QUASI-CONCAVITY FOR SEMILINEAR ELLIPTIC EQUATIONS WITH NON-MONOTONE AND ANISOTROPIC NONLINEARITIES

ANTONIO GRECO

Received 24 August 2004; Revised 25 May 2005; Accepted 9 August 2005

A boundary-value problem for a semilinear elliptic equation in a convex ring is considered. Under suitable structural conditions, any classical solution u lying between its (constant) boundary values is shown to decrease along each ray starting from the origin, and to have convex level surfaces.

1. Introduction

An interesting field of modern mathematical research is the study of geometric properties of solutions to elliptic problems. Remarkably, this is often done without any explicit representation of the solution.

This paper concentrates on the problem of *convexity of level sets* for solutions to some elliptic semilinear boundary-value problems in convex rings. More precisely, let Ω_0, Ω_1 be convex, bounded domains in $\mathbb{R}^N$, $N \geq 2$, satisfying $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_0$ (mnemonic: $0 =$ outer, $1 =$ inner). The domain $\Omega = \Omega_0 \setminus \overline{\Omega}_1$ is said a *convex ring*. Consider the following problem:

$$\begin{aligned} \Delta u &= f(x,u,Du) \quad \text{in } \Omega, \\ u &= a_0 \quad \text{on } \partial\Omega_0, \\ u &= a_1 \quad \text{on } \partial\Omega_1, \end{aligned} \tag{1.1}$$

where the boundary values a_0, a_1 are constants satisfying $a_0 < a_1$. The function $f(x,u,Du)$ is assumed to be locally Lipschitz continuous in (u,Du), locally uniformly in x. The question is whether the set $\overline{\Omega}_1 \cup \{x \in \Omega \mid u(x) \geq c\}$ is convex for every $c \in \mathbb{R}$. If this occurs, then u is said *quasi-concave*. The main result is the following.

Hindawi Publishing Corporation
Boundary Value Problems
Volume 2006, Article ID 80347, Pages 1–15
DOI 10.1155/BVP/2006/80347

Theorem 1.1. *Let Ω and f be as above. Suppose that for every $(x,u,Du) \in \mathbb{R}^N \times (a_0,a_1) \times \mathbb{R}^N$ we have:*

$$s^2 f(sx,u,Du/s) \text{ is non-decreasing in } s > 0 \text{ as long as } sx \in \Omega. \tag{1.2}$$

If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a classical solution to problem (1.1) satisfying

$$a_0 < u(x) < a_1 \quad \text{in } \Omega, \tag{1.3}$$

then

$$x \cdot Du(x) < 0 \quad \text{in } \Omega. \tag{1.4}$$

If, furthermore, for every $(u,Du) \in (a_0,a_1) \times \mathbb{R}^N$ we have:

$$s^3 f(x,u,Du/s) \text{ is convex with respect to } (s,x) \in \mathbb{R}^+ \times \Omega, \tag{1.5}$$

then the level surfaces of u are convex, and the intersection of any level surface (apart from the boundary) with any tangent hyperplane has an empty interior with respect to the canonical topology of the surface.

Monotonicity of f in u is not required. The theorem is also applicable to equations with anisotropic nonlinearities, like, for instance, the equation $\Delta u = |\partial u/\partial x_1|$ as well as $\Delta u = \partial u/\partial x_1 \partial u/\partial x_2$.

Related results are found in [1–4, 11]. For instance, if $f = f(u)$ then both (1.2) and (1.5) reduce to $f \geq 0$. If f takes the form $f = f(u,|Du|)$, then (1.5) is equivalent to the convexity of $t^{-2} f(u,t)$ with respect to the variable t (which was assumed in [11]). If, instead, f is positive and does not depend on Du, then (1.5) is equivalent to the *concavity* of the function $f(x,u)^{-1/2}$ with respect to the variable x (this is an assumption of [1]). The last two equivalences are proved in the appendix.

Observe that there are convex $f = f(x)$ not satisfying (1.5): if we take, for instance, $f(x) = |x|$ then the restriction of $g(s,x) = s^3 f(x)$ to the segment $x(s) = (1-s)x_0$, $s \in (0,1)$, $x_0 \neq 0$, is not convex, hence (1.5) does not hold.

Note that the conclusion (1.4) fails if the bound (1.3) on u is dropped, and a counterexample can be readily constructed with the equation $\Delta u = 1$ in an annulus. Indeed, the solution may attain its minimum at interior points.

Non-degeneracy (1.4) is proved in Section 2 by constructing an elliptic inequality satisfied in the weak sense by the function $\varphi(x) = x \cdot Du(x)$, thus making the result applicable to a non-C^3 solution u.

The method of proof of quasi-concavity, instead, is a generalization of the Gabriel-Lewis technique, devised by Gabriel (see [5]) for harmonic functions in $\mathbb{R}^3$, and then by Lewis, who proved in [12] quasi-concavity for p-harmonic functions.

Classically, the technique is based on the *quasi-concavity function* Q given by $Q(x,y) = u(z) - \min(u(x),u(y))$, $z = (x+y)/2$. The aim is to prove that $Q \geq 0$ in the set $G = \{(x,y) \mid x,y,z \in \Omega\}$. A limitation of such method is that the nonlinearity f is required to be non-decreasing in u.

The new idea to avoid such limitation is to work with a *vanishing* minimum of the function Q_s defined below. To be more precise, take $s \in (0,1]$ and define

$$G_s = \{(x,y) \mid x,y,sz \in \Omega\}, \quad z = (x+y)/2. \tag{1.6}$$

Let

$$s' = \inf\{s \mid G_s \neq \varnothing\}, \tag{1.7}$$

and consider the function

$$Q_s(x,y) = u(sz) - \min\left(u(x),u(y)\right) \tag{1.8}$$

for $(x,y) \in G_s$ and $s \in (s',1]$. By the results of continuity and localization of the minimum $m_s = \min_{\overline{G_s}} Q_s$, established in Sections 3 and 4, we may restrict our attention to the case when $m_s = 0$ and it is attained on the manifold

$$M_s = \{(x,y) \mid x,y,sz \in \Omega,\ u(x) = u(y)\}. \tag{1.9}$$

Since the restriction of Q_s to M_s satisfies an elliptic inequality (Section 5), the conclusion of Theorem 1.1 follows (see Section 6).

2. Non-degeneracy

Recall that a *convex* domain containing the origin is *strictly star-shaped* with respect to the origin. Theorem 2.1 below shows that if the domains Ω_0, Ω_1 have the latter property, then (1.4) follows. Related results are found in [1, 4, 10, 12, 13, 15]. Equations not satisfying (1.2) are considered in [7–9].

