Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 262854, 17 pages
doi:10.1155/2010/262854

Research Article

Positive and Dead-Core Solutions of Two-Point Singular Boundary Value Problems with ϕ-Laplacian

Svatoslav Staněk

Department of Mathematical Analysis, Faculty of Science, Palacký University, Tř. 17. listopadu 12, 771 46 Olomouc, Czech Republic

Correspondence should be addressed to Svatoslav Staněk, stanek@inf.upol.cz

Received 18 December 2009; Accepted 15 March 2010

Academic Editor: Leonid Berezansky

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The paper discusses the existence of positive solutions, dead-core solutions, and pseudo-dead-core solutions of the singular problem $(\phi(u'))' = \lambda f(t,u,u')$, $u(0)-\alpha u'(0) = A$, $u(T)+\beta u'(0)+\gamma u'(T) = A$. Here λ is a positive parameter, $\alpha > 0$, $A > 0$, $\beta \geq 0$, $\gamma \geq 0$, f is singular at $u = 0$, and f may be singular at $u' = 0$.

1. Introduction

Consider the singular boundary value problem

$$(\phi(u'(t)))' = \lambda f(t, u(t), u'(t)), \quad \lambda > 0, \tag{1.1}$$

$$u(0) - \alpha u'(0) = A, \quad u(T) + \beta u'(0) + \gamma u'(T) = A, \quad \alpha, A > 0, \ \beta, \gamma \geq 0, \tag{1.2}$$

depending on the parameter λ. Here $\phi \in C(\mathbb{R})$, f satisfies the Carathéodory conditions on $[0,T]\times\mathfrak{D}$, $\mathfrak{D} = (0,(1+\beta/\alpha)A]\times(\mathbb{R}\setminus\{0\})$ ($f \in \mathrm{Car}([0,T]\times\mathfrak{D})$), f is positive, $\lim_{x\to 0+} f(t,x,y) = \infty$ for a.e. $t \in [0,T]$ and each $y \in \mathbb{R}\setminus\{0\}$, and f may be singular at $y = 0$.

Throughout the paper $\mathrm{AC}[0,T]$ denotes the set of absolutely continuous functions on $[0,T]$ and $\|x\| = \max\{|x(t)| : t \in [0,T]\}$ is the norm in $C[0,T]$.

We investigate positive, dead-core, and pseudo-dead-core solutions of problem (1.1), (1.2).

A function $u \in C^1[0,T]$ is *a positive solution of problem* (1.1), (1.2) if $\phi(u') \in \mathrm{AC}[0,T]$, $u > 0$ on $[0,T]$, u satisfies (1.2), and (1.1) holds for a.e. $t \in [0,T]$.

We say that $u \in C^1[0,T]$ satisfying (1.2) is *a dead-core solution of problem* (1.1), (1.2) if there exist $0 < t_1 < t_2 < T$ such that $u = 0$ on $[t_1, t_2]$, $u > 0$ on $[0,T] \setminus [t_1, t_2]$, $\phi(u') \in \mathrm{AC}[0,T]$ and (1.1) holds for a.e. $t \in [0,T] \setminus [t_1, t_2]$. The interval $[t_1, t_2]$ is called the *dead-core of* u. If $t_1 = t_2$, then u is called *a pseudo-dead-core solution of problem* (1.1), (1.2).

The existence of positive and dead core solutions of singular second-order differential equations with a parameter was discussed for Dirichlet boundary conditions in [1, 2] and for mixed and Robin boundary conditions in [3–5]. Papers [6, 7] discuss also the existence and multiplicity of positive and dead core solutions of the singular differential equation $u'' = \lambda g(u)$ satisfying the boundary conditions $u'(0) = 0$, $\beta u'(1) + \alpha u(1) = A$ and $u(0) = 1$, $u(1) = 1$, respectively, and present numerical solutions. These problems are mathematical models for steady-state diffusion and reactions of several chemical species (see, e.g., [4, 5, 8, 9]). Positive and dead-core solutions to the third-order singular differential equation

$$\left(\phi(u'')\right)' = \lambda f\left(t, u, u', u''\right), \quad \lambda > 0, \tag{1.3}$$

satisfying the nonlocal boundary conditions $u(0) = u(T) = A$, $\min\{u(t) : t \in [0,T]\} = 0$, were investigated in [10].

We work with the following conditions on the functions ϕ and f in the differential equation (1.1). Without loss of generality we can assume that $1/n < A$ for each $n \in \mathbb{N}$ (otherwise $\mathbb{N}$ is replaced by $\mathbb{N}' := \{n \in \mathbb{N} : 1/n < A\}$), where A is from (1.2).

(H_1) $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing and odd homeomorphism such that $\phi(\mathbb{R}) = \mathbb{R}$.

(H_2) $f \in \mathrm{Car}([0,T] \times \mathfrak{D})$, where $\mathfrak{D} = (0, (1+\beta/\alpha)A] \times (\mathbb{R} \setminus \{0\})$, and

$$\lim_{x \to 0+} f(t, x, y) = \infty \quad \text{for a.e.} t \in [0,T] \text{ and each } y \in \mathbb{R} \setminus \{0\}. \tag{1.4}$$

(H_3) for a.e. $t \in [0,T]$ and all $(x, y) \in \mathfrak{D}$,

$$\varphi(t) \le f(t, x, y) \le \left(p_1(x) + p_2(x)\right)\left(\omega_1(|y|) + \omega_2(|y|)\right) + \psi(t), \tag{1.5}$$

where $\varphi, \psi \in L^1[0,T]$, $p_1 \in C(0, (1+\beta/\alpha)A] \cap L^1[0, (1+\beta/\alpha)A]$, $\omega_1 \in C(0,\infty)$, $p_2 \in C[0, (1+\beta/\alpha)A]$, and $\omega_2 \in C[0,\infty)$ are positive, p_1, ω_1 are nonincreasing, p_2, ω_2 are nondecreasing, $\omega_2(u) \ge u$ for $u \in [0,\infty)$, and

$$\int_0^\infty \frac{\phi^{-1}(s)}{\omega_2\left(\phi^{-1}(s)\right)} \mathrm{d}s = \infty. \tag{1.6}$$

The aim of this paper is to discuss the existence of positive, dead-core, and pseudo-dead-core solutions of problem (1.1), (1.2). Since problem (1.1), (1.2) is singular we use regularization and sequential techniques.

For this end for $n \in \mathbb{N}$, we define $f_n^* \in \mathrm{Car}([0,T] \times \mathfrak{D}_*)$, where $\mathfrak{D}_* = (0, (1+(\beta/\alpha))A] \times \mathbb{R}$, and $f_n \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$ by the formulas

$$f_n^*(t,x,y) = \begin{cases} f(t,x,y) & \text{for } (x,y) \in \left(0, \left(1+\dfrac{\beta}{\alpha}\right)A\right] \\ & \times\left(\mathbb{R} \setminus \left[-\dfrac{1}{n}, \dfrac{1}{n}\right]\right), \\ \dfrac{n}{2}\left[f\left(t,x,\dfrac{1}{n}\right)\left(y+\dfrac{1}{n}\right)\right. & \text{for } (x,y) \in \left(0, \left(1+\dfrac{\beta}{\alpha}\right)A\right] \\ \left.-f\left(t,x,-\dfrac{1}{n}\right)\left(y-\dfrac{1}{n}\right)\right] & \times\left[-\dfrac{1}{n}, \dfrac{1}{n}\right], \end{cases}$$

$$f_n(t,x,y) = \begin{cases} f_n^*\left(t, \left(1+\dfrac{\beta}{\alpha}\right)A, y\right) & \text{for } (x,y) \in \left(\left(1+\dfrac{\beta}{\alpha}\right)A, \infty\right) \times \mathbb{R}, \\ f_n^*(t,x,y) & \text{for } (x,y) \in \left(\dfrac{1}{n}, \left(1+\dfrac{\beta}{\alpha}\right)A\right] \times \mathbb{R}, \\ \left[\phi\left(\dfrac{1}{n}\right)\right]^{-1} \phi(x) f_n^*\left(t, \dfrac{1}{n}, y\right) & \text{for } (x,y) \in \left[0, \dfrac{1}{n}\right] \times \mathbb{R}, \\ x & \text{for } (x,y) \in (-\infty, 0) \times \mathbb{R}. \end{cases} \tag{1.7}$$

