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Three Ways to Solve for Bond Prices in the Vasicek Model

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Abstract. Three approaches in obtaining the closed-form solution of the Vasicek bond pricing problem are discussed in this exposition. A derivation based solely on the distribution of the short rate process is reviewed. Solving the bond price partial differential equation (PDE) is another method. In this paper, this PDE is derived via a martingale approach and the bond price is determined by integrating ordinary differential equations. The bond pricing problem is further considered within the Heath-Jarrow-Morton (HJM) framework in which the analytic solution follows directly from the short rate dynamics under the forward measure.

Keywords: Bond pricing, Vasicek model, Martingales, HJM methodology, Forward measure.

1. Introduction

Vasicek's pioneering work (1977) is the first account of a bond pricing model that incorporates stochastic interest rate. The short rate dynamics is modeled as a diffusion process with constant parameters. When the bond price is based on this assumption, it has the feature that on a given date, the ratio of expected excess return per unit of volatility (the market price of risk) is the same, regardless of bond's maturity. Vasicek's model is a special version of Ornstein-Uhlenbeck (O-U) process, with constant volatility. This implies that the short rate is both Gaussian and Markovian. The model also exhibits mean-reversion and is therefore able to capture monetary authority's behavior of setting target rates. Furthermore, historical experience of interest rates justifies the O-U specification.

Given the pedagogical value of the Vasicek model in stochastic term structure modeling, the purpose of this paper is to present alternative derivation of the bond price solution. From the bond price the entire yield curve can be constructed at any given time. Thus, in turn, the term structure dynamics is characterized by the evolution of the short rate.

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Vasicek model's tractability property in bond pricing and the model's interesting stochastic characteristics make this classical model quite popular. In this paper a review of short rate's stochastic properties relevant to the derivation of the closed-form solution of the bond price within the Vasicek framework is presented. These properties become the basis for the first method examined in section 2. Under this technique, the bond price is derived from the implications of the interest rate's probability distribution. The development of the theory under this set-up follows from the outline of Lamberton and Lapeyre (1995).

The orginal derivation of the explicit formula for the bond price was based on solving the PDE that must be satisfied by the bond price. This is done by constructing a locally riskless portfolio and using the no-arbitrage arguments. Duffie and Kan (1996) provide a further characterization of this PDE. They prove that, if some Ricatti equations have solutions to the required maturity, the bond price has an exponential affine form. Vasicek's model belongs to this exponential affine class because the specification of its drift and volatility gives rise to a solvable set of equations in accordance with the Duffie-Kan descriptions. The second approach discussed in section 3 relies on the solution of the bond price PDE. However, unlike the traditional approach, this paper presents a martingale-oriented derivation of this PDE. This is motivated by the equivalence of the no-arbitrage pricing technique and the risk-neutral valuation which is a martingale-based method. Recently, Elliott and Van der Hoek (2001) offer a new method of solving the problem studied by Duffie and Kan. In their paper, it is shown that, when the short rate process is given by Gaussian dynamics or square root processes, the bond price is an exponential affine function. Their technique determines the bond price by integrating linear ODE and Ricatti equations are not needed. A similar idea is applied here to provide a solution to the bond pricing problem in the Vasicek model.

Section 4 presents a third alternative that considers the Heath-Jarrow-Morton (HJM) pricing paradigm. The equivalence between the forward rate and the conditional expectation of the short rate under the forward measure is discussed. Elaborating on the work of Geman, El Karoui and Rochet (1995) using the bond price as a numéraire, the short rate's dynamics is obtained under the forward measure. Consequently, the Vasicek forward rate dynamics is explicitly determined and therefore the analytic bond price follows immediately from the HJM bond pricing formula.

2. Bond Price Implied by the Short Rate Distribution

In modeling the uncertainty of interest rates, assume that there is an underlying probability space $(\Omega, \mathcal{F}, P)$ equipped with a standard filtration $\{\mathcal{F}_t\}$. Under the risk-neutral measure P, the short rate dynamics is given by

$$dr_t = a(b - r_t)dt + \sigma dW_t \tag{1}$$

where a, b and σ are all positive constants.