Theorem 2.1 (non-degeneracy). *Let Ω_0,Ω_1 be two bounded domains in $\mathbb{R}^N$, $N \geq 2$, strictly star-shaped with respect to the origin and satisfying $\overline{\Omega}_1 \subset \Omega_0$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a classical solution to problem (1.1), bounded by its boundary values as in (1.3). Suppose that $f(x,u,Du)$ is locally Lipschitz continuous in (u,Du), locally uniformly in x. If f also possesses property (1.2), then u satisfies (1.4).*

Proof. From [7, Theorem 2.1] it follows that $\varphi(x) = x \cdot Du(x) \leq 0$ in Ω. To complete the proof we have to check that such inequality is indeed strict. This is done by means of the maximum principle, after having constructed a suitable elliptic inequality in the weak form (namely, inequality (2.8) below) satisfied by φ. The argument is the following.

Since the Laplacian of u evaluated at sx is given by $\Delta u(sx) = f(sx,u(sx),Du(sx))$, assumption (1.2) on f implies that

$$\frac{s^2\Delta u(sx) - \Delta u(x)}{s-1} \geq \frac{f\left(x,u(sx),sDu(sx)\right) - f\left(x,u(x),Du(x)\right)}{s-1} \tag{2.1}$$

for $s < 1$, $x \in \Omega^s = \Omega \cap (1/s)\Omega$. What happens when $s \to 1^-$? Let us consider for first the left-hand side. Choose a *non-negative* test function $\psi \in C_0^\infty(\Omega)$. Observe that $\operatorname{supp}\psi \subset \Omega^s$

for s close to 1. Multiplication by ψ, followed by integration by parts, yields:

$$\int_\Omega \frac{s^2\Delta u(sx)-\Delta u(x)}{s-1}\psi(x) = -\int_\Omega \frac{sDu(sx)-Du(x)}{s-1}\cdot D\psi(x) \xrightarrow[s\to 1^-]{} -\int_\Omega D\varphi(x)\cdot D\psi(x). \tag{2.2}$$

Let us now turn to examine the right-hand side $R_s(x)$ of (2.1). In order to obtain a more suitable expression, let us introduce the vector-valued functions $\eta_s^i:\Omega^s\to\mathbb{R}^N$, $i=0,\dots,N$, whose components η_s^{ij} are given as follows:

$$\eta_s^{ij}(x)=\begin{cases} u_j(x), & \text{if } 1\le j\le i;\\ su_j(sx), & \text{if } i<j\le N,\end{cases} \tag{2.3}$$

where $u_j=\partial u/\partial x_j$. In particular, we have $\eta_s^0(x)=sDu(sx)$ and $\eta_s^N(x)=Du(x)$. With such notation, we may write:

$$R_s(x)=\frac{f(x,u(sx),\eta_s^0(x))-f(x,u(x),\eta_s^0(x))}{s-1}+\sum_{i=1}^N\frac{f(x,u(x),\eta_s^{i-1}(x))-f(x,u(x),\eta_s^i(x))}{s-1}. \tag{2.4}$$

Define the functions $b_s^i:\Omega\to\mathbb{R}$, $i=0,\dots,N$, by setting

$$\begin{aligned} b_s^0(x)&=\frac{f(x,u(sx),\eta_s^0(x))-f(x,u(x),\eta_s^0(x))}{u(sx)-u(x)},\\ b_s^i(x)&=\frac{f(x,u(x),\eta_s^{i-1}(x))-f(x,u(x),\eta_s^i(x))}{su_i(sx)-u_i(x)}, \quad \text{for } i=1,\dots,N,\end{aligned} \tag{2.5}$$

whenever $x\in\Omega^s$ and the denominator does not vanish; let $b_s^i(x)=0$ for all $x\in\Omega$ not matching the previous conditions. Now expression (2.4) may be rewritten as:

$$R_s(x)=b_s^0(x)\frac{u(sx)-u(x)}{s-1}+\sum_{i=1}^N b_s^i(x)\frac{su_i(sx)-u_i(x)}{s-1}. \tag{2.6}$$

For each compact subset $K\subset\Omega$ there exists $\varepsilon_K>0$ such that if $s\in\overline{I}_K=[1-\varepsilon_K,1]$ then $K\subset\Omega^s$. By the local Lipschitz continuity of f, the functions $b_s^i(x)$ are bounded in K, uniformly with respect to $s\in\overline{I}_K$. Now let K invade Ω. Using the Banach-Alaoglu-Bourbaki theorem, and by standard diagonal process, we conclude that there exist locally bounded functions $b^i\in L^\infty_{\text{loc}}(\Omega)$ and a sequence $\{s_n\}_{n\in\mathbb{N}}$ such that $s_n\nearrow 1$ as $n\to+\infty$ and $b_{s_n}^i$ converges to b^i in the weak-$*$ topology $\sigma(L^\infty(K),L^1(K))$ for *every* compact $K\subset\Omega$ and for each $i=0,\dots,N$. This is denoted by $b_{s_n}^i \overset{*}{\underset{\text{loc}}{\rightharpoonup}} b^i$.

Define $B(x)=(b^1(x),\dots,b^N(x))$. Since $u\in C^2(\Omega)$, the difference quotients in (2.6) converge uniformly on compact subsets of Ω. This and the boundedness of the coefficients $b_s^i(x)$, which has been noticed before, imply

$$R_{s_n} \overset{*}{\underset{\text{loc}}{\rightharpoonup}} b^0\varphi+B\cdot D\varphi. \tag{2.7}$$

Hence, if we replace s by s_n in (2.1), then, after multiplication by $\psi \geq 0$ and integration over Ω, we can pass to the limit for $n \to +\infty$. Taking into account (2.2), we obtain:

$$-\int_{\Omega} D\varphi(x) \cdot D\psi(x) \geq \int_{\Omega} \left(b^0(x)\varphi(x) + B(x) \cdot D\varphi(x)\right)\psi(x) \tag{2.8}$$

for every non-negative $\psi \in C_0^\infty(\Omega)$. Since we already know that $\varphi \leq 0$, we may also write the positive part $(b^0)^+$ in place of b^0. By the strong maximum principle for weak solutions of elliptic inequalities (see, for instance, [6, Section 8.7]), and since φ is continuous, we must have either $\varphi < 0$ in Ω or $\varphi \equiv 0$ in Ω. Since $a_0 < a_1$, the last case is impossible and the conclusion follows. □

3. Boundary values

This section deals with the *boundary values* of the function Q_s. Since u is continuous in $\overline{\Omega}$, and by (1.8), Q_s is defined not only in G_s but in the whole closed set F_s given by

$$F_s = \{(x,y) \mid x, y, sz \in \overline{\Omega}\} \tag{3.1}$$

for every $s \leq 1$ such that $F_s \neq \emptyset$. Of course, we have $\overline{G}_s \subset F_s$. However, such inclusion is *not* an equality, in general, and a counter-example may be constructed by letting Ω_0, Ω_1 be two rectangles with parallel edges. The *exceptional set* $F_s \setminus \overline{G}_s$ can be characterized as follows.