Then (H_2) and (H_3) give

$$\varphi(t) \le f_n(t,x,y) \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in \left[\frac{1}{n}, \infty\right) \times \mathbb{R}, \tag{1.8}$$

$$0 < f_n(t,x,y) \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in (0,\infty) \times \mathbb{R}, \tag{1.9}$$

$$x = f_n(t,x,y) \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in (-\infty, 0] \times \mathbb{R}, \tag{1.10}$$

$$\begin{gathered} f_n(t,x,y) \le (p_1(x) + \tilde{p}_2(x))(\omega_1(|y|) + \tilde{\omega}_2(|y|)) + \varphi(t) \\ \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in \left(0, \left(1+\frac{\beta}{\alpha}\right)A\right] \times (\mathbb{R} \setminus \{0\}), \text{ where} \\ \tilde{p}_2(x) = \max\{p_2(x), p_2(1)\}, \qquad \tilde{\omega}_2(|y|) = \max\{\omega_2(|y|), \omega_2(1)\}. \end{gathered} \tag{1.11}$$

Consider the auxiliary regular differential equation

$$(\phi(u'(t)))' = \lambda f_n(t, u(t), u'(t)), \quad \lambda > 0. \tag{1.12}$$

A function $u \in C^1[0,T]$ is a *solution of problem* (1.12), (1.2) if $\phi(u') \in AC[0,T]$, u fulfils (1.2), and (1.12) holds for a.e. $t \in [0,T]$.

We introduce also the notion of a sequential solution of problem (1.1), (1.2). We say that $u \in C^1[0,T]$ is *a sequential solution of problem* (1.1), (1.2) if there exists a sequence $\{k_n\} \subset \mathbb{N}$, $\lim_{n\to\infty} k_n = \infty$, such that $u = \lim_{n\to\infty} u_{k_n}$ in $C^1[0,T]$, where u_{k_n} is a solution of problem

(1.12), (1.2) with n replaced by k_n. In Section 3 (see Theorem 3.1) we show that any sequential solution of problem (1.1), (1.2) is either a positive solution or a pseudo-dead-core solution or a dead-core solution of this problem.

The next part of our paper is divided into two sections. Section 2 is devoted to the auxiliary regular problem (1.12), (1.2). We prove the solvability of this problem by the existence principle in [11] and investigate the properties of solutions. The main results are given in Section 3. We prove that under assumptions (H_1)–(H_3), for each $\lambda > 0$, problem (1.1), (1.2) has a sequential solution and that any sequential solution is either a positive solution or a pseudo-dead-core solution or a dead-core solution (Theorem 3.1). Theorem 3.2 shows that for sufficiently small values of λ all sequential solutions of problem (1.1), (1.2) are positive solutions while, by Theorem 3.3, all sequential solutions are dead-core solutions if λ is sufficiently large. An example demonstrates the application of our results.

2. Auxiliary Regular Problems

The properties of solutions of problem (1.12), (1.2) are given in the following lemma.

Lemma 2.1. *Let (H_1)–(H_3) hold. Let u_n be a solution of problem* (1.12), (1.2). *Then*

$$0 < u_n(t) \le \left(1 + \frac{\beta}{\alpha}\right) A \quad \textit{for } t \in [0,T], \tag{2.1}$$

$$u_n(0) < A, \quad u_n(T) < \left(1 + \frac{\beta}{\alpha}\right) A, \tag{2.2}$$

$$u_n' \textit{ is increasing on } [0,T] \textit{ and } u_n'(\gamma_n) = 0 \textit{ for a } \gamma_n \in (0,T). \tag{2.3}$$

Proof. Suppose that $u_n'(0) \ge 0$. Then $u_n(0) = A + \alpha u_n'(0) \ge A > 0$. Let

$$\tau = \sup\{t \in (0,T] : u(s) > 0 \text{ for } s \in [0,t]\}. \tag{2.4}$$

Then $\tau \in (0,T]$ and, by (1.9), $(\phi(u_n'))' > 0$ a.e. on $[0,\tau]$. Hence $\phi(u_n')$ is increasing on $[0,\tau]$, and therefore, u_n' is also increasing on this interval since ϕ is increasing on $\mathbb{R}$ by (H_1). Consequently, $\tau = T$ and $u_n' > 0$ on $(0,T]$. Then $u(T) > u(0)$, which contradicts $u_n(0) - u_n(T) = (\alpha + \beta)u_n'(0) + \gamma u_n'(T) \ge 0$. Hence $u_n'(0) < 0$. Let $u_n(0) \le 0$. Then $u_n < 0$ on a right neighbourhood of $t = 0$. Put

$$\nu = \sup\{t \in (0,T] : u_n(s) < 0 \text{ for } s \in (0,t]\}. \tag{2.5}$$

Then $u_n < 0$ on $(0,\nu)$, and therefore, $(\phi(u'_n))' = \lambda u_n < 0$ a.e. on $[0,\nu]$, which implies that u_n' is decreasing on $[0,\nu]$. Now it follows from $u_n(0) \le 0$ and $u'_n(0) < 0$ that $\nu = T$, $u_n < 0$ on $(0,T]$ and $u_n' < 0$ on $[0,T]$. Consequently, $u_n(0) > u_n(T)$, which contradicts $u_n(0) - u_n(T) = (\alpha+\beta)u_n'(0) + \gamma u_n'(T) < 0$. To summarize, $u_n(0) > 0$ and $u_n'(0) < 0$. Suppose that $\min\{u_n(t) : t \in [0,T]\} < 0$. Then there exist $0 < a < b \le T$ such that $u_n(a) = 0$, $u_n'(a) \le 0$ and $u_n < 0$ on (a,b). Hence $(\phi(u_n'))' = \lambda u_n < 0$ a.e. on $[a,b]$ and arguing as in the above part of the proof we can verify that $b = T$ and $u_n < 0$, $u_n' < 0$ on $(a,T]$. Consequently, $u_n(T) = A - \beta u_n'(0) - \gamma u_n'(T) \ge A$, which is impossible. Hence $u_n \ge 0$ on $[0,T]$. New it follows from (1.9) and (1.10) that

