightarrow 0 \quad (n \longrightarrow \infty), \end{aligned} \tag{2.10}$$

where $M = \max_{t\in[0,1],x\in[0,R]} f(t,x)$, and we have used the facts $g(s)a(s) \in L^1(0,1)$ and (2.8). So we conclude that A_n converges uniformly to A on B_R as $n \to \infty$. Thus, A is completely continuous. □

3. Existence of a positive solution

Lemma 2.1 will help us obtain the following existence results of positive solution of BVP (1.1).

THEOREM 3.1. *Assume that conditions* (H_1)*–*(H_4) *hold. Then BVP (1.1) has at least one positive solution.*

Proof. By the first inequality of (H_4), there exist $M_1 > 0$ and $0 < \varepsilon_1 < L$ such that $f(t,u) \le (L-\varepsilon_1)u$ for $0 \le t \le 1$, $0 < u \le M_1$. Set $\Omega_1 = \{u \in E : \|u\| < M_1\}$. So for any $u \in P \cap \partial\Omega_1$,

$$Au(t) \le \int_0^1 g(s)a(s)f(s,u(s))ds \le (L-\varepsilon_1)\|u\| \int_0^1 g(s)a(s)ds < \|u\|, \quad t \in [0,1]. \tag{3.1}$$

Thus,

$$\|Au\| < \|u\|, \quad u \in P \cap \partial\Omega_1. \tag{3.2}$$

Next, by $l < f_\infty \le \infty$, there exist $\overline{M}_2 > 0$ and $\varepsilon_2 > 0$ such that $f(t,u) \ge (l+\varepsilon_2)u$ for $u \ge \overline{M}_2$, $t \in [\eta_1,\eta_2]$. Let $M_2 = \max\{2M_1, \overline{M}_2/\gamma\}$ and $\Omega_2 = \{u \in E : \|u\| < M_2\}$. Then $u \in P \cap \partial\Omega_2$ implies that $\min_{t\in[\eta_1,\eta_2]} u(t) \ge \gamma\|u\| = \gamma M_2 \ge \overline{M}_2$. So, by (1.10), we obtain

$$\begin{aligned} Au(\eta) &= \int_0^1 |G(\eta,s)|a(s)f(s,u(s))ds \ge \gamma \int_0^1 g(s)a(s)f(s,u(s))ds \\ &\ge \gamma \int_{\eta_1}^{\eta_2} g(s)a(s)f(s,u(s))ds \ge \gamma^2(l+\varepsilon_2)\|u\| \int_{\eta_1}^{\eta_2} g(s)a(s)ds > \|u\|. \end{aligned} \tag{3.3}$$

Thus,

$$\|Au\| > \|u\|, \quad u \in P \cap \partial\Omega_2. \tag{3.4}$$

By (3.2), (3.4) and Lemma 1.1, A has at least one fixed point $u^* \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$ with $0 < M_1 \le \|u^*\| \le M_2$. On the other hand, for any $t \in (0,1)$ we have that $u^*(t) = Au^*(t) = \int_0^1 |G(t,s)|a(s)f(s,u^*(s))ds \ge \int_{\eta_1}^{\eta_2} |G(t,s)|a(s)f(s,u^*(s))ds > 0$, and hence u^* is a positive solution of BVP (1.1). □

Theorem 3.2. *Assume that conditions (H_1)–(H_3) and (H_5) hold. Then BVPs (1.1) has at least one positive solution.*

Proof. By $l < f_0 \le \infty$, there exist $M_3 > 0$ and $\varepsilon_3 > 0$ such that $f(t,u) \ge (l+\varepsilon_3)u$ for $0 < u \le M_3$, $t \in [\eta_1,\eta_2]$. Let $\Omega_3 = \{u \in E : \|u\| < M_3\}$. Following the procedure used in the second part of Theorem 3.1, we have

$$Au(\eta) \ge \gamma^2(l+\varepsilon_3)\|u\| \int_{\eta_1}^{\eta_2} g(s)a(s)ds > \|u\|. \tag{3.5}$$