Lemma 3.1 (exceptional set). *Let $\Omega = \Omega_0 \setminus \overline{\Omega}_1$ be a convex ring. For $s \in (s', 1]$, define F_s as above and G_s as in (1.6). If there exists $(x_0, y_0) \in F_s \setminus \overline{G}_s$, then $s(x_0 + y_0)/2 \in \partial\Omega_1$, both x_0 and y_0 lie on $\partial\Omega$, and at least one of the last two points lies on $\partial\Omega_0$.*

Proof. If at least one point among x_0, y_0, sz_0 were *interior* to Ω, then it would be possible to move the other two points slightly, one after the other, and reach points x, y, sz that are *all* interior to Ω. This is equivalent to say that $(x,y) \in G_s$, and therefore (x_0, y_0) is a cluster point for G_s, contrary to the assumptions. Hence, all the three points x_0, y_0, sz_0 belong to $\partial\Omega$.

To complete the proof, observe that if $x_0, y_0, sz_0 \in \partial\Omega_0$ (or, alternatively, $\in \partial\Omega_1$), then we have $\lambda x_0, \lambda y_0, \lambda sz_0 \in \Omega$ for every $\lambda < 1$ (resp., $\lambda > 1$) such that $|\lambda - 1|$ is sufficiently small. Hence this case is excluded by the same argument as before.

In particular, sz_0 cannot be on $\partial\Omega_0$, because this would imply that ($s = 1$ and) also $x_0, y_0 \in \partial\Omega_0$. Similarly, it is not possible that both x_0 and y_0 are on $\partial\Omega_1$. The proof is complete. □

The behaviour of $\min_{\overline{G}_s} Q_s$ with respect to s is clarified by the following lemma.

Lemma 3.2 (continuity of the minimum). *Let $\Omega = \Omega_0 \setminus \overline{\Omega}_1$ be a convex ring. Let $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ satisfy (1.4) and attain the boundary values $u = a_0$ on $\partial\Omega_0$ and $u = a_1 > a_0$ on $\partial\Omega_1$. Then the minimum*

$$m_s = \min_{\overline{G}_s} Q_s \tag{3.2}$$

is strictly decreasing and continuous in $s \in (s', 1]$, and tends to $a_1 - a_0$ as $s \searrow s'$.

Proof. To prove *monotonicity*, take s,t satisfying $s' < s < t \le 1$. Since Ω is a convex ring, if $x, y, sz \in \Omega$ then $tz \in \Omega$. Hence, $G_s \subset G_t$. By (1.4), it follows that

$$Q_s(x,y) - Q_t(x,y) = u(sz) - u(tz) > 0 \tag{3.3}$$

in the closure $\overline{G}_s$, which immediately implies that $m_s > m_t$. Hence m_s is strictly decreasing in s.

Limit as $s \searrow s'$. Observe that if s is sufficiently small then for every $(x,y) \in G_s$ the point sz must lie near $\partial\Omega_1$, while z, together with x and y, stays close to $\partial\Omega_0$. Thus, the boundary conditions on u imply that $Q_s(x,y)$ approaches $a_1 - a_0$, uniformly in (x,y), hence

$$\lim_{s \searrow s'} m_s = a_1 - a_0. \tag{3.4}$$

Continuity. Let us prove that m_s is continuous from the left with respect to s. If this is not the case at some $s_0 \in (s',1]$, then $m_{s_0} < \lim_{s \nearrow s_0} m_s =: m_{s_0^-}$. The minimum m_{s_0} of Q_{s_0} may, in principle, be attained on the boundary ∂G_{s_0}. However, we may take an approximating, *interior* $(x_0, y_0) \in G_{s_0}$ such that $Q_{s_0}(x_0,y_0) < m_{s_0^-}$. Since u is continuous, we still have $Q_s(x_0,y_0) < m_{s_0^-}$ for every $s < s_0$ and sufficiently close to s_0. Hence $m_s < m_{s_0^-}$, which contradicts monotonicity.

To check that m_s is also continuous from the right with respect to s, it suffices to show that $m_{s_0} \le m_{s_0^+}$ for an arbitrary $s_0 \in (s',1)$. Let $\{t_n\}_{n\in\mathbb{N}}$ be a decreasing sequence in the interval $(s_0,1)$, approaching s_0 as $n \to +\infty$. For each $n \in \mathbb{N}$, let (x_n, y_n) be a point in $\overline{G}_{t_n}$ such that

$$Q_{t_n}(x_n,y_n) = m_{t_n} < m_{s_0} < a_1 - a_0. \tag{3.5}$$

By compactness, (x_n,y_n) converges (up to a subsequence) to some $(x_\infty, y_\infty) \in \overline{\Omega} \times \overline{\Omega}$ such that $s_0 z_\infty \in \overline{\Omega}$, where $z_\infty = (x_\infty + y_\infty)/2$. In the notation of (3.1), we have $(x_\infty, y_\infty) \in F_{s_0}$ but we do not know, for the moment, whether $(x_\infty, y_\infty) \in \overline{G}_{s_0}$. To check this, let us pass to the limit in the inequality above and find

$$Q_{s_0}(x_\infty,y_\infty) = m_{s_0^+} \le m_{s_0} < a_1 - a_0. \tag{3.6}$$

This and the boundary values of u show that it is impossible to have $s_0 z_\infty \in \partial\Omega_1$ and x_∞ (or y_∞) on $\partial\Omega_0$. By Lemma 3.1, we deduce that $(x_\infty, y_\infty) \in \overline{G}_{s_0}$, and therefore $m_{s_0} = \min_{\overline{G}_{s_0}} Q_{s_0} \le Q_{s_0}(x_\infty,y_\infty) = m_{s_0^+}$, as claimed. ☐

We can finally prove that any non-positive minimum of Q_s must be *interior*, provided that $s < 1$.

Lemma 3.3 (non-positive minima are interior). *Let Ω and u be as in Lemma 3.2. If for some $s < 1$ we have $\min_{\overline{G}_s} Q_s = m_s \le 0$, then $Q_s > m_s$ on the boundary ∂G_s.*

Proof. Observe, firstly, that since u is non-degenerate, then it is bounded as in (1.3). The study of an arbitrary $(x_0,y_0) \in \partial G_s$ reduces to the following three cases:

(1) At least one of x_0, y_0 is on $\partial\Omega_0$. In this case, since $u = a_0$ on $\partial\Omega_0$, we have $\min(u(x_0), u(y_0)) = a_0$. By convexity, the point $z_0 = (x_0 + y_0)/2$ is in $\overline{\Omega}_0$. Since $s < 1$, the point sz_0 is *interior* to Ω_0, and therefore $u(sz_0) > a_0$. Hence, $Q_s(x_0,y_0) > 0$.

(2) The point sz_0 is on $\partial\Omega_1$. Since $s < 1$, the points x_0, y_0 cannot lie both on $\partial\Omega_1$. By a similar argument as before, and since $u < a_1$ in Ω, we still see that $Q_s(x_0, y_0) > 0$.