$(\phi(u_n'))' \geq 0$ a.e. on $[0,T]$, which together with (H_1) gives that u_n' is nondecreasing on $[0,T]$. Suppose that $u_n(\xi) = 0$ for some $\xi \in (0,T]$. If $\xi = T$, then $u_n'(T) \leq 0$, which contradicts $\beta u_n'(0) + \gamma u_n'(T) = A$ since $u_n'(0) < 0$. Hence $\xi \in (0,T)$ and $u_n'(\xi) = 0$. Let

$$\eta = \min\{t \in [0,T] : u_n(t) = 0\}. \tag{2.6}$$

Then $0 < \eta \leq \xi < T$, $u_n'(\eta) = 0$ and u_n' is increasing on $[0,\eta]$ since $(\phi(u'))' > 0$ a.e. on this interval by (1.9). Hence there exists $t_1 \in (0,\eta)$, $\eta - t_1 \leq 1$, such that $0 < u_n < 1/n$ on (t_1,η) and it follows from the definition of the function f_n that

$$\left(\phi\left(u_n'(t)\right)\right)' = Q\phi(u_n(t))p(t) \quad \text{for a.e. } t \in \left[t_1,\eta\right], \tag{2.7}$$

where $Q = \lambda[\phi(1/n)]^{-1}$, $p(t) = f_n^*(t,1/n,u_n'(t)) \in L^1[t_1,\eta]$, and $p > 0$ a.e. on $[t_1,\eta]$. Integrating (2.7) over $[t,\eta] \subset [t_1,\eta]$ yields

$$\phi\left(-u_n'(t)\right) = -\phi\left(u_n'(t)\right) = Q\int_t^\eta \phi(u_n(s))p(s)\mathrm{d}s, \quad t \in \left[t_1,\eta\right]. \tag{2.8}$$

From this equality, from (H_1) and from $u_n(t) = u_n(t) - u_n(\eta) = u_n'(\mu)(t-\eta) \leq u_n'(t)(t-\eta)$, where $\mu \in [t,\eta]$, we obtain

$$\begin{aligned}\phi\left(-u_n'(t)\right) &\leq Q\phi(u_n(t))\int_t^\eta p(s)\mathrm{d}s \leq Q\phi\left(-u_n'(t)(\eta-t)\right)\int_t^\eta p(s)\mathrm{d}s \\ &\leq Q\phi\left(-u_n'(t)\right)\int_t^\eta p(s)\mathrm{d}s\end{aligned} \tag{2.9}$$

for $t \in [t_1,\eta]$. Since $\phi(-u_n'(t)) > 0$ for $t \in [t_1,\eta)$, we have

$$1 \leq Q\int_t^\eta p(s)\mathrm{d}s \quad \text{for } t \in \left[t_1,\eta\right), \tag{2.10}$$

which is impossible. We have proved that

$$u_n(t) > 0 \quad \text{for } t \in [0,T]. \tag{2.11}$$

Hence $(\phi(u_n'))' > 0$ a.e. on $[0,T]$ by (1.9), and therefore, u_n' is increasing on $[0,T]$. If $u_n'(T) \leq 0$, then $u_n' < 0$ on $[0,T)$, and so $u_n(0) > u_n(T)$, which is impossible since $u_n(0) - u_n(T) = (\alpha + \beta)u_n'(0) + \gamma u_n'(T) \leq \alpha u_n'(0) < 0$. Consequently, $u_n'(T) > 0$ and u_n' vanishes at a unique point $\gamma_n \in (0,T)$. Hence (2.3) is true.

Next, we deduce from $u_n(0) > 0$, $u_n'(0) < 0$ and from $u_n(0) = A + \alpha u_n'(0)$ that $u_n(0) < A$ and $u_n'(0) > -(A/\alpha)$. Consequently, $u_n(T) = A - \beta u_n'(0) - \gamma u_n'(T) \leq A - \beta u_n'(0) < (1+\beta/\alpha)A$. Hence (2.2) holds. Inequality (2.1) follows from (2.2), (2.3), and (2.11). □

Remark 2.2. Let u be a solution of problem (1.12), (1.2) with $\lambda = 0$. Then $(\phi(u'))' = 0$ a.e. on $[0, T]$, and so u' is a constant function. Let $u(t) = a + bt$. Now, it follows from (1.2) that $A = a - \alpha b$ and $A = a + bT + (\beta + \gamma)b$. Consequently, $(\alpha + \beta + \gamma)b = -bT$, and since $\alpha + \beta + \gamma > 0$, we have $b = 0$. Hence $A = a$, and $u = A$ is the unique solution of problem (1.12), (1.2) for $\lambda = 0$.

The following lemma gives a priori bounds for solutions of problem (1.12), (1.2).

Lemma 2.3. *Let* (H_1)*–*(H_3) *hold. Then there exists a positive constant* S *independent of* n(*and depending on* λ) *such that*

$$\|u_n'\| < S \tag{2.12}$$

for any solution u_n *of problem* (1.12), (1.2).

Proof. Let u_n be a solution of problem (1.12), (1.2). By Lemma 2.1, u_n satisfies (2.1)–(2.3). Hence

$$\|u_n'\| = \max\{|u_n'(0)|, u_n'(T)\}. \tag{2.13}$$

In view of (1.11),

$$(\phi(u_n'(t)))' u_n'(t) \geq \lambda\left[(p_1(u_n(t)) + \tilde{p}_2(u_n(t)))(\omega_1(-u_n'(t)) + \tilde{\omega}_2(-u_n'(t))) + \psi(t)\right] u_n'(t) \tag{2.14}$$

for a.e. $t \in [0, \gamma_n]$ and

$$(\phi(u'_n(t)))' u_n'(t) \leq \lambda\left[(p_1(u_n(t)) + \tilde{p}_2(u_n(t)))(\omega_1(u_n'(t)) + \tilde{\omega}_2(u_n'(t))) + \psi(t)\right] u_n'(t) \tag{2.15}$$

for a.e. $t \in [\gamma_n, T]$. Since $\tilde{\omega}_2(u) \geq u$ for $u \in [0, \infty)$ by (H_3), we have

$$\begin{aligned} \frac{u_n'(t)}{\omega_1(-u_n'(t)) + \tilde{\omega}_2(-u'_n(t))} &\geq -1 \quad \text{for } t \in [0, \gamma_n), \\ \frac{u_n'(t)}{\omega_1(u_n'(t)) + \tilde{\omega}_2(u_n'(t))} &\leq 1 \quad \text{for } t \in (\gamma_n, T]. \end{aligned} \tag{2.16}$$

Therefore,

$$\frac{(\phi(u_n'(t)))' u_n'(t)}{\omega_1(-u_n'(t)) + \tilde{\omega}_2(-u_n'(t))} \geq \lambda\left[(p_1(u_n(t)) + \tilde{p}_2(u_n(t))) u_n'(t) - \psi(t)\right] \tag{2.17}$$

for a.e. $t \in [0, \gamma_n]$ and

$$\frac{(\phi(u_n'(t)))' u_n'(t)}{\omega_1(u_n'(t)) + \tilde{\omega}_2(u_n'(t))} \leq \lambda\left[(p_1(u_n(t)) + \tilde{p}_2(u_n(t))) u_n'(t) + \psi(t)\right] \tag{2.18}$$

for a.e. $t \in [\gamma_n, T]$. Integrating (2.17) over $[0, \gamma_n]$ and (2.18) over $[\gamma_n, T]$ gives

$$\begin{aligned}\int_0^{\phi(|u_n'(0)|)} \frac{\phi^{-1}(s)}{\omega_1(\phi^{-1}(s)) + \tilde{\omega}_2(\phi^{-1}(s))}\mathrm{d}s &\le \lambda\left(\int_{u_n(\gamma_n)}^{u_n(0)} (p_1(s) + \tilde{p}_2(s))\mathrm{d}s + \int_0^{\gamma_n} \varphi(t)\mathrm{d}t\right) \\ &< \lambda\left(\int_0^{A} (p_1(s) + \tilde{p}_2(s))\mathrm{d}s + \int_0^{T} \varphi(t)\mathrm{d}t\right),\end{aligned} \tag{2.19}$$