Thus,

$$\|Au\| > \|u\|, \quad u \in P \cap \partial\Omega_3. \tag{3.6}$$

By $0 \le f^\infty < L$, there exist $\overline{M}_4 > 0$ and $0 < \varepsilon_4 < L$ such that $f(t,u) \le (L-\varepsilon_4)u$ for $u \ge \overline{M}_4$, $t \in [0,1]$. Set $M = \max_{0\le t\le 1, 0\le x\le \overline{M}_4} f(t,x)$, then

$$f(t,u) \le M + (L-\varepsilon_4)u, \quad (t,u) \in [0,1] \times [0,+\infty). \tag{3.7}$$

Choose $M_4 > \max\{M_3, M/\varepsilon_4\}$ and $\Omega_4 = \{u \in E : \|u\| < M_4\}$, then for any $u \in P \cap \partial\Omega_4$, by (3.7), we have

$$\begin{aligned}\|Au\| &\le \int_0^1 g(s)a(s)f(s,u(s))\,ds \le \int_0^1 g(s)a(s)[M + (L-\varepsilon_4)M_4]\,ds\\ &\le LM_4\int_0^1 g(s)a(s)\,ds - (\varepsilon_4 M_4 - M)\int_0^1 g(s)a(s)\,ds < M_4 = \|u\|.\end{aligned} \tag{3.8}$$

Thus,

$$\|Au\| < \|u\|, \quad u \in P \cap \partial\Omega_4. \tag{3.9}$$

Applying Lemma 1.1 to (3.6) and (3.9), it follows that A has at least one positive solution $u^{**} \in P \cap (\overline{\Omega}_4 \backslash \Omega_3)$. This completes the proof of Theorem 3.2. □

The following corollaries are direct consequences of Theorems 3.1 and 3.2.

Corollary 3.3. *Assume that conditions (H_1)–(H_4) are satisfied. Then for each $\lambda \in (l/f_\infty, L/f^0)$, there exists at least one positive solution for the eigenvalue problems*

$$\begin{gathered}u^{(n)}(t) + \lambda a(t)f(t,u) = 0, \quad t \in (0,1),\\ u(0) = 0, \qquad u'(0) = 0,\dots,u^{(n-2)}(0) = 0, \qquad \alpha u(\eta) = u(1),\end{gathered} \tag{3.10}$$

where $0 < \eta < 1$, $0 < \alpha\eta^{n-1} < 1$.

Corollary 3.4. *Assume that conditions (H_1)–(H_3) and (H_5) are satisfied. Then for each $\lambda \in (l/f_0, L/f^\infty)$, there exists at least one positive solution for (3.10).*

4. Existence of multiple positive solutions

Theorem 4.1. *Assume that conditions (H_1)–(H_3), (H_6) and (H_7) hold. Then BVP (1.1) has at least two positive solutions.*

Proof. Firstly, by $f_0 = \infty$, there exists $R_1 : 0 < R_1 < \rho$ such that $f(t,u) > lu$ for $0 < u \le R_1$, $t \in [\eta_1, \eta_2]$. Set $\Omega_1 = \{u \in E : \|u\| < R_1\}$, then for any $u \in P \cap \partial\Omega_1$,

$$\begin{aligned}Au(\eta) &= \int_0^1 |G(\eta,s)|a(s)f(s,u(s))\,ds \ge \gamma\int_0^1 g(s)a(s)f(s,u(s))\,ds\\ &\ge \gamma\int_{\eta_1}^{\eta_2} g(s)a(s)f(s,u(s))\,ds > \gamma^2 l\|u\|\int_{\eta_1}^{\eta_2} g(s)a(s)\,ds = \|u\|.\end{aligned} \tag{4.1}$$