(3) At least one of x_0, y_0 is on $\partial\Omega_1$. This is the less immediate case. Assume, without loss of generality, that $x_0 \in \partial\Omega_1$. Suppose, further, that $sz_0 \notin \partial\Omega_1$, otherwise we are in the previous case. Since Ω_1 is convex, we have $y_0 \notin \partial\Omega_1$ and therefore $u(y_0) < a_1$. Hence, $\min(u(x_0), u(y_0)) = u(y_0) < u(x_0)$. Let $\gamma : [0, T) \to \Omega$ be a maximal integral curve of the continuous field $-Du$ starting from sz_0. By (1.4), the modulus $|\gamma(t)|$ increases in t. Furthermore, since γ is maximal, the distance $\operatorname{dist}(\gamma(t), \partial\Omega_0)$ approaches 0 as $t \to T$. Now keep y_0 fixed and let $x(t)$ be such that

$$s\frac{x(t) + y_0}{2} = \gamma(t). \tag{3.7}$$

In particular, we have $x(0) = x_0 \in \partial\Omega_1$. As t increases, the corresponding $x(t)$ must reach (passing through Ω_1, if necessary), some $x_1 = x(t_1)$ which is *interior* to Ω but still so close to $\partial\Omega_1$ that the inequality $u(y_0) < u(x_1)$ is satisfied. Since $u(sz_0) > u(\gamma(t_1))$, we have $Q_s(x_0, y_0) > Q_s(x_1, y_0) \geq m_s$. □

4. Interior extremal condition

If Q_s attained an interior minimum at $(x, y) \in G_s$ with $u(x) < u(y)$, then by differentiation in y we would find $0 = D_y Q_s(x, y) = sDu(sz)/2$, but this is impossible if u satisfies (1.4).

Therefore, in the search for an interior minimum of Q_s, we are led to restrict our attention from the set G_s to the $(2N - 1)$-dimensional manifold M_s defined in (1.9). Let $(\tilde{x}, \tilde{y})$ be a point of M_s, and define $\tilde{z} = (\tilde{x} + \tilde{y})/2$. Assuming that $Du(s\tilde{z})$ is neither orthogonal to $Du(\tilde{x})$ nor to $Du(\tilde{y})$, let us construct convenient local coordinates on M_s in a neighborhood of $(\tilde{x}, \tilde{y})$. Let $(e_1, \ldots, e_N)$ be an orthonormal frame in $\mathbb{R}^N$ such that

$$Du(s\tilde{z}) = \left|Du(s\tilde{z})\right| e_N. \tag{4.1}$$

The derivatives of u with respect to such frame are denoted by subscripts. Let ξ, η be two variables in $\mathbb{R}^{N-1}$, and let t be a scalar one. Since $u_N(\tilde{x}), u_N(\tilde{y}) \neq 0$, by the implicit function theorem there exist smooth functions $\sigma(\xi, t), \varsigma(\eta, t)$ vanishing at $(0,0)$ and such that

$$\begin{aligned} u\left(\tilde{x} + \sum_{i=1}^{N-1} \xi_i e_i + \sigma(\xi, t) e_N\right) &= u(\tilde{x}) + t, \\ u\left(\tilde{y} + \sum_{i=1}^{N-1} \eta_i e_i + \varsigma(\eta, t) e_N\right) &= u(\tilde{y}) + t \end{aligned} \tag{4.2}$$

for all $(\xi, t), (\eta, t)$ in a conveniently small neighborhood of the origin. Now the mapping

$$\begin{aligned} x(\xi, t) &= \tilde{x} + \sum_{i=1}^{N-1} \xi_i e_i + \sigma(\xi, t) e_N, \\ y(\eta, t) &= \tilde{y} + \sum_{i=1}^{N-1} \eta_i e_i + \varsigma(\eta, t) e_N \end{aligned} \tag{4.3}$$

provides a local coordinate system on the manifold M_s. Define

$$\mathfrak{Q}_s(\xi,\eta,t) = Q_s\big(x(\xi,t),y(\eta,t)\big). \tag{4.4}$$

Since $u(x(\xi,t)) = u(\tilde{x})+t = u(y(\eta,t))$, we have:

$$\mathfrak{Q}_s(\xi,\eta,t) = u(sz) - u(\tilde{x}) - t, \tag{4.5}$$

where z is given by

$$z = \tilde{z} + \sum_{i=1}^{N-1} \frac{\xi_i+\eta_i}{2} e_i + \frac{\sigma(\xi,t)+\varsigma(\eta,t)}{2} e_N. \tag{4.6}$$

In particular, $\mathfrak{Q}_s$ is smooth with respect to (ξ,η,t). In order to characterize a minimum of $\mathfrak{Q}_s$, let us compute its derivatives with respect to such coordinates. By differentiation of (4.5) we find:

$$\begin{aligned}\frac{\partial\mathfrak{Q}_s}{\partial\xi_i}(\xi,\eta,t) &= \frac{s}{2}u_i(sz) + \frac{s}{2}u_N(sz)\sigma_i(\xi,t),\\ \frac{\partial\mathfrak{Q}_s}{\partial t}(\xi,\eta,t) &= \frac{s}{2}u_N(sz)\big(\sigma_t(\xi,t)+\varsigma_t(\eta,t)\big) - 1.\end{aligned} \tag{4.7}$$

The expression of $\partial\mathfrak{Q}_s/\partial\eta_i$ is analogous to the first one. The derivatives $\sigma_i = \partial\sigma/\partial\xi_i$, $\sigma_t = \partial\sigma/\partial t$, $\varsigma_t = \partial\varsigma/\partial t$, which occur above, can be computed by differentiating (4.2). We obtain:

$$\begin{gathered}u_i\big(x(\xi,t)\big) + u_N\big(x(\xi,t)\big)\sigma_i(\xi,t) = 0,\\ u_N\big(x(\xi,t)\big)\sigma_t(\xi,t) = 1.\end{gathered} \tag{4.8}$$

With this replacement, (4.7) become:

$$\frac{\partial\mathfrak{Q}_s}{\partial\xi_i}(\xi,\eta,t) = \frac{s}{2}u_i(sz) - \frac{s}{2}u_N(sz)\frac{u_i(x)}{u_N(x)}, \tag{4.9}$$

$$\frac{\partial\mathfrak{Q}_s}{\partial t}(\xi,\eta,t) = \frac{s}{2}u_N(sz)\left(\frac{1}{u_N(x)}+\frac{1}{u_N(y)}\right) - 1. \tag{4.10}$$

We can now characterize an interior minimum of Q_s. When $s=1$, the following lemma reduces to corresponding results of [2, 4, 10].