It can be verified using Itô's formula that

$$r_t = e^{-at}\left[r_0 + \int_0^t abe^{au}du + \sigma\int_0^t e^{au}dW_u\right]$$

is a solution to the stochastic differential equation (SDE) in (1). Note further that

$$\begin{aligned} r_t &= e^{-at}\left[r_0 + b(e^{at} - 1) + \int_0^t \sigma e^{au}dW_u\right] \\ &= \mu_t + \sigma\int_0^t e^{a(u-t)}dW_u, \end{aligned}$$

where μ_t is a deterministic function. Clearly, $E[r_t] = \mu_t$.

Observe further that r_t is a Gaussian random variable. This follows from the definition of the stochastic integral term, which is $\lim_{|\pi|\to 0}\sum_{i=0}^{n-1} e^{a(u_i - t)}(W_{u_{i+1}} - W_{u_i})$ and the increment $(W_{u_{i+1}} - W_{u_i}) \sim N(0, u_{i+1} - u_i)$. In general, if δ is deterministic (i.e., a function only of t), $\int_0^t \delta(u)dW_u$ is Gaussian.

While the expectation follows immediately from the solution for r_t given above, $E[r_t]$ can be determined without necessarily solving explicitly the SDE. Consider the integral form of (1). That is,

$$r_t = r_0 + \int_0^t (a(b - r_u)du + \sigma dW_u).$$

Hence,

$$\mu_t := E[r_t] = r_0 + \int_0^t a(b - E[r_u])du. \tag{2}$$

From (2),

$$\frac{d}{dt}\mu_t = a(b - \mu_t),$$

which is a linear ordinary differential equation (ODE). Consequently, using the integrating factor e^{at}

$$E[r_t] = e^{-at}[r_0 + b(e^{at} - 1)] = \mu_t. \tag{3}$$

In this model, b is some kind of level r is trying to attain. We call this the *mean-reverting level.* Similarly, define

$$\begin{aligned}\sigma_t^2 :&= Var[r_t] = E\left[\left(\sigma e^{-at}\int_0^t e^{au}dW_u)^2\right)\right]\\ &= \sigma^2 e^{-2at}E\left[\int_0^t e^{2au}du\right] \text{ by Itô's isometry}\\ &= \sigma^2\left(\frac{1-e^{-2at}}{2a}\right). \end{aligned} \tag{4}$$

Therefore, $r_t \sim N(\mu_t, \sigma_t^2)$ with mean and variance given in (3) and (4), respectively.

Since normal random variables can become negative with positive probability, this is considered to be the weakness of the Vasicek model. Nevertheless, the simplicity and tractability of the model validate its discussion.

Using the risk-neutral valuation framework, the price of a zero-coupon bond with maturity T at time t is

$$B(t,T) = E\left[exp\left(-\int_t^T r_u du\right)\middle|\mathcal{F}_t\right].$$

Write

$$X(u) = r_u - b. \tag{5}$$

Here, $X(u)$ is the solution of the Ornstein-Uhlenbeck equation

$$dX(t) = -aX(t) + \sigma dW_t \tag{6}$$

with $X(0) = r_0 - b$. Applying Itô's lemma, the $X(u)$ process is given by

$$X(u) = e^{-au}\left(X(0) + \int_0^u \sigma e^{as}dW_s\right). \tag{7}$$

Clearly, $X(u)$ is a Gaussian process with continuous sample paths. If $X(u)$ is Gaussian then $\int_0^t X(u)du$ is also Gaussian. Using (7), we obtain

$$E[X(u)] = X(0)e^{-au}.$$

Thus,

$$E\left[\int_0^t X(u)du\right] = \frac{X(0)}{a}(1-e^{-at}). \tag{8}$$

Similarly,

$$\begin{aligned} Cov[X(t),\ X(u)] &= \sigma^2 e^{-a(u+t)} E\left[\int_0^t e^{as} dW_s \int_0^u e^{as} dW_s\right] \\ &= \sigma^2 e^{-a(u+t)} \int_0^{u\wedge t} e^{2as} ds = \frac{\sigma^2}{2a} e^{-a(u+t)}(e^{2a(u\wedge t)} - 1). \end{aligned}$$