$$\begin{aligned}\int_0^{\phi(u_n'(T))} \frac{\phi^{-1}(s)}{\omega_1(\phi^{-1}(s)) + \tilde{\omega}_2(\phi^{-1}(s))}\mathrm{d}s &\le \lambda\left(\int_{u_n(\gamma_n)}^{u_n(T)} (p_1(s) + \tilde{p}_2(s))\mathrm{d}s + \int_{\gamma_n}^{T} \varphi(t)\mathrm{d}t\right) \\ &< \lambda\left(\int_0^{(1+\beta/\alpha)A} (p_1(s) + \tilde{p}_2(s))\mathrm{d}s + \int_0^{T} \varphi(t)\mathrm{d}t\right),\end{aligned} \tag{2.20}$$

respectively. We now show that condition (1.6) implies

$$\int_0^{\infty} \frac{\phi^{-1}(s)}{\omega_1(\phi^{-1}(s)) + \tilde{\omega}_2(\phi^{-1}(s))}\mathrm{d}s = \infty. \tag{2.21}$$

Since $\lim_{y\to\infty}\tilde{\omega}_2(y) = \infty$ by (H_3), we have $\lim_{y\to\infty}(\omega_1(y) + \tilde{\omega}_2(y))/\tilde{\omega}_2(y) = 1$. Therefore, there exists $y_* \in (\phi(1), \infty)$ such that

$$\omega_1\left(\phi^{-1}(y)\right) + \tilde{\omega}_2\left(\phi^{-1}(y)\right) \le 2\tilde{\omega}_2\left(\phi^{-1}(y)\right) = 2\omega_2\left(\phi^{-1}(y)\right) \quad \text{for } y \in [y_*, \infty). \tag{2.22}$$

Then

$$\begin{aligned}\int_0^{\infty} \frac{\phi^{-1}(s)}{\omega_1(\phi^{-1}(s)) + \tilde{\omega}_2(\phi^{-1}(s))}\mathrm{d}s &> \int_{y_*}^{\infty} \frac{\phi^{-1}(s)}{\omega_1(\phi^{-1}(s)) + \tilde{\omega}_2(\phi^{-1}(s))}\mathrm{d}s \\ &\ge \frac{1}{2}\int_{y_*}^{\infty} \frac{\phi^{-1}(s)}{\omega_2(\phi^{-1}(s))}\mathrm{d}s,\end{aligned} \tag{2.23}$$

and (2.21) follows from (1.6). Since $\int_0^{(1+\beta/\alpha)A}(p_1(t) + \tilde{p}_2(t))\mathrm{d}t < \infty$, inequality (2.21) guarantees the existence of a positive constant M such that

$$\int_0^{y} \frac{\phi^{-1}(s)}{\omega_1(\phi^{-1}(s)) + \tilde{\omega}_2(\phi^{-1}(s))}\mathrm{d}s \ge \lambda\left(\int_0^{(1+\beta/\alpha)A} (p_1(s) + \tilde{p}_2(s))\mathrm{d}s + \int_0^{T} \varphi(t)\mathrm{d}t\right) \tag{2.24}$$

for all $y \ge M$. Hence (2.19) and (2.20) imply $\max\{\phi(|u_n'(0)|), \phi(u_n'(T))\} < M$. Consequently, $\max\{|u_n'(0)|, u_n'(T)\} < \phi^{-1}(M)$ and equality (2.13) shows that (2.12) is true for $S = \phi^{-1}(M)$. □

Remark 2.4. By Lemma 2.3, estimate (2.12) is true for any solution u_n of problem (1.12), (1.2), where S is a positive constant independent of n and depending on λ. Fix $\lambda > 0$ and consider the differential equation

$$(\phi(u'))' = \mu\lambda f_n(t,u,u'), \quad \mu \in [0,1]. \tag{2.25}$$

It follows from the proof of Lemma 2.3 that $\|u'\| < S$ for each $\mu \in (0,1]$ and any solution u of problem (2.25), (1.2). Since $u = A$ is the unique solution of this problem with $\mu = 0$ by Remark 2.2, we have $\|u\| < S$ for each $\mu \in [0,1]$ and any solution u of problem (2.25), (1.2).

We are now in the position to show that problem (1.12), (1.2) has a solution. Let $\chi_j : C^1[0,T] \to \mathbb{R}$, $j = 1,2$, be defined by

$$\chi_1(x) = x(0) - \alpha x'(0) - A, \qquad \chi_2(x) = x(T) + \beta x'(0) + \gamma u'(T) - A, \tag{2.26}$$

where α, β, γ, and A are as in (1.2). We say that the functionals χ_1 and χ_2 are *compatible* if for each $\rho \in [0,1]$ the system

$$\chi_j(a+bt) - \rho\chi_j(-a-bt) = 0, \quad j = 1,2, \tag{2.27}$$

has a solution $(a,b) \in \mathbb{R}^2$. We apply the following existence principle which follows from [11–13] to prove the solvability of problem (1.12), (1.2).

Proposition 2.5. *Let* (H_1)*–*(H_3) *hold. Let there exist positive constants* S_0, S_1 *such that*

$$\|u\| < S_0, \qquad \|u'\| < S_1 \tag{2.28}$$

for each $\mu \in [0,1]$ *and any solution* u *of problem* (2.25), (1.2)*. Also assume that* χ_1 *and* χ_2 *are compatible and there exist positive constants* Λ_0, Λ_1 *such that*

$$|a| < \Lambda_0, \qquad |b| < \Lambda_1 \tag{2.29}$$

for each $\rho \in [0,1]$ *and each solution* $(a,b) \in \mathbb{R}^2$ *of system* (2.27).

Then problem (1.12), (1.2) *has a solution.*

Lemma 2.6. *Let* (H_1)*–*(H_3) *hold. Then problem* (1.12), (1.2) *has a solution.*

Proof. By Lemmas 2.1 and 2.3 and Remark 2.4, there exists a positive constant S such that

$$0 < u(t) \le \left(1 + \frac{\beta}{\alpha}\right)A \quad \text{for } t \in [0,T], \ \|u'\| < S \tag{2.30}$$

for each $\mu \in [0,1]$ and any solution u of problem (2.25), (1.2). Hence (2.28) is true for $S_0 = (1+\beta/\alpha)A$ and $S_1 = S$. System (2.27) has the form of

$$(1+\rho)(a-\alpha b) = (1-\rho)A, \quad (1+\rho)(a+bT+\beta b+\gamma b) = (1-\rho)A. \tag{2.31}$$

Subtracting the first equation from the second, we get $(1+\rho)(T+\alpha+\beta+\gamma)b = 0$. Due to $(1+\rho)(T+\alpha+\beta+\gamma) > 0$ for $\rho \in [0,1]$, we have $b = 0$, and consequently, $a = (1-\rho)A/(1+\rho)$. Hence $(a,b) = ((1-\rho)A/(1+\rho), 0)$ is the unique solution of system (2.31). Therefore, χ_1 and χ_2 are compatible and (2.29) is fulfilled for $\Lambda_0 = A+1$ and $\Lambda_1 = 1$. The result now follows from Proposition 2.5. □

The following result deals with the sequences of solutions of problem (1.12), (1.2).

Lemma 2.7. *Let* (H_1)*–*(H_3) *hold and let* u_n *be a solution of problem* (1.12), (1.2). *Then* $\{u_n'\}$ *is equicontinuous on* $[0,T]$.