Thus,

$$\|Au\| > \|u\|, \quad u \in P \cap \partial\Omega_1. \tag{4.2}$$

Secondly, since $f_\infty = \infty$, there exists $\overline{R}_2 > \rho$ such that $f(t,u) > lu$ for $u \geq \overline{R}_2$, $t \in [\eta_1, \eta_2]$. Set $R_2 = \overline{R}_2/\gamma$, $\Omega_2 = \{u \in E : \|u\| < R_2\}$. Then for $u \in P \cap \partial\Omega_2$, we have $\min_{t\in[\eta_1,\eta_2]} u(t) \geq \gamma\|u\| = \overline{R}_2$. Hence,

$$\begin{aligned} Au(\eta) &= \int_0^1 |G(\eta,s)| a(s) f(s,u(s))\,ds \geq \gamma \int_0^1 g(s)a(s)f(s,u(s))\,ds \\ &\geq \gamma \int_{\eta_1}^{\eta_2} g(s)a(s)f(s,u(s))\,ds > \gamma^2 l\|u\| \int_{\eta_1}^{\eta_2} g(s)a(s)\,ds = \|u\|, \end{aligned} \tag{4.3}$$

which indicates

$$\|Au\| > \|u\|, \quad u \in P \cap \partial\Omega_2. \tag{4.4}$$

Thirdly, let $\Omega_3 = \{u \in E : \|u\| < \rho\}$. For any $u \in P \cap \partial\Omega_3$, we get from (H_7) that $f(t,u(t)) < L\rho$ for $t \in [0,1]$, then

$$\|Au\| \leq \int_0^1 g(s)a(s)f(s,u(s))\,ds < L\rho \int_0^1 g(s)a(s)\,ds = \rho = \|u\|. \tag{4.5}$$

Therefore,

$$\|Au\| < \|u\|, \quad u \in P \cap \partial\Omega_3. \tag{4.6}$$

Finally, (4.2), (4.4), (4.6), and $0 < R_1 < \rho < R_2$ imply that A has fixed points $u^* \in P \cap (\overline{\Omega}_3 \backslash \Omega_1)$ and $u^{**} \in P \cap (\overline{\Omega}_2 \backslash \Omega_3)$ such that $0 < \|u^*\| < \rho < \|u^{**}\|$. This completes the proof. □

Theorem 4.2. *Assume that conditions (H_1)–(H_3), (H_8) and (H_9) hold. Then BVP (1.1) has at least two positive solutions.*

The proof of Theorem 4.2 is similar to that of Theorem 4.1, so we omit it.

5. Examples

Example 5.1. Let $a(t) = (1-\alpha\eta^{n-1})(n-1)!/(1-t)^{n-1}$, $f(t,u) = \lambda t\ln(1+u) + u^2$, fix $\lambda > 0$ sufficiently small. By tedious compute,

$$0 < \int_0^1 g(s)a(s)\,ds \leq \int_0^1 \frac{(1-s)^{n-1}}{(1-\alpha\eta^{n-1})(n-1)!} a(s)\,ds = 1 < +\infty, \tag{5.1}$$

but $\int_0^1 a(s)\,ds = +\infty$. On the other hand, $f^0 = \lambda$, $f_\infty = \infty$. By Theorem 3.1, BVPs (1.1) have at least one positive solution. But the result of [13] is not suitable for this problem.

Example 5.2. Let $a(t)$ be as in Example 5.1 and let $f(t,u) = f(u) = u^2 e^{-u} + \mu \sin u$, fix $\mu > 0$ sufficiently large. Then $\lim_{u\to 0}(f(t,u)/u) = \mu$, $\lim_{u\to\infty}(f(t,u)/u) = 0$. By Theorem 3.2, BVP (1.1) has at least one positive solution. But the result of [13] is not suitable for this problem because of $\lim_{u\to 0}(f(t,u)/u) = \mu < \infty$.

Example 5.3. Let $a(t) = (1-\alpha\eta^{n-1})(n-1)!/10(1-t)^{n-1}$, $f(t,u) = u^2 + 1 + (t + 1/2)(\sin u)^{2/3}$. Then $f_0 = +\infty$, $f_\infty = +\infty$, $0 < 1/L = \int_0^1 g(s)a(s)ds \le \int_0^1 ((1-s)^{n-1}/(1-\alpha\eta^{n-1})(n-1)!)a(s)ds = 1/10$, $L \ge 10$. On the other hand, we could choose $\rho = 1$, then $f(t,u) \le 1^2 + 1 + 3/2 < L\rho$ for $(t,u) \in [0,1] \times [0,\rho]$. By Theorem 4.1, BVP (1.1) has at least two positive solutions.