Consequently,

$$\begin{aligned} &Var\left[\int_0^t X(u)du\right] = Cov\left[\int_0^t X(u)du,\ \int_0^t X(s)ds\right] \\ &= E\left[\left(\int_0^t X(u)du - E\left[\int_0^t X(u)du\right]\right)\left(\int_0^t X(s)ds - E\left[\int_0^t X(s)ds\right]\right)\right] \\ &= \int_0^t \int_0^t E[(X(u) - E[X(u)])(X(s) - E[X(s)])]duds \\ &= \int_0^t \int_0^t Cov[X(u), X(s)]duds = \int_0^t \int_0^t \frac{\sigma^2}{2a} e^{-a(u+s)}(e^{2a(u\wedge s)} - 1)duds \\ &= \frac{\sigma^2}{2a^3}(2at - 3 + 4e^{-at} - e^{-2at}). \end{aligned} \tag{9}$$

From (5), we have

$$E\left[-\int_0^t r_u du\right] = E\left[-\int_0^t (X(u) + b)du\right].$$

Therefore, together with equation (8)

$$E\left[-\int_t^T r_u du\right] = -\frac{r_t - b}{a}(1 - e^{-a(T-t)}) - b(T-t). \tag{10}$$

Furthermore,

$$\begin{aligned} Var\left[-\int_t^T r_u du\right] &= Var\left[\int_t^T X(u)du\right] \\ &= \frac{\sigma^2}{2a^3}(2a(T-t) - 3 + 4e^{-a(T-t)} - e^{-2a(T-t)}) \end{aligned} \tag{11}$$

by the result from (9).

From the Itô integral representation of r_t, we also note that the defining process for the short rate is also Markov. For proof, see Karatzas and Shreve, p. 355.

Thus,

$$B(t,T) = E\left[exp\left(-\int_t^T r_u du\right)\middle|\mathcal{F}_t\right] = E\left[exp\left(-\int_t^T r_u du\right)\middle| r_t\right].$$

We write

$$B(t,T,r_t) := E\left[exp\left(-\int_t^T r_u du\right)\middle| r_t\right] = E\left[exp\left(-\int_t^T r_u(r_t) du\right)\right].$$

That is, r_u is a function of r_t.
Combining (10) and (11), the bond price is given by

$$\begin{aligned}
&B(t,T,r_t) = exp\left(E\left[-\int_t^T r_u(r_t)du\right] + \frac{1}{2}Var\left[-\int_t^T r_u(r_t)du\right]\right)\\
&= exp\left(-\frac{r_t-b}{a}(1-e^{-a(T-t)}) - b(T-t)\right.\\
&\quad \left. + \frac{\sigma^2}{4a^3}(2a(T-t) - 3 + 4e^{-a(T-t)} - e^{-2a(T-t)})\right) \qquad (12)\\
&= exp\left[-\left(\frac{1-e^{-a(T-t)}}{a}\right) r_t + b\left(\frac{1-e^{-a(T-t)}}{a} - (T-t)\right)\right.\\
&\quad \left. -\frac{\sigma^2}{2a^2}\left(\frac{1-e^{-a(T-t)}}{a}\right) + \frac{\sigma^2}{2a^2}(T-t) - \frac{\sigma^2}{4a}\left(\frac{1-2e^{-a(T-t)}+e^{-2a(T-t)}}{a^2}\right)\right]\\
&= exp\left[-A(t,T)r_t + bA(t,T) - b(T-t) - \frac{\sigma^2}{2a^2}A(t,T)\right.\\
&\quad \left. +\frac{\sigma^2}{2a^2}(T-t) - \frac{\sigma^2}{4a}A(t,T)^2\right] = exp(-A(t,T)r_t + D(t,T)), \qquad (13)
\end{aligned}$$

where

$$A(t,T) = \frac{1-e^{-a(T-t)}}{a} \quad \text{and} \qquad (14)$$

$$D(t,T) = \left(b - \frac{\sigma^2}{2a^2}\right)[A(t,T) - (T-t)] - \frac{\sigma^2 A(t,T)^2}{4a}. \qquad (15)$$

Since for all t, the yield $-\frac{\log\ B(t,T,r_t)}{T-t}$ obtained from (13) is affine in r_t, equation (13) is called an *affine term structure model* or an *exponential affine bond price.*