Proof. By Lemmas 2.1 and 2.3, relations (2.1)–(2.3) and (2.12) hold, where S is a positive constant. Let $H \in C[0,\infty)$, $H^* \in C(\mathbb{R})$, and $P \in AC[0,(1+\beta/\alpha)A]$ be defined by the formulas

$$H(v) = \int_0^{\phi(v)} \frac{\phi^{-1}(v)}{\omega_1(\phi^{-1}(s)) + \tilde{\omega}_2(\phi^{-1}(s))} ds \quad \text{for } v \in [0,\infty),$$
$$H^*(v) = \begin{cases} H(v) & \text{for } v \in [0,\infty), \\ -H(-v) & \text{for } v \in (-\infty,0), \end{cases} \tag{2.32}$$
$$P(v) = \int_0^{v} (p_1(s) + \tilde{p}_2(s)) ds \quad \text{for } v \in \left[0, \left(1+\frac{\beta}{\alpha}\right)A\right],$$

where $\tilde{p}_2$ and $\tilde{\omega}_2$ are given in (1.11). Then H^* is an increasing and odd function on $\mathbb{R}$, $H^*(\mathbb{R}) = \mathbb{R}$ by (2.21), and P is increasing on $[0,(1+(\beta/\alpha))A]$. Since $\{u_n'\}$ is bounded in $C[0,T]$, $\{u_n\}$ is equicontinuous on $[0,T]$, and consequently, $\{P(u_n)\}$ is equicontinuous on $[0,T]$, too. Let us choose an arbitrary $\varepsilon > 0$. Then there exists $\rho > 0$ such that

$$|P(u_n(t_1)) - P(u_n(t_2))| < \varepsilon, \quad \left|\int_{t_1}^{t_2} \psi(t) dt\right| < \varepsilon \quad \text{for } t_1, t_2 \in [0,T],\ |t_1 - t_2| < \rho,\ n \in \mathbb{N}. \tag{2.33}$$

In order to prove that $\{u_n'\}$ is equicontinuous on $[0,T]$, let $0 \le t_1 < t_2 \le T$ and $t_2 - t_1 < \rho$. If $t_2 \le \gamma_n$, then integrating (2.17) from t_1 to t_2 gives

$$0 < H^*(u_n'(t_2)) - H^*(u_n'(t_1)) \le \lambda\left(P(u_n(t_1)) - P(u_n(t_2)) + \int_{t_1}^{t_2} \psi(t) dt\right) < 2\lambda\varepsilon. \tag{2.34}$$

If $t_1 \ge \gamma_n$, then integrating (2.18) over $[t_1,t_2]$ yields

$$0 < H^*(u'_n(t_2)) - H^*(u'_n(t_1)) \le \lambda\left(P(u_n(t_2)) - P(u_n(t_1)) + \int_{t_1}^{t_2} \psi(t) dt\right) < 2\lambda\varepsilon. \tag{2.35}$$

Finally, if $t_1 < \gamma_n < t_2$, then one can check that

$$0 < H^*(u_n'(t_2)) - H^*(u_n'(t_1)) < 3\lambda\varepsilon. \tag{2.36}$$

To summarize, we have

$$0 \le H^*(u_n'(t_2)) - H^*(u_n'(t_1)) < 3\lambda\varepsilon, \quad n \in \mathbb{N}, \tag{2.37}$$

whenever $0 \le t_1 < t_2 \le T$ and $t_2 - t_1 < \rho$. Hence $\{H^*(u_n')\}$ is equicontinuous on $[0,T]$ and, since $\{u_n'\}$ is bounded in $C[0,T]$ and H^* is continuous and increasing on $\mathbb{R}$, $\{u_n'\}$ is equicontinuous on $[0,T]$. □

The results of the following two lemmas we use in the proofs of the existence of positive and dead-core solutions to problem (1.1), (1.2).

Lemma 2.8. *Let* (H_1)*–*(H_3) *hold. Then there exist* $\lambda_* > 0$ *and* $\varepsilon > 0$ *such that*

$$u_n(t) > \varepsilon \quad \text{for } t \in [0,T], \ n \in \mathbb{N}, \tag{2.38}$$

where u_n *is any solution of problem* (1.12), (1.2) *with* $\lambda \in (0, \lambda_*]$.

Proof. Suppose that the lemma was false. Then we could find sequences $\{k_m\} \subset \mathbb{N}$ and $\{\lambda_m\} \subset (0,\infty)$, $\lim_{m\to\infty}\lambda_m = 0$, and a solution u_m of the equation $(\phi(u'))' = \lambda_m f_{k_m}(t,u,u')$ satisfying (1.2) such that $\lim_{m\to\infty} u_m(\xi_m) = 0$, where $u_m(\xi_m) = \min\{u_m(t) : t \in [0,T]\}$. Note that $u_m > 0$ on $[0,T]$, $u_m' < 0$ on $[0,\xi_m)$, $u_m'(\xi_m) = 0$, and $u_m' > 0$ on $(\xi_m, T]$ for each $m \in \mathbb{N}$ by Lemma 2.1. Then, by (1.11),

$$(\phi(u_m'(t)))' \le \lambda_m\left[(p_1(u_m(t)) + \tilde{p}_2(u_m(t)))(\omega_1(-u_m'(t)) + \tilde{\omega}_2(-u_m'(t))) + \varphi(t)\right] \tag{2.39}$$

for a.e. $t \in [0, \xi_m]$,

$$(\phi(u_m'(t)))' \le \lambda_m\left[(p_1(u_m(t)) + \tilde{p}_2(u_m(t)))(\omega_1(u_m'(t)) + \tilde{\omega}_2(u_m'(t))) + \varphi(t)\right] \tag{2.40}$$

for a.e. $t \in [\xi_m, T]$, and (cf. (2.13))

$$\|u_m'\| = \max\{|u_m'(0)|, u_m'(T)\}. \tag{2.41}$$

Essentially, the same reasoning as in the proof of Lemma 2.3 gives that for $m \in \mathbb{N}$ (cf. (2.19) and (2.20))

$$\begin{aligned} \int_0^{\phi(|u_m'(0)|)} \frac{\phi^{-1}(s)}{\omega_1(\phi^{-1}(s)) + \tilde{\omega}_2(\phi^{-1}(s))} \mathrm{d}s &< \lambda_m \left(\int_0^A (p_1(s) + \tilde{p}_2(s)) \mathrm{d}s + \int_0^T \varphi(t) \mathrm{d}t \right), \\ \int_0^{\phi(u_m'(T))} \frac{\phi^{-1}(s)}{\omega_1(\phi^{-1}(s)) + \tilde{\omega}_2(\phi^{-1}(s))} \mathrm{d}s &< \lambda_m \left(\int_0^{(1+\beta/\alpha)A} (p_1(s) + \tilde{p}_2(s)) \mathrm{d}s + \int_0^T \varphi(t) \mathrm{d}t \right). \end{aligned} \tag{2.42}$$

In view of $\lim_{m\to\infty}\lambda_m = 0$, we have $\lim_{m\to\infty} u_m'(0) = 0$, $\lim_{m\to\infty} u_m'(T) = 0$. Consequently, $\lim_{m\to\infty}\|u_m'\| = 0$ by (2.41). We now deduce from $u_m(t) = u_m(\xi_m) + \int_{\xi_m}^t u_m'(t)\, \mathrm{d}t$ for $t \in [0,T]$

and $m \in \mathbb{N}$, and from $\lim_{m\to\infty} u_m(\xi_m) = 0$ that $\lim_{m\to\infty}\|u_m\| = 0$. Hence $\lim_{m\to\infty}(u_m(0) - \alpha u_m'(0)) = 0$, $\lim_{m\to\infty}(u_m(T) + \beta u_m'(0) + \gamma u_m'(T)) = 0$, which contradicts $u_m(0) - \alpha u_m'(0) = A$, $u_m(T) + \beta u_m'(0) + \gamma u_m'(T) = A$ for $m \in \mathbb{N}$. □

Lemma 2.9. *Let* (H_1)*–*(H_3) *hold. Then for each* $c \in (0, T)$ *there exists* $\lambda_c > 0$ *such that*

$$\lim_{n\to\infty} u_n(c) = 0, \tag{2.43}$$

where u_n *is any solution of problem* (1.12), (1.2) *with* $\lambda > \lambda_c$.