Lemma 4.1 (interior extremal condition). *Let Ω be a domain in $\mathbb{R}^N$, $N \ge 2$. Let $s \in (s',1]$, and let u be a function in the class $C^1(\Omega)$ satisfying $Du \neq 0$ in Ω. Suppose that Q_s attains a local minimum at $(\bar{x},\bar{y}) \in G_s$, and let $\bar{z} = (\bar{x}+\bar{y})/2$. Then $u(\bar{x}) = u(\bar{y})$ and the vectors $Du(\bar{x})$, $Du(\bar{y})$, $Du(s\bar{z})$ are parallel, have the same orientation, and their moduli are related as follows:*

$$\frac{s}{2}\left(\frac{1}{|Du(\bar{x})|}+\frac{1}{|Du(\bar{y})|}\right) = \frac{1}{|Du(s\bar{z})|}. \tag{4.11}$$

Proof. The equality $u(\bar{x}) = u(\bar{y})$ has been already noticed. To complete the proof, observe that $Du(s\bar{z})$ is neither orthogonal to $Du(\bar{x})$ nor to $Du(\bar{y})$: indeed, if $Du(\bar{x})$ were orthogonal to $Du(s\bar{z})$ then the plane tangent at $\bar{x}$ to the level surface $u = u(\bar{x})$ would contain the direction of $Du(s\bar{z})$. Hence we could pass from $\bar{x}$ to a neighbor point x such that $u(x) = u(\bar{x})$ and $u(sz) < u(s\bar{z})$, thus contradicting the minimality of $Q_s(\bar{x}, \bar{y})$.

We can, therefore, use the local coordinates (ξ, η, t) introduced before and let $(\tilde{x}, \tilde{y}) = (\bar{x}, \bar{y})$. Since the derivatives of $\mathfrak{Q}_s$ must vanish there, by (4.1)–(4.9) we deduce that $Du(\bar{x})$ is parallel to $Du(s\bar{z})$. The same conclusion holds for $Du(\bar{y})$. Furthermore, equality (4.11) follows from (4.10).

Now if $Du(\bar{x})$ were opposite to $Du(s\bar{z})$, then we could move $\bar{x}$ to a close $x = \bar{x} + \varepsilon Du(\bar{x})$ and contradict the minimality of $Q_s(\bar{x}, \bar{y})$. Hence $Du(\bar{x})$ has the same orientation of $Du(s\bar{z})$. Interchanging the role of $\bar{x}$ and $\bar{y}$ we see that $Du(\bar{y})$ and $Du(s\bar{z})$ also have the same orientation, and the proof is complete. □

5. An elliptic inequality

The purpose of this section is to construct the elliptic inequality (5.4) below, which is satisfied by $\mathfrak{Q}_s$ in a neighborhood of a given $(\bar{x}, \bar{y}) \in M_s$, $s \leq 1$, provided that

$$Du(\bar{x}) \cdot Du(s\bar{z}),\ Du(\bar{y}) \cdot Du(s\bar{z}) > 0. \tag{5.1}$$

Denote by $(\tilde{x}, \tilde{y})$ an arbitrary point of M_s, so close to $(\bar{x}, \bar{y})$ that $Du(\tilde{x}) \cdot Du(s\tilde{z})$, $Du(\tilde{y}) \cdot Du(s\tilde{z}) > 0$, by continuity. By rotating the frame in $\mathbb{R}^N$, we may assume that (4.1) holds. Note that assumptions (1.2)–(1.5) are preserved under rotations. Consider the local coordinates (ξ, η, t) introduced in Section 4, and let L be the degenerate elliptic operator whose characteristic matrix A at $(\tilde{x}, \tilde{y})$ is:

$$A(\tilde{x}, \tilde{y}) = \frac{1}{s\,|\,Du(s\tilde{z})\,|}\begin{pmatrix} \alpha^2 I & \alpha\beta I & 0 \\ \alpha\beta I & \beta^2 I & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.2}$$

where

$$\alpha = \frac{1}{u_N(\tilde{x})} = \frac{|\,Du(s\tilde{z})\,|}{Du(\tilde{x}) \cdot Du(s\tilde{z})}, \qquad \beta = \frac{1}{u_N(\tilde{y})} = \frac{|\,Du(s\tilde{z})\,|}{Du(\tilde{y}) \cdot Du(s\tilde{z})}, \tag{5.3}$$

and I stands for the identity matrix of order $N - 1$.

Theorem 5.1 (an elliptic inequality). *Let Ω be a domain in $\mathbb{R}^N$, $N \geq 2$, and let $u \in C^2(\Omega)$ be a solution to the equation in (1.1) such that $Du \neq 0$ in Ω. Suppose that the nonlinearity f satisfies all the assumptions of Theorem 1.1. Denote by M_s the manifold (1.9) for $s \in (s', 1]$, where s' is as in (1.7). If (5.1) holds at some $(\bar{x}, \bar{y}) \in M_s$, then at every $(\tilde{x}, \tilde{y}) \in M_s$ sufficiently close to $(\bar{x}, \bar{y})$, the function $\mathfrak{Q}_s$ given by (4.5) satisfies the inequality*

$$L\mathfrak{Q}_s \leq b\mathfrak{Q}_s + B \cdot D\mathfrak{Q}_s, \tag{5.4}$$

where L is as above and the coefficients b, B are bounded with respect to $(\tilde{x}, \tilde{y})$.

Proof. In order to compute $L\mathcal{Q}_s$ we need some entries of the Hessian matrix of $\mathcal{Q}_s$ at $(\tilde{x},\tilde{y})$. Before proceeding further, observe that by (4.1) and (4.7) we have

$$\sigma_i(0,0) = \frac{2}{su_N(s\tilde{z})}\frac{\partial \mathcal{Q}_s}{\partial \xi_i}(0,0,0), \tag{5.5}$$

and similarly for ς. From now on, we will collect terms in the first derivatives of $\mathcal{Q}_s$, since they are not relevant for Hopf's lemma provided that their coefficients remain bounded. This will simplify the next computations. For instance, by differentiating (4.9) we may write:

$$\sigma_{ii}(0,0) = -\frac{u_{ii}(\tilde{x})}{u_N(\tilde{x})} + b_i^\sigma \frac{\partial \mathcal{Q}_s}{\partial \xi_i}(0,0,0), \tag{5.6}$$

for a suitable b_i^σ which is bounded with respect to $(\tilde{x},\tilde{y})$ near $(\bar{x},\bar{y})$. The expression of $\varsigma_{ii}(0,0)$ is analogous. By (4.9) we also find $\sigma_{tt}(0,0) = -\alpha^3 u_{NN}(\tilde{x})$ and $\varsigma_{tt}(0,0) = -\beta^3 u_{NN}(\tilde{y})$. Making use of such expressions, and differentiating (4.7), we find that the second derivatives of $\mathcal{Q}_s$ at $(0,0,0)$ are as follows:

$$\begin{aligned}
\frac{\partial^2 \mathcal{Q}_s}{(\partial \xi_i)^2}(0,0,0) &= \frac{s^2}{4}u_{ii}(s\tilde{z}) - \frac{s}{2}\alpha u_N(s\tilde{z})u_{ii}(\tilde{x}) + b_i \frac{\partial \mathcal{Q}_s}{\partial \xi_i},\\
\frac{\partial^2 \mathcal{Q}_s}{\partial \xi_i \partial \eta_i}(0,0,0) &= \frac{s^2}{4}u_{ii}(s\tilde{z}) + c_i\frac{\partial \mathcal{Q}_s}{\partial \xi_i} + d_i \frac{\partial \mathcal{Q}_s}{\partial \eta_i}\\
\frac{\partial^2 \mathcal{Q}_s}{\partial t^2}(0,0,0) &= \frac{s^2}{4}(\alpha+\beta)^2 u_{NN}(s\tilde{z}) - \frac{s}{2}u_N(s\tilde{z})\big(\alpha^3 u_{NN}(\tilde{x}) + \beta^3 u_{NN}(\tilde{y})\big),
\end{aligned} \tag{5.7}$$

where the coefficients b_i, c_i, d_i are bounded with respect to $(\tilde{x},\tilde{y})$. The expression of $\partial^2\mathcal{Q}_s/(\partial \eta_i)^2$ is similar to the first one above, with x replaced by y and α replaced by β. Therefore we obtain:

$$L\mathcal{Q}_s = \frac{s}{4}(\alpha+\beta)^2\frac{\Delta u(s\tilde{z})}{u_N(s\tilde{z})} - \frac{\alpha^3 \Delta u(\tilde{x}) + \beta^3 \Delta u(\tilde{y})}{2} + B\cdot D\mathcal{Q}_s, \tag{5.8}$$

where the vector B is bounded with respect to $(\tilde{x},\tilde{y})$. By (4.10) we have:

$$\frac{1}{u_N(s\tilde{z})} = \frac{s}{2}(\alpha+\beta) - \frac{1}{u_N(s\tilde{z})}\frac{\partial \mathcal{Q}_s}{\partial t}(0,0,0), \tag{5.9}$$

and therefore, slightly changing the value of B, we may write:

$$L\mathcal{Q}_s = s^2\left(\frac{\alpha+\beta}{2}\right)^3 \Delta u(s\tilde{z}) - \frac{\alpha^3 \Delta u(\tilde{x}) + \beta^3 \Delta u(\tilde{y})}{2} + B\cdot D\mathcal{Q}_s. \tag{5.10}$$

Define the vectors X, Y, Z as follows:

$$X = \alpha^{-1}e_N, \qquad Y = \beta^{-1}e_N, \qquad Z = \left(\frac{\alpha+\beta}{2}\right)^{-1}e_N. \tag{5.11}$$

Let us express $Du(\tilde{x})$ in terms of X. Since $X_i = 0$ for $i = 1,\dots,N-1$, and by (4.9), we have:

$$\begin{gathered} u_i(\tilde{x}) = X_i - \frac{2u_N(\tilde{x})}{su_N(s\tilde{z})}\frac{\partial \mathcal{Q}_s}{\partial \xi_i}, \quad i = 1,\dots,N-1, \\ u_N(\tilde{x}) = X_N. \end{gathered} \tag{5.12}$$

A similar representation holds for $Du(\tilde{y})$. Furthermore, by (4.10) we have $sDu(s\tilde{z}) = (1 + \partial\mathcal{Q}_s/\partial t)Z$. Since $f(x,u,Du)$ is Lipschitz continuous in Du, and by redefining suitably the vector B, we arrive at:

$$\begin{aligned} L\mathcal{Q}_s = {} & s^2\left(\frac{\alpha+\beta}{2}\right)^3 f(s\tilde{z},u(s\tilde{z}),Z/s) \\ & - \frac{\alpha^3 f(\tilde{x},\tilde{u},X) + \beta^3 f(\tilde{y},\tilde{u},Y)}{2} + B\cdot D\mathcal{Q}_s, \end{aligned} \tag{5.13}$$

where $\tilde{u} = u(\tilde{x}) = u(\tilde{y})$. By (1.2) we have $s^2 f(s\tilde{z},u(s\tilde{z}),Z/s) \le f(\tilde{z},u(s\tilde{z}),Z)$. Since f is Lipschitz continuous in u, and since $\mathcal{Q}_s(0,0,0) = u(s\tilde{z}) - \tilde{u}$, we also have $f(\tilde{z},u(s\tilde{z}),Z) = f(\tilde{z},\tilde{u},Z) + b\mathcal{Q}_s$, where the coefficient b is bounded with respect to $(\tilde{x},\tilde{y})$. Recalling the definition of X, Y, Z we finally obtain:

$$\begin{aligned} L\mathcal{Q}_s \le {} & \left(\frac{\alpha+\beta}{2}\right)^3 f\left(\tilde{z},\tilde{u},e_N/\frac{\alpha+\beta}{2}\right) \\ & - \frac{\alpha^3 f(\tilde{x},\tilde{u},e_N/\alpha) + \beta^3 f(\tilde{y},\tilde{u},e_N/\beta)}{2} + b\mathcal{Q}_s + B\cdot D\mathcal{Q}_s, \end{aligned} \tag{5.14}$$

and the conclusion follows from assumption (1.5). □

Remark 5.2. L is an operator with continuous coefficients in the local coordinates (ξ',η',t') centred at $(\tilde{x},\tilde{y})$. Indeed, since $Du \in C^1(\Omega)$, the frame field $(e_1,\dots,e_N)$ may be chosen of class C^1 with respect to $(\tilde{x},\tilde{y})$. By the implicit function theorem, the Jacobian and the Hessian of the change of variables $\Phi : (\xi,\eta,t) \mapsto (\xi',\eta',t')$ depend continuously on $(\tilde{x},\tilde{y})$. This implies the claim.

6. Proof of Theorem 1.1

The non-degeneracy of u was proved in Theorem 2.1. The remainder of the proof is divided into two parts.

Part 1. The level surfaces of u are convex. Assume, contrary to the claim, that $u(z) < \min(u(x),u(y))$ at some (x,y) in G_1. By Lemmas 3.2 and 3.3, there exists $s < 1$ such that the function $Q_s(x,y) = u(sz) - \min(u(x),u(y))$ attains an interior, *vanishing* minimum at $(\bar{x},\bar{y}) \in G_s$, with $\bar{x} \ne \bar{y}$.

By Lemma 4.1 we have $u(\bar{x}) = u(\bar{y})$ and $Du(\bar{x}) \cdot Du(\bar{z})$, $Du(\bar{x}) \cdot Du(\bar{z}) > 0$. Hence, inequality (5.4) holds in a neighborhood of $(\bar{x},\bar{y})$. Since $Q_s \ge 0$, we may also write the positive part b^+ in place of b, and Hopf's lemma for degenerate operators holds (see [14, page 67, Remark (iv)]).