Remark 5.4. Note that if f is superlinear or sublinear, our conclusions hold. In particular, if $f(t,u) = f(u)$ and a has no singularity, the conclusions of Theorems 3.1 and 3.2 still hold. So our conclusions extend and improve the corresponding results of [13].

Remark 5.5. Under suitable conditions, the multiplicity results for the more general equations are established. The multiplicity of positive solutions of Theorems 4.1 and 4.2 still holds for nonlocal BVP (1.2) and (1.3) and they are new results.

Acknowledgments

The first and second authors were supported financially by the National Natural Science Foundation of China (10471075) and the State Ministry of Education Doctoral Foundation of China (20060446001). The third author was supported financially by the Australian Research Council through an ARC Discovery Project Grant. The authors are grateful to the referees for their valuable suggestions and comments.

References

[1] C P. Gupta, "Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation," *Journal of Mathematical Analysis and Applications*, vol. 168, no. 2, pp. 540–551, 1992.

[2] V. A. Il'in and E. I. Moiseev, "A nonlocal boundary value problem of the first kind for the Sturm-Liouville operator in differential and difference interpretations," *Differential Equations*, vol. 23, no. 7, pp. 803–810, 1987.

[3] P. W. Eloe and J. Henderson, "Positive solutions for higher order ordinary differential equations," *Electronic Journal of Differential Equations*, vol. 1995, no. 3, pp. 8 pages, 1995.

[4] R. Y. Ma, "Positive solutions for a nonlinear three-point boundary-value problem," *Electronic Journal of Differential Equations*, vol. 1999, no. 34, pp. 8 pages, 1999.

[5] D. O'Regan, "Singular second order boundary value problems," *Nonlinear Analysis*, vol. 15, no. 12, pp. 1097–1109, 1990.

[6] Z.-C. Hao and A. M. Mao, "A necessary and sufficient condition for the existence of positive solutions to a class of singular second-order boundary value problems," *Journal of Systems Science and Mathematical Sciences*, vol. 21, no. 1, pp. 93–100, 2001 (Chinese).

[7] R. P. Agarwal and D. O'Regan, "Positive solutions to superlinear singular boundary value problems," *Journal of Computational and Applied Mathematics*, vol. 88, no. 1, pp. 129–147, 1998.

[8] R. Y. Ma, "Positive solutions of singular second-order boundary value problems," *Acta Mathematica Sinica*, vol. 41, no. 6, pp. 1225–1230, 1998 (Chinese).

[9] Z.-C. Hao, J. Liang, and T.-J. Xiao, "Positive solutions of operator equations on half-line," *Journal of Mathematical Analysis and Applications*, vol. 314, no. 2, pp. 423–435, 2006.

[10] I. T. Kiguradze and A. G. Lomtatidze, "On certain boundary value problems for second-order linear ordinary differential equations with singularities," *Journal of Mathematical Analysis and Applications*, vol. 101, no. 2, pp. 325–347, 1984.

[11] A. G. Lomtatidze, "A boundary value problem for second-order nonlinear ordinary differential equations with singularities," *Differential Equations*, vol. 22, no. 3, pp. 416–426, 1986.

[12] A. G. Lomtatidze, "Positive solutions of boundary value problems for second-order ordinary differential equations with singularities," *Differential Equations*, vol. 23, no. 10, pp. 1146–1152, 1987.
[13] P. W. Eloe and B. Ahmad, "Positive solutions of a nonlinear *n*th order boundary value problem with nonlocal conditions," *Applied Mathematics Letters*, vol. 18, no. 5, pp. 521–527, 2005.
[14] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5 of *Notes and Reports in Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1988.

Xin'an Hao: Department of Mathematics, Qufu Normal University, Qufu 273165, Shandong, China
Email address: haoxinan@eyou.com

Lishan Liu: Department of Mathematics, Qufu Normal University, Qufu 273165, Shandong, China
Current address: Department of Mathematics and Statistics, Curtin University of Technology, Perth 6845, WA, Australia
Email address: lls@mail.qfnu.edu.cn

Yonghong Wu: Department of Mathematics and Statistics, Curtin University of Technology, Perth 6845, WA, Australia
Email address: yhwu@maths.curtin.edu.au