3. Solution via Bond Price PDE

Under this approach, the derivation is based on the fact that the r_u process is Markov. In other words, to determine how r_u evolves from t we need know only the value of r_t, $u \geq t$. Thus,

$$B(t,T,r_t) = E\left[exp\left(-\int_t^T r_u(r_t)du\right)\middle| r_t\right]$$

and

$$r_u = e^{-a(u-t)}\left[r_t + b(e^{a(u-t)}-1) + \sigma\int_t^u e^{a(v-t)}dW_v\right].$$

With r_t as a parameter,

$$\frac{\partial r_u(r_t)}{\partial r_t} = e^{-a(u-t)}.$$

So

$$\int_t^T \frac{\partial r_u(r_t)}{\partial r_t}du = \int_t^T e^{-a(u-t)}du = \frac{1}{a}(1-e^{-a(T-t)}),$$

which is deterministic.

Also,

$$\begin{aligned}\frac{\partial B(t,T,r_t)}{\partial r_t} &= E\left[-\left(\int_t^T \frac{\partial r_u(r_t)}{\partial r_t}du\right)exp\left(-\int_t^T r_u(r_t)du\right)\right]\\ &= -\frac{1}{a}(1-e^{-a(T-t)})E\left[exp\left(-\int_t^T r_u(r_t)du\right)\right]\\ &= -A(t,T)B(t,T,r_t),\end{aligned}$$

where $A(t,T)$ is given as in (14).

Thus, $\frac{\partial B}{\partial r_t} = -AB$. So,

$$B(t,T,r_t) = C(t,T)exp(-A(t,T)r_t),$$

for some function C independent of r_t.

Consider

$$exp\left(-\int_0^t r_u(r_t)du\right)B(t,T,r_t) = E\left[exp\left(-\int_0^T r_u du\right)\middle|\mathcal{F}_t\right].$$

Note that this is a $P-$martingale by the tower property. By Itô's lemma, we obtain

$$\begin{aligned}
&exp\left(-\int_0^t r_u(r_t)du\right)B(t,T,r_t)\\
&\quad = B(0,T,r_0)+\int_0^t -r_u exp\left(-\int_0^u r_v dv\right)B(u,T,r_u)du\\
&\quad + \int_0^t exp\left(-\int_0^u r_v dv\right)\frac{\partial}{\partial u}B(u,T,r_u)du\\
&\quad + \int_0^t exp\left(-\int_0^u r_v dv\right)\frac{\partial}{\partial r_u}B(u,T,r_u)(a(b-r_u)du+\sigma dW_u)\\
&\quad + \frac{1}{2}\int_0^t exp\left(-\int_0^u r_v dv\right)\frac{\partial^2}{\partial r_u^2}B(u,T,r_u)\sigma^2 du.
\end{aligned}$$

Since this is a martingale, all the du terms must sum to zero. So,

$$\begin{aligned}
-r_t B(t,T,r_t) \;+\; &\frac{\partial}{\partial t}B(t,T,r_t)+\frac{\partial}{\partial r_t}B(t,T,r_t)(a(b-r_t))\\
&+\;\frac{\sigma^2}{2}\frac{\partial^2}{\partial r_t^2}B(t,T,r_t)=0. \qquad (16)
\end{aligned}$$

Equation (16) is the PDE for the bond price in the Vasicek model. Moreover, this is a backward parabolic equation with $B(T,T,r_t)=1$ for every r_t.

So far we know

$$B(t,T,r_t)=C(t,T)exp(-A(t,T)r_t).$$

Therefore, we get the following partial derivatives.

$$\begin{aligned}
\frac{\partial B}{\partial t} &= \frac{\partial C}{\partial t}exp(-A(t,T)r_t)-C\frac{\partial A}{\partial t}r_t exp(-A(t,T)r_t)\\
\frac{\partial B}{\partial r_t} &= -AC exp(-A(t,T)r_t)\\
\frac{\partial^2 B}{\partial r_t^2} &= A^2 C exp(-A(t,T)r_t)
\end{aligned}$$