By Lemma 3.3, all vanishing minima are far from the boundary ∂G_s, and therefore we may assume that $\bar{u} = u(\bar{x})$ is the maximum of $u(x)$ for all $(x,y) \in G_s$ such that $Q_s(x,y) = 0$.

Hence, if we take any couple $(x,y) \in G_s$ such that $u(x) = u(y) > \bar{u}$ then $Q_s(x,y) > 0$. As a consequence, in the local coordinates (ξ,η,t) centred at $(\bar{x},\bar{y})$ we have $\mathcal{Q}_s(\xi,\eta,t) > 0$ for $t > 0$, and $\mathcal{Q}_s(0,0,0) = 0$. The outer normal $n = (0,0,-1)$ to the domain $\{t > 0\}$ at $(0,0,0)$ does not belong to the kernel of the matrix $A(\bar{x},\bar{y})$. Hence, Hopf's lemma applies and we find a contradiction with the fact that the gradient of $\mathcal{Q}_s$ vanishes at $(0,0,0)$. This proves that the level surfaces of u are convex.

Part 2. The intersection of any level surface (apart from the boundary) with any tangent hyperplane has an empty interior. Assume, contrary to the claim, that for some $\bar{u} \in (a_0,a_1)$ the intersection F between the level surface $\Sigma = \{x \mid u(x) = \bar{u}\}$ and some tangent hyperplane contains an interior point $\bar{y}$. Pick $\bar{x} \in F$ as far as possible from $\bar{y}$, that is,

$$|\bar{x} - \bar{y}| = \max_{x \in F} |x - \bar{y}|. \tag{6.1}$$

The segment $\bar{x}\bar{y}$ lies on F, hence $Q_1(\bar{x},\bar{y}) = 0$. By Part 1, $Q_1(x,y) \geq 0$ for all $(x,y) \in G_1$. Thus, Q_1 attains its minimum at $(\bar{x},\bar{y})$.

The local coordinates (4.3), centred at $(\bar{x},\bar{y})$, map an open neighborhood $U \subset \mathbb{R}^{2N-1}$ of the origin onto a subset of the manifold M_1. To reach a contradiction with Hopf's lemma, we construct a ball $B \subset U$ such that: (1) $\mathcal{Q}_1$ is positive in B; (2) $(0,0,0) \in \partial B$; (3) the outer normal n to ∂B at $(0,0,0)$ does not belong to the kernel of the matrix $A(\bar{x},\bar{y})$.

By rotating the coordinate frame if necessary, we may assume that $\bar{y} - \bar{x} = |\bar{y} - \bar{x}|e_1$. Consider the point $c = (x_c,\bar{y}) \in M_1$ where $x_c = x(\xi_c,0)$, $\xi_c = (-r,0,\dots,0) \in \mathbb{R}^{N-1}$, and $r > 0$ is so small that the ball $B = B(c,r) \subset \mathbb{R}^{2N-1}$,

$$B(c,r) = \left\{(\xi,\eta,t) \mid \left|\xi - \xi_c\right|^2 + |\eta|^2 + t^2 < r^2\right\}, \tag{6.2}$$

is contained in U. Since $\bar{x} \neq \bar{y}$ and $\bar{y}$ is interior to F, we may also assume that for every $(\xi,\eta,t) \in B$ the inequality $x(\xi,t) \neq y(\eta,t)$ holds, and $y(\eta,0)$ is interior to F.

We have $Q_1(x_c,\bar{y}) = \mathcal{Q}_1(\xi_c,0,0) > 0$: indeed, if we had $Q_1(x_c,\bar{y}) = 0$ then the segment $x_c\bar{y}$ would lie on the (convex) surface Σ and would pass through $\bar{x}$, contradicting (6.1). By continuity, we still have $\mathcal{Q}_1 > 0$ in the ball $B(c,\varepsilon)$ whose radius $\varepsilon \leq r$ is maximal in the sense that there exists $P = (\xi_P,\eta_P,t_P) \in \partial B(c,\varepsilon)$ such that $\mathcal{Q}_1(P) = 0$.

Let us check that $\varepsilon = r$ and $P = (0,0,0)$. Define $x_P = x(\xi_P,t_P)$ and $y_P = y(\eta_P,t_P)$. If t_P were different from zero, then the outer normal n at P would have a non-vanishing component in the direction of $\partial/\partial t$, hence n could not belong to the kernel of $A(x_P,y_P)$ and we would contradict Hopf's lemma. Hence, $t_P = 0$ and x_P, y_P lie on the level surface Σ. Since $\mathcal{Q}_1(P) = 0$, the whole segment x_Py_P lies on Σ. Furthermore, since $|\eta_P| \leq \varepsilon \leq r$, and since r has been chosen small enough, the point $y_P = y(\eta_P,0)$ is interior to F. Hence, x_P belongs to F. By (6.1) and by the definition of x_c, the point of F closest to x_c is $\bar{x}$. Since $|\xi_c| = r$, this and (6.2) imply $\varepsilon = r$, $x_P = \bar{x}$, and $y_P = \bar{y}$, as claimed.

We have thus proved that $\mathcal{Q}_1 > 0$ in the ball $B(c,r)$, and we know that $\mathcal{Q}_1(0,0,0) = 0$. The components of the outer normal n at $P = (0,0,0)$ are $(e_1',0,0)$, where e_1' is the first element of the canonical frame in $\mathbb{R}^{N-1}$. Since n does not belong to the kernel of the matrix $A(\bar{x},\bar{y})$, we reach again a contradiction with Hopf's lemma. Hence, the intersection of any level surface Σ with any tangent hyperplane must have an empty interior.

Appendix

When the nonlinearity f does not have a full dependence on (x,u,Du), condition (1.5) admits some alternative formulations, which are found in the literature. For instance, in the paper [11], Korevaar considered a C^2 function $f = f(u,|Du|)$ such that

$$\frac{f(u,t)}{t^2} \text{ is convex in } t > 0 \text{ for each } u. \tag{A.1}$$

In [1], Acker considered $f = f(x,u) > 0$ such that

$$\frac{1}{\sqrt{f(x,u)}} \text{ is concave in } x \text{ for each } u. \tag{A.2}$$

The next lemma puts into evidence the relation between such assumptions and (1.5). Condition (1.5) has been used in [3] for the special case $f = f(x,u,|Du|)$.