Proof. Fix $c \in (0, T)$ and let φ be as in (H_3). Put $\rho = \min\{c, T - c\}$,

$$\Lambda = \min\left\{\int_{c/2}^{c} \varphi(t)\mathrm{d}t, \int_{c}^{(T+c)/2} \varphi(t)\mathrm{d}t\right\} > 0, \quad \lambda_c = \frac{1}{\Lambda}\phi\left(\frac{2(\alpha+\beta)A}{\alpha\rho}\right). \tag{2.44}$$

Let $\lambda \in (\lambda_c, \infty)$ and choose $\varepsilon \in (0, \rho)$. If we prove that

$$u_n(c) < \varepsilon \quad \forall n > \frac{1}{\varepsilon}, \tag{2.45}$$

where u_n is any solution of problem (1.12), (1.2), then (2.43) is true since $u_n > 0$ by Lemma 2.1. In order to prove (2.45), suppose the contrary, that is suppose that there is some $n_0 > 1/\varepsilon$ such that $u_{n_0}(c) \geq \varepsilon$. The next part of the proof is broken into two cases if $u_{n_0}'(c) \leq 0$ or $u_{n_0}'(c) > 0$.

Case 1. Suppose $u_{n_0}'(c) \leq 0$. By Lemma 2.1, u_{n_0}' is increasing on $[0, T]$. Consequently, if $u_{n_0}'(c/2) < -2A/c$, then $u_{n_0}'(t) < -2A/c$ for $t \in [0, c/2]$, and so

$$u_{n_0}(0) = u_{n_0}\left(\frac{c}{2}\right) - \int_0^{c/2} u_{n_0}'(t)\mathrm{d}t > u_{n_0}\left(\frac{c}{2}\right) + A > A, \tag{2.46}$$

which contradicts $u_{n_0}(0) < A$ by Lemma 2.1. Therefore,

$$u_{n_0}'\left(\frac{c}{2}\right) \geq -\frac{2A}{c}, \qquad 0 \geq u_{n_0}'(t) \geq -\frac{2A}{c} \quad \text{for } t \in \left[\frac{c}{2}, c\right]. \tag{2.47}$$

Keeping in mind that $n_0 u_{n_0}(t) \geq n_0\varepsilon > 1$ for $t \in [0, c]$, we have, by (1.8),

$$f_{n_0}(t, u_{n_0}(t), u_{n_0}'(t)) \geq \varphi(t) \quad \text{for a.e. } t \in [0, c], \tag{2.48}$$

and therefore,

$$(\phi(u_{n_0}'(t)))' \geq \lambda\varphi(t) > \lambda_c\varphi(t) \quad \text{for a.e. } t \in [0, c]. \tag{2.49}$$

Then

$$\phi(u'_{n_0}(c)) - \phi\left(u'_{n_0}\left(\frac{c}{2}\right)\right) > \lambda_c \int_{c/2}^{c} \varphi(t)\mathrm{d}t \geq \lambda_c \Lambda, \tag{2.50}$$

which yields

$$\begin{aligned} \phi\left(-u'_{n_0}\left(\frac{c}{2}\right)\right) &= -\phi\left(u'_{n_0}\left(\frac{c}{2}\right)\right) > -\phi(u'_{n_0}(c)) + \lambda_c \Lambda \\ &\geq \lambda_v \Lambda = \phi\left(\frac{2(\alpha+\beta)A}{\alpha\rho}\right) \geq \phi\left(\frac{2A}{c}\right). \end{aligned} \tag{2.51}$$

Hence $-u'_{n_0}(c/2) > 2A/c$, which contradicts the first inequality in (2.47).

Case 2. Suppose $u'_{n_0}(c) > 0$. Then u'_{n_0} is positive and increasing on $[c,T]$ by Lemma 2.1. If $u'_{n_0}((T+c)/2) \geq 2(\alpha+\beta)A/\alpha(T-c)$, then $u'_{n_0} > 2(\alpha+\beta)A/\alpha(T-c)$ on $((T+c)/2, T]$, and consequently,

$$u_{n_0}(T) = u_{n_0}\left(\frac{T+c}{2}\right) + \int_{(T+c)/2}^{T} u'_{n_0}(t)\mathrm{d}t > u_{n_0}\left(\frac{T+c}{2}\right) + \left(1+\frac{\beta}{\alpha}\right)A > \left(1+\frac{\beta}{\alpha}\right)A, \tag{2.52}$$

which contradicts $u_{n_0}(T) \leq (1+\beta/\alpha)A$ by Lemma 2.1. Hence

$$0 < u'_{n_0}(t) < \frac{2(\alpha+\beta)A}{\alpha(T-c)} \quad \text{for } t \in \left[c, \frac{T+c}{2}\right]. \tag{2.53}$$

Since $n_0 u_{n_0}(t) \geq n_0\varepsilon > 1$ for $t \in [c,T]$, the inequality in (2.48) holds a.e. on $[c,T]$, and therefore, the inequality in (2.49) is true for a.e. $t \in [c,T]$. Integrating $(\phi(u'_{n_0}(t)))' > \lambda_c \varphi(t)$ over $[c,(T+c)/2]$ gives

$$\phi\left(u'_{n_0}\left(\frac{T+c}{2}\right)\right) > \phi(u'_{n_0}(c)) + \lambda_c \int_{c}^{(T+c)/2} \varphi(t)\mathrm{d}t. \tag{2.54}$$

Then

$$\phi\left(u'_{n_0}\left(\frac{T+c}{2}\right)\right) > \lambda_c \int_{c}^{(T+c)/2} \varphi(t)\mathrm{d}t \geq \lambda_c \Lambda \geq \phi\left(\frac{2(\alpha+\beta)A}{\alpha(T-c)}\right). \tag{2.55}$$

Hence $u'_{n_0}((T+c)/2) > 2(\alpha+\beta)A/\alpha(T-c)$, which contradicts (2.53) with $t = (T+c)/2$. □

3. Main Results and an Example

Theorem 3.1. *Suppose there are* (H_1)*–*(H_3)*, then the following assertions hold.*

(i) *For each* $\lambda > 0$ *problem* (1.1), (1.2) *has a sequential solution.*

(ii) *Any sequential solution of problem* (1.1), (1.2) *is either a positive solution, a pseudo-dead-core solution, or a dead-core solution.*

Proof. (i) Fix $\lambda > 0$. By Lemma 2.6, for each $n \in \mathbb{N}$ problem (1.12), (1.2) has a solution u_n. Lemmas 2.1 and 2.7 guarantee that $\{u_n\}$ is bounded in $C^1[0,T]$ and $\{u_n'\}$ is equicontinuous on $[0,T]$. By the Arzelà-Ascoli theorem, there exist $u \in C^1[0,T]$ and a subsequence $\{u_{k_n}\}$ of $\{u_n\}$ such that $u = \lim_{n\to\infty} u_{k_n}$ in $C^1[0,T]$. Hence u is a sequential solution of problem (1.1), (1.2).