So, substituting to the PDE in (16) we have

$$\begin{aligned}
&-r_t C exp(-Ar_t)+\frac{\partial C}{\partial t}exp(-Ar_t)-C\frac{\partial A}{\partial t}r_t exp(-Ar_t)\\
&-AC exp(-Ar_t)(a(b-r_t))+\frac{\sigma^2}{2}A^2 C exp(-Ar_t)=0.
\end{aligned}$$

Therefore,

$$-r_t C + \frac{\partial C}{\partial t} - C\frac{\partial A}{\partial t} r_t - AC(a(b - r_t)) + \frac{\sigma^2}{2} A^2 C = 0.$$

Now, $B(t, T, 0) = C(t, T)$ and by putting $r_t = 0$ we get

$$\frac{\partial C}{\partial t} - abAC + \frac{\sigma^2}{2} A^2 C = 0.$$

Noting again that we are solving a backward ODE with $C(T, T) = 1$, we get

$$\begin{aligned} C(t,T) &= exp\left[-\frac{ab}{a}\int_t^T (1 - e^{-a(T-u)})du + \frac{\sigma^2}{2a^2}\int_t^T (1 - e^{-a(T-u)})^2 du\right] \\ &= exp\left[-b(T-t) + \frac{b}{a}(1 - e^{-a(T-t)}) + \frac{\sigma^2}{2a^2}(T-t)\right. \\ &\quad \left. + \frac{\sigma^2}{4a^3}(1 - e^{-2a(T-t)}) - \frac{\sigma^2}{a^3}(1 - e^{-a(T-t)})\right]. \end{aligned}$$

Write

$$D(t, T) := \log C(t, T).$$

We see that this reconciles with the second to the last terms of equation (12) and hence with the expression of equation (15). Under this approach, we have

$$B(t, T, r_t) = exp(-A(t, T)r_t + D(t, T))$$

where $A(t, T)$ is given by (14).

4. Bond Pricing by HJM Methodology

Following the terminology and notation of Heath, Jarrow and Morton (1992), this pricing paradigm is based on the concept of forward rate. The instantaneous forward rate at time t for date $T > t$ is defined by

$$f(t, T) = -\left.\frac{\partial \log B(t, t')}{\partial t'}\right|_{t'=T>t}. \tag{17}$$

This refers to the rate of interest that must be paid between t' and T. It is known at time t and therefore $\mathcal{F}_t-$measurable. Solving the differential equation in (17), yields

$$B(t, T) = exp\left(-\int_t^T f(t, u)du\right). \tag{18}$$

The short rate at time t, r_t, is the instantaneous rate at time t, i.e., $r_t = f(t,t)$ for every $t \in [0,T]$. From equation (18), it is clear that once $f(t,T)$ is completely determined the bond price immediately follows. The dynamics of the forward rate and that of the short rate are related via the forward measure. Invoking the insights of Geman, El Karoui and Rochet (1995), the forward measure P^T is defined on $\mathcal{F}_T$ by setting

$$\left.\frac{dP^T}{dP}\right|_{\mathcal{F}_T} = \Lambda_T := \frac{exp(-\int_0^T r_u du)}{B(0,T,r_0)}.$$

Consider the Radon-Nikodým process

$$\Lambda_t := E[\Lambda_T|\mathcal{F}_t] := \frac{exp(-\int_0^t r_u du)B(t,T)}{B(0,T)}, \quad t \in [0,T].$$

For any $\mathcal{F}_T-$measurable random variable X we have

$$\begin{aligned} E^T[X|\mathcal{F}_t] &= {\Lambda_t}^{-1} E[X \cdot \Lambda_T|\mathcal{F}_t] \\ &= E\left[\left.\frac{X exp\left(-\int_t^T r_u du\right)}{B(t,T)}\right| \mathcal{F}_t\right]. \end{aligned} \tag{19}$$

Now, the bond price in terms of the short rate is given by

$$B(t,T) = E\left[\left. exp\left(-\int_t^T r_u du\right)\right| \mathcal{F}_t\right].$$

Differentiating with respect to T, we get

$$\frac{\partial B(t,T)}{\partial T} = E\left[\left. -r_T exp\left(-\int_t^T r_u du\right)\right| \mathcal{F}_t\right] = -E^T[r_T|\mathcal{F}_t]B(t,T), \tag{20}$$

where the last equality follows from (19) with $X = r_T$. The bond price in terms of the forward rate is given in equation (18). Thus, differentiating $B(t,T)$ with respect to T, we obtain

$$\frac{\partial B(t,T)}{\partial T} = -B(t,T)f(t,T). \tag{21}$$

Comparing (20) and (21), in terms of the short rate model, the forward rate is given by

$$f(t,T) = E^T[r_T|\mathcal{F}_t] \tag{22}$$

where E^T denotes the expectation under P^T.