Lemma A.1. *If f has the form $f = f(u,|Du|)$ and is continuous, then (1.5) is equivalent to (A.1). If, instead, f is positive and does not depend on Du, then (1.5) is equivalent to (A.2).*

Proof. In the first case, condition (1.5) reduces to

$$(\lambda s_1 + (1-\lambda)s_2)^3 f\left(u, \frac{\tau}{\lambda s_1 + (1-\lambda)s_2}\right) \le \lambda s_1^3 f(u,\tau/s_1) + (1-\lambda)s_2^3 f(u,\tau/s_2) \tag{A.3}$$

for every $s_1,s_2 > 0$, $\lambda \in (0,1)$, $u \in (a_0,a_1)$, $\tau \ge 0$. Define $\sigma_1 = 1/s_1$ and $\sigma_2 = 1/s_2$. Since the mapping $s \mapsto s^{-1}$ is strictly monotone, for each $\lambda \in (0,1)$ there exists $\mu \in (0,1)$ such that

$$(\lambda s_1 + (1-\lambda)s_2)^{-1} = \mu\sigma_1 + (1-\mu)\sigma_2. \tag{A.4}$$

More precisely, if $s_1 \ne s_2$ then μ is unique and by an elementary computation we see that

$$\begin{aligned} \mu &= (\mu\sigma_1 + (1-\mu)\sigma_2)\frac{\lambda}{\sigma_1}, \\ 1-\mu &= (\mu\sigma_1 + (1-\mu)\sigma_2)\frac{1-\lambda}{\sigma_2}. \end{aligned} \tag{A.5}$$

If $s_1 = s_2$ then we take $\mu = \lambda$, so that the equalities above continue to hold. Inequality (A.3), after multiplication by $\mu\sigma_1 + (1-\mu)\sigma_2$, and by (A.4), becomes

$$\frac{f(u,(\mu\sigma_1 + (1-\mu)\sigma_2)\tau)}{(\mu\sigma_1 + (1-\mu)\sigma_2)^2} \le (\mu\sigma_1 + (1-\mu)\sigma_2)\left(\frac{\lambda}{\sigma_1}\frac{f(u,\sigma_1\tau)}{\sigma_1^2} + \frac{1-\lambda}{\sigma_2}\frac{f(u,\sigma_2\tau)}{\sigma_2^2}\right). \tag{A.6}$$

Using (A.5), this can be rewritten as

$$\frac{f(u,(\mu\sigma_1 + (1-\mu)\sigma_2)\tau)}{(\mu\sigma_1 + (1-\mu)\sigma_2)^2} \le \mu\frac{f(u,\sigma_1\tau)}{\sigma_1^2} + (1-\mu)\frac{f(u,\sigma_2\tau)}{\sigma_2^2}, \tag{A.7}$$

which expresses the convexity of $f(u,\sigma\tau)/\sigma^2$ with respect to $\sigma > 0$ for every fixed $u \in (a_0,a_1)$, $\tau \ge 0$. Choosing $\tau = 1$ we deduce (A.1) from (1.5). If, instead, we know that

$f(u,t)/t^2$ is convex in $t>0$, then, letting $t=\sigma\tau$, we get that for every $\tau>0$ the function $\tau^{-2}f(u,\sigma\tau)/\sigma^2$ is convex in $\sigma>0$. Thus we arrive at (A.7) for $\tau>0$. To prove (A.7) also for $\tau=0$ we have just to check that $f(u,0)\geq 0$. But if this were not the case, then, since f is continuous, (A.1) would cease to hold for $t\to 0$. This proves the first claim.

Suppose, now, that f is positive and does not depend on Du. After a straightforward computation we find that the concavity of $1/\sqrt{f(x,u)}$ with respect to the first variable is expressed by the inequality

$$f(\lambda x+(1-\lambda)y,u)\leq \frac{f(x,u)f(y,u)}{\left(\lambda\sqrt{f(y,u)}+(1-\lambda)\sqrt{f(x,u)}\right)^2}. \tag{A.8}$$

Condition (1.5), instead, is expressed by

$$f(\lambda x+(1-\lambda)y,u)\leq \frac{\lambda s_1^3 f(x,u)+(1-\lambda)s_2^3 f(y,u)}{\left(\lambda s_1+(1-\lambda)s_2\right)^3}, \tag{A.9}$$

which must hold for all positive s_1,s_2. Minimizing with respect to s_1,s_2 we find that the right-hand side of the inequality above (which is homogeneous in (s_1,s_2)) takes its minimum for $s_1/s_2=\sqrt{f(y,u)/f(x,u)}$, and the least value equals the right-hand side of (A.8). The second claim follows and the proof is complete. □

Acknowledgment

I am grateful to the referee, whose striking remarks have helped me to improve the paper.

References

[1] A. Acker, *On the uniqueness, monotonicity, starlikeness, and convexity of solutions for a nonlinear boundary value problem in elliptic PDEs*, Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal. Series A: Theory and Methods **22** (1994), no. 6, 697–705.

[2] L. A. Caffarelli and J. Spruck, *Convexity properties of solutions to some classical variational problems*, Communications in Partial Differential Equations **7** (1982), no. 11, 1337–1379.

[3] A. Colesanti and P. Salani, *Quasi-concave envelope of a function and convexity of level sets of solutions to elliptic equations*, Mathematische Nachrichten **258** (2003), no. 1, 3–15.

[4] J. I. Diaz and B. Kawohl, *On convexity and starshapedness of level sets for some nonlinear elliptic and parabolic problems on convex rings*, Journal of Mathematical Analysis and Applications **177** (1993), no. 1, 263–286.

[5] R. M. Gabriel, *A result concerning convex level surfaces of* 3*-dimensional harmonic functions*, Journal of the London Mathematical Society. Second Series **32** (1957), 286–294.

[6] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer, Berlin, 1998.

[7] A. Greco and M. Lucia, *Gamma-star-shapedness for semilinear elliptic equations*, Communications on Pure and Applied Analysis **4** (2005), no. 1, 93–99.

[8] A. Greco and W. Reichel, *Existence and starshapedness for the Lane-Emden equation*, Applicable Analysis. An International Journal **78** (2001), no. 1-2, 21–32.

[9] M. Grossi and R. Molle, *On the shape of the solutions of some semilinear elliptic problems*, Communications in Contemporary Mathematics **5** (2003), no. 1, 85–99.

[10] B. Kawohl, *Rearrangements and Convexity of Level Sets in PDE*, Lecture Notes in Mathematics, vol. 1150, Springer, Berlin, 1985.

[11] N. J. Korevaar, *Convexity of level sets for solutions to elliptic ring problems*, Communications in Partial Differential Equations **15** (1990), no. 4, 541–556.

[12] J. L. Lewis, *Capacitary functions in convex rings*, Archive for Rational Mechanics and Analysis **66** (1977), no. 3, 201–224.

[13] M. Longinetti, *A maximum principle for the starshape of solutions of nonlinear Poisson equations*, Unione Matematica Italiana. Bollettino. A. Serie VI **4** (1985), no. 1, 91–96.

[14] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, New Jersey, 1967.

[15] P. Salani, *Starshapedness of level sets of solutions to elliptic PDEs*, Abstract and Applied Analysis **84** (2005), no. 12, 1185–1197.

Antonio Greco: Dipartimento di Matematica e Informatica, via Ospedale 72, 09124 Cagliari, Italy
E-mail address: greco@unica.it