(ii) Let u be a sequential solution of problem (1.1), (1.2). Then $u \in C^1[0,T]$ and $u = \lim_{n\to\infty} u_{k_n}$ in $C^1[0,T]$, where u_{k_n} is a solution of problem (1.12), (1.2) with n replaced by k_n. Hence $u(0) - \alpha u'(0) = A$ and $u(T) + \beta u'(0) + \gamma u'(T) = A$, that is, u fulfils the boundary condition (1.2). It follows from the properties of u_{k_n} given in Lemmas 2.1 and 2.3 that $0 \le u(t) \le (1+\beta/\alpha)A$ for $t \in [0,T]$, u' is nondecreasing on $[0,T]$ and $\|u_{k_n}'\| < S$ for $n \in \mathbb{N}$, where S is a positive constant. The next part of the proof is divided into two cases if $\min\{u(t) : t \in [0,T]\}$ is positive, or is equal to zero.

Case 1. Suppose that $\min\{u(t) : t \in [0,T]\} > 0$. Then there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, $n_0 > 1/\varepsilon$ such that

$$u_{k_n}(t) \ge \varepsilon \quad \text{for } t \in [0,T],\ n \ge n_0. \tag{3.1}$$

Hence (cf. (1.8)) $(\phi(u_{k_n}'(t)))' = \lambda f_{k_n}(t, u_{k_n}(t), u_{k_n}'(t)) \ge \lambda\varphi(t)$ for a.e. $t \in [0,T]$ and all $n \ge n_0$. Since $u_{k_n}'(\gamma_{k_n}) = 0$ for some $\gamma_{k_n} \in (0,T)$ by Lemma 2.1, we have $-\phi(u_{k_n}'(t)) \ge \lambda\int_t^{\gamma_{k_n}} \varphi(s)\, ds$ for $t \in [0, \gamma_{k_n}]$, and therefore,

$$u_{k_n}'(t) \le -\phi^{-1}\left(\lambda\int_t^{\gamma_{k_n}} \varphi(s)\mathrm{d}s\right) \quad \text{for } t \in [0, \gamma_{k_n}], \quad n \ge n_0. \tag{3.2}$$

Essentially, the same reasoning shows that

$$u_{k_n}'(t) \ge \phi^{-1}\left(\lambda\int_{\gamma_{k_n}}^t \varphi(s)\mathrm{d}s\right) \quad \text{for } t \in [\gamma_{k_n}, T], n \ge n_0. \tag{3.3}$$

Passing if necessary to a subsequence, we may assume that $\{\gamma_{k_n}\}$ is convergent, and let $\lim_{n\to\infty}\gamma_{k_n} = \theta$. Letting $n \to \infty$ in (3.2) and (3.3) gives

$$\begin{aligned} u'(t) &\le -\phi^{-1}\left(\lambda\int_t^{\theta} \varphi(s)\mathrm{d}s\right) \quad \text{for } t \in [0,\theta], \\ u'(t) &\ge \phi^{-1}\left(\lambda\int_{\theta}^t \varphi(s)\mathrm{d}s\right) \quad \text{for } t \in [\theta,T]. \end{aligned} \tag{3.4}$$

Hence θ is the unique zero of u', $\theta \in (0,T)$ since u fulfils (1.2), and

$$\lim_{n\to\infty} f_{k_n}\left(t, u_{k_n}(t), u'_{k_n}(t)\right) = f(t,u(t),u'(t)) \quad \text{for a.e. } t \in [0,T]. \tag{3.5}$$

In addition, it follows from the Fatou lemma and from the relation

$$\lambda \int_0^T f_{k_n}\left(t, u_{k_n}(t), u'_{k_n}(t)\right) \mathrm{d}t = \phi\left(u'_{k_n}(T)\right) - \phi\left(u'_{k_n}(0)\right) < 2\phi(S), \quad n \in \mathbb{N}, \tag{3.6}$$

that $\int_0^T f(t,u(t),u'(t))\mathrm{d}t \le 2\phi(S)/\lambda$. Therefore, $f(t,u(t),u'(t)) \in L^1[0,T]$. We now show that $\phi(u') \in AC[0,T]$ and u fulfils (1) a.e. on $[0,T]$. Let us choose $0 \le t_1 < (\theta/2) < t_2 < \theta$. In view of (3.1), (3.4), (3.5) and Lemma 2.1, there exist $\nu > 0$ and $n_1 \ge n_0$ such that

$$\varepsilon \le u_{k_n}(t) \le \left(1 + \frac{\beta}{\alpha}\right)A, \quad -S < u'_{k_n}(t) \le -\nu \quad \text{for } t \in [t_1, t_2], \ n \ge n_1. \tag{3.7}$$

Then (cf. (1.11))

$$f_{k_n}\left(t, u_{k_n}(t), u'_{k_n}(t)\right) \le \left(p_1(\varepsilon) + \tilde{p}_2\left(\left(1 + \frac{\beta}{\alpha}\right)A\right)\right)(\omega_1(\nu) + \tilde{\omega}_2(S)) + \varphi(t) \tag{3.8}$$

for a.e. $t \in [t_1, t_2]$ and $n \ge n_1$. Letting $n \to \infty$ in

$$\phi\left(u'_{k_n}(t)\right) = \phi\left(u'_{k_n}\left(\frac{\theta}{2}\right)\right) + \lambda \int_{\theta/2}^t f_{k_n}\left(s, u_{k_n}(s), u'_{k_n}(s)\right) \mathrm{d}s \tag{3.9}$$

yields

$$\phi(u'(t)) = \phi\left(u'\left(\frac{\theta}{2}\right)\right) + \lambda \int_{\theta/2}^t f(s,u(s),u'(s))\mathrm{d}s \tag{3.10}$$

for $t \in [t_1, t_2]$ by the Lebesgue dominated convergence theorem. Since t_1, t_2 satisfying $0 \le t_1 < \theta/2 < t_2 < \theta$ are arbitrary and $f(t,u(t),u'(t)) \in L^1[0,T]$, equality (3.10) holds for $t \in [0,\theta]$. Essentially, the same reasoning which is now applied to t_1, t_2 satisfying $\theta < t_1 < (T+\theta)/2 < t_2 \le T$ gives

$$\phi(u'(t)) = \phi\left(u'\left(\frac{T+\theta}{2}\right)\right) + \lambda \int_{(T+\theta)/2}^t f(s,u(s),u'(s))\mathrm{d}s \tag{3.11}$$

for $t \in [\theta, T]$. Hence $\phi(u') \in AC[0,T]$ and u fulfills (1.1) a.e. on $[0,T]$. Consequently, u is a positive solution of problem (1.1), (1.2).