Invoking the change of probability measures and numéraire technique, under the forward measure P^T, the stochastic dynamics for r_t is given by

$$dr_t = (ab - A(t,T)\sigma^2 - ar_t)dt + \sigma dW_t^T, \tag{23}$$

where W_t^T is the P^T- Brownian motion defined by

$$dW_t^T = dW_t + \sigma A(t,T)dt,$$

and $A(t,T)$ is the function defined in equation (14). See Appendix for the proof of (23). By Itô's lemma, for $t \leq T$, the solution to (23) is given by

$$\begin{aligned} r_T = r_t e^{-a(T-t)} &+ \left(b - \frac{\sigma^2}{a^2}\right)(1 - e^{-b(T-t)}) + \frac{\sigma^2}{2a^2}[e^{-a(T-t)} - e^{-2a(T-t)}] \\ &+ \sigma \int_t^T e^{-a(T-u)} dW_u^T. \end{aligned}$$

Thus,

$$\begin{aligned} E^u[r_u|\mathcal{F}_t] &= r_t e^{-a(u-t)} + \left(b - \frac{\sigma^2}{2a^2}\right)(1 - e^{-a(u-t)}) \\ &+ \frac{\sigma^2}{2a^2}(e^{-a(u-t)} - e^{-2a(u-t)}). \end{aligned}$$

So,

$$\begin{aligned} \int_t^T E^u[r_u|\mathcal{F}_t]du &= \frac{r_t}{a}\left[-e^{-a(u-t)}\right]_t^T + \left(b - \frac{\sigma^2}{2a^2}\right)(T-t) \\ &\quad - \left(b - \frac{\sigma^2}{2a^2}\right)\left[-\frac{1}{a}e^{-a(u-t)}\right]_t^T \\ &\quad + \frac{\sigma^2}{2a^3}\left[-e^{-a(u-t)}\right]_t^T - \frac{\sigma^2}{4a^3}\left[-e^{-2a(u-t)}\right]_t^T \\ &= \frac{r_t}{a}(1 - e^{-a(T-t)}) + \left(b - \frac{\sigma^2}{2a^2}\right)(T-t) \\ &\quad - \left(b - \frac{\sigma^2}{2a^2}\right)\left(\frac{1 - e^{-a(T-t)}}{a}\right) + \frac{\sigma^2}{2a^3}(1 - e^{-a(T-t)}) \\ &\quad - \frac{\sigma^2}{4a^3}(1 - e^{-2a(T-t)}) \\ &= r_t A(t,T) + \left(b - \frac{\sigma^2}{2a^3}\right)[(T-t) + A(t,T)] \\ &\quad + \sigma^2\left[\frac{1 - e^{-a(T-t)}}{a}\right]^2. \end{aligned}$$

Therefore,

$$\begin{aligned} B(t,T,r_t) &= exp\left(-\int_t^T f(t,u)du\right) = exp\left(-\int_t^T E^u[r_u|\mathcal{F}_t]du\right) \\ &= exp(-r_t A(t,T) + D(t,T)) \end{aligned}$$

and the $A(t,T)$ and $D(t,T)$ values are in agreement with that of equations (14) and (15), respectively.

5. Conclusion

The pedagogical value of the Vasicek model is well-known in stochastic interest rate modeling. This paper contributes to the development of the available mathematical techniques in obtaining the closed-form solution of the bond price under the Vasicek framework. A discussion for each of the three different methods was provided. The first derivation considers the distributional properties of the short rate proces r_t. The simple Gaussian structure of r_t leads to a closed-form solution of the bond price. The bond price backward PDE is also derived using a martingale-oriented methodology. This PDE together with the Vasicek dynamics is the basis of the second method which integrates ordinary differential equations to get the bond price. Turning to the HJM pricing framework, the third approach employs the dynamics of the forward rate to fully describe the bond price process. The forward rate is linked to the short rate via the forward measure. When the short rate dynamics is determined under the forward measure, the HJM bond price is obtained and this reconciles with the prices computed from the other two approaches.