Case 2. Suppose that $\min\{u(t) : t \in [0,T]\} = 0$, and let $u(\rho_1) = u(\rho_2) = 0$ for some $\rho_1 \le \rho_2$ and $u > 0$ on $[0,T] \setminus [\rho_1, \rho_2]$. Since u' is nondecreasing on $[0,T]$, we have $u' < 0$ on $[0, \rho_1)$, $u' = 0$ on $[\rho_1, \rho_2]$ and $u' > 0$ on $(\rho_2, T]$. Consequently, $u = 0$ on $[\rho_1, \rho_2]$ and

$$\lim_{n\to\infty} f_{k_n}\Big(t, u_{k_n}(t), u'_{k_n}(t)\Big) = f(t, u(t), u'(t)) \quad \text{for a.e.}\, t \in [0,T] \setminus \big[\rho_1, \rho_2\big]. \tag{3.12}$$

Furthermore, it follows from

$$\begin{aligned} \lambda \int_0^{\rho_1} f_{k_n}\Big(t, u_{k_n}(t), u'_{k_n}(t)\Big)\mathrm{d}t &= \phi\Big(u'_{k_n}(\rho_1)\Big) - \phi\Big(u'_{k_n}(0)\Big) < 2\phi(S), \\ \lambda \int_{\rho_2}^{T} f_{k_n}\Big(t, u_{k_n}(t), u'_{k_n}(t)\Big)\mathrm{d}t &= \phi\Big(u'_{k_n}(T)\Big) - \phi\Big(u'_{k_n}(\rho_2)\Big) < 2\phi(S) \end{aligned} \tag{3.13}$$

that $f(t, u(t), u'(t))$ is integrable on the intervals $[0, \rho_1]$ and $[\rho_2, T]$ by the Fatou lemma. We can now proceed analogously to Case 1 with $0 \le t_1 < \rho_1/2 < t_2 < \rho_1$ and with $\rho_2 < t_1 < (T+\rho_2)/2 < t_2 \le T$ and obtain

$$\begin{aligned} \phi(u'(t)) &= \phi\Big(u'\Big(\frac{\rho_1}{2}\Big)\Big) + \lambda \int_{\rho_1/2}^{t} f(s, u(s), u'(s))\mathrm{d}s \quad \text{for } t \in \big[0, \rho_1\big], \\ \phi(u'(t)) &= \phi\left(u'\left(\frac{T+\rho_2}{2}\right)\right) + \lambda \int_{(T+\rho_2)/2}^{t} f(s, u(s), u'(s))\mathrm{d}s \quad \text{for } t \in \big[\rho_2, T\big]. \end{aligned} \tag{3.14}$$

It follows from these equalities and from $u' = 0$ on $[\rho_1, \rho_2]$ that $\phi(u') \in AC[0,T]$ and that u fulfils (1.1) a.e. on $[0,T] \setminus [\rho_1, \rho_2]$. Hence u is a dead-core solution of problem (1.1), (1.2) if $\rho_1 < \rho_2$, and u is a pseudo-dead-core solution if $\rho_1 = \rho_2$. □

Theorem 3.2. *Let* (H_1)*–*(H_3) *hold. Then there exists* $\lambda_* > 0$ *such that for each* $\lambda \in (0, \lambda_*]$*, all sequential solutions of problem* (1.1), (1.2) *are positive solutions.*

Proof. Let $\lambda_* > 0$ and $\varepsilon > 0$ be given in Lemma 2.8. Let us choose an arbitrary $\lambda \in (0, \lambda_*]$. Then (2.38) holds, where u_n is any solution of problem (1.12), (1.2). Let u be a sequential solution of problem (1.1), (1.2). Then $u = \lim_{n\to\infty} u_{k_n}$ in $C^1[0,T]$, where u_{k_n} is a solution of (1.12), (1.2) with n replaced by k_n. Consequently, $u \ge \varepsilon$ on $[0,T]$ by (2.38), which means that u is a positive solution of problem (1.1), (1.2) by Theorem 3.1. □

Theorem 3.3. *Let* (H_1)*–*(H_3) *hold. Then for each* $0 < c_1 < c_2 < T$*, there exists* $\lambda^* > 0$ *such that any sequential solution* u *of problem* (1.1), (1.2) *with* $\lambda > \lambda^*$ *satisfies the equality*

$$u(t) = 0 \quad \textit{for } t \in [c_1, c_2], \tag{3.15}$$

which means that the dead-core of u *contains the interval* $[c_1, c_2]$*. Consequently, all sequential solutions of problem* (1.1), (1.2) *are dead-core solutions for sufficiently large value of* λ*.*

Proof. Fix $0 < c_1 < c_2 < T$. Then, by Lemma 2.9, there exists $\lambda^* > 0$ such that

$$\lim_{n\to\infty} u_n(c_j) = 0 \quad \text{for } j = 1, 2, \tag{3.16}$$

where u_n is any solution of problem (1.12), (1.2) with $\lambda > \lambda^*$. Let us choose $\lambda > \lambda^*$ and let u be a sequential solution of problem (1.1), (1.2). Then $u = \lim_{n\to\infty} u_{k_n}$ in $C^1[0,T]$, where u_{k_n} is a solution of problem (1.12), (1.2) with n replaced by k_n. It follows from (3.16) that $u(c_j) = 0$ for $j = 1, 2$, and since u' is nondecreasing on $[0,T]$, (3.15) holds. Consequently, u is a dead-core solution of problem (1.1), (1.2) by Theorem 3.1. □

Example 3.4. Let $p \in (1,\infty)$, $\gamma_1 \in [1,p)$, $\delta_1, \gamma_2, \gamma_3 \in (0,\infty)$, $\delta_2, \delta_3 \in (0,1)$ and $\varphi \in L^1[0,T]$ be positive. Consider the differential equation

$$\left(|u'|^{p-2}u'\right)' = \lambda\left(u^{\delta_1} + \frac{1}{u^{\delta_2}} + |u'|^{\gamma_1} + \frac{1}{|u'|^{\gamma_2}} + \frac{1}{u^{\delta_3}|u'|^{\gamma_3}} + \varphi(t)\right). \tag{3.17}$$

Equation (3.17) is the special case of (1.1) with $\phi(y) = |y|^{p-2}y$ and $f(t,x,y) = x^{\delta_1} + 1/x^{\delta_2} + |y|^{\gamma_1} + 1/|y|^{\gamma_2} + 1/x^{\delta_3}|y|^{\gamma_3} + \varphi(t)$. Since

$$\varphi(t) \le f(t,x,y) \le \left(1 + x^{\delta_1} + \frac{1}{x^{\delta_2}} + \frac{1}{x^{\delta_3}}\right)\left(1 + y^{\gamma_1} + \frac{1}{|y|^{\gamma_2}} + \frac{1}{|y|^{\gamma_3}}\right) + \varphi(t) \tag{3.18}$$

for $(t,x,t) \in [0,T] \times \mathfrak{D}_*$, where $\mathfrak{D}_* = (0,\infty) \times (\mathbb{R} \setminus \{0\})$, f fulfils (H_3) with $\varphi = \psi$, $p_1(x) = 1/x^{\delta_2} + 1/x^{\delta_3}$, $p_2(x) = 1 + x^{\delta_1}$, $\omega_1(y) = 1/y^{\gamma_2} + 1/y^{\gamma_3}$, and $\omega_2(y) = 1 + y^{\gamma_1}$. Hence, by Theorem 3.1, problem (3.17), (1.2) has a sequential solution for each $\lambda > 0$, and any sequential solution is either a positive solution or a pseudo-dead-core solution or a dead-core solution. If the values of λ are sufficiently small, then all sequential solutions of problem (3.17), (1.2) are positive solutions by Theorem 3.2. Theorem 3.3 guarantees that all sequential solutions of problem (3.17), (1.2) are dead-core solutions for sufficiently large values of λ.

Acknowledgment

This work was supported by the Council of Czech Government MSM 6198959214.

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