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Appendix : Proof of Result in Equation 23

Let P be an equivalent martingale measure (EMM) for the numéraire H_t and Q an EMM for the numéraire J_t. Then for any $V_T \in L^2(\Omega, \mathcal{F}_T, P)$ and $V_T \in L^2(\Omega, \mathcal{F}_T, Q)$

$$V_t := E^P\left[V_T \frac{H_t}{H_T}\middle|\, \mathcal{F}_t\right] = E^Q\left[V_T \frac{J_t}{J_T}\middle|\, \mathcal{F}_t\right].$$

Assume that P and Q are equivalent and denote the Radon-Nikodým derivative of Q with respect to P by Γ_t. We then have

$$E^Q\left[V_T \frac{J_t}{J_T}\middle|\, \mathcal{F}_t\right] = E^P\left[V_T \frac{H_t}{H_T}\Gamma_t\middle|\, \mathcal{F}_t\right].$$

In particular, $\Gamma_t = \frac{H_T}{H_t} \cdot \frac{J_t}{J_T}$ for $t < T$. Suppose that the process under some measure P associated with numéraire H_t is given by $dX_t = m(X_t, t)dt + \sigma(X_t, t)dW_t$ for some functions $m(X_t, t)$ and $\sigma(X_t, t)$. We are interested on the process followed by X_t under another measure Q with numéraire J_t.

Consider $\Gamma_{t,T} = \frac{H_T}{H_t} \cdot \frac{J_t}{J_T}$. From Girsanov's theorem, if W_t^Q is a Wiener process under Q, $W_t^Q = W_t^P - \int_0^t \theta_u du$ where $d\Gamma_{t,T} = \Gamma_{t,T}\theta_T dW_T^P$ and θ_t can be determined. Moreover, conditional upon $\mathcal{F}_t$, $\Gamma_{t,T}$ is a process in T.

Let H_t and J_t have dynamics under P given by

$$dH_t = m_H dt + \sigma_H dW_t^P \quad \text{and} \quad dJ_t = m_J dt + \sigma_J dW_t^P.$$

Using these dynamics and noting that Γ_t is a martingale under P, it can be verified that

$$\frac{d\Gamma_T}{\Gamma_T} = \left(\frac{\sigma_J}{J_T} - \frac{\sigma_H}{H_T}\right) dW_T^P.$$

So, $\theta_t = \frac{\sigma_J}{J_t} - \frac{\sigma_H}{H_t}$.

Applied to our current situation, suppose P is the EMM under the bank account numéraire and Q is the forward measure with bond as the associated numéraire. For $s < t < T$

$$\Gamma_t := \Gamma_{s,t} = \frac{J_t}{J_s} \cdot \frac{H_s}{H_T} = \frac{B(t,T)}{B(s,T)} exp\left(-\int_s^t r_u du\right).$$

Under measure P, $\frac{dB(t,T)}{B(t,T)} = r_t dt + \sigma_B(t) dW_t$ for some function $\sigma_B(t)$.

It is a straightforward calculation to show that the process $\Gamma_t = \Gamma_{s,t}$, conditional upon $\mathcal{F}_s$, satifies

$$\frac{d\Gamma_t}{\Gamma_t} = \frac{dB(t,T)}{B(t,T)} - r_t dt = \sigma_B(t) dW_t.$$

This implies that $W_t^Q = W_t^P - \int_0^t \sigma_B(u) du$. Hence, if under P we have the dynamics $dX_t = m(X_t,t)dt + \sigma(X_t,t)dW_t^P$ then the $Q-$process for X_t is $dX_t = (m(X_t,t) + \sigma_B(t)\sigma(X_t,t))dt + \sigma(X_t,t)dW_t^Q$.

Equation 23 follows from this result with $X = r$, $Q = P^T$, $\sigma(X_t,t) = \sigma$, $\sigma_B(t) = -A(t,T)\sigma$ and $m(X_t,t) = a(b - r_t)$. □