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Research Article

Dynamics for Nonlinear Difference Equation $x_{n+1} = (\alpha x_{n-k})/(\beta + \gamma x_{n-l}^{p})$

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We mainly study the global behavior of the nonlinear difference equation in the title, that is, the global asymptotical stability of zero equilibrium, the existence of unbounded solutions, the existence of period two solutions, the existence of oscillatory solutions, the existence, and asymptotic behavior of non-oscillatory solutions of the equation. Our results extend and generalize the known ones.

1. Introduction

Consider the following higher order difference equation:

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma x_{n-l}^{p}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $k, l, \in \{0, 1, 2, \ldots\}$, the parameters α, β, γ and p, are nonnegative real numbers and the initial conditions $x_{-\max\{k,\, l\}}, \ldots, x_{-1}$ and x_0 are nonnegative real numbers such that

$$\beta + \gamma x_{n-l}^{p} > 0, \quad \forall n \geq 0. \tag{1.2}$$

It is easy to see that if one of the parameters α, γ, p is zero, then the equation is linear. If $\beta = 0$, then (1.1) can be reduced to a linear one by the change of variables $x_n = e^{y_n}$. So in the sequel we always assume that the parameters α, β, γ, and p are positive real numbers.

The change of variables $x_n = (\beta/\gamma)^{1/p} y_n$ reduces (1.1) into the following equation:

$$y_{n+1} = \frac{r y_{n-k}}{1 + y_{n-l}^p}, \quad n = 0, 1, \ldots, \tag{1.3}$$

where $r = \alpha/\beta > 0$.

Note that $\overline{y}_1 = 0$ is always an equilibrium point of (1.3). When $r > 1$, (1.3) also possesses the unique positive equilibrium $\overline{y}_2 = (r-1)^{1/p}$.

The linearized equation of (1.3) about the equilibrium point $\overline{y}_1 = 0$ is

$$z_{n+1} = r z_{n-k}, \quad n = 0, 1, \ldots, \tag{1.4}$$

so, the characteristic equation of (1.3) about the equilibrium point $\overline{y}_1 = 0$ is either, for $k \geq l$,

$$\lambda^{k+1} - r = 0, \tag{1.5}$$

or, for $k < l$,

$$\lambda^{l-k}\left(\lambda^{k+1} - r\right) = 0. \tag{1.6}$$

The linearized equation of (1.3) about the positive equilibrium point $\overline{y}_2 = (r-1)^{1/p}$ has the form

$$z_{n+1} = -\frac{p(r-1)}{r} z_{n-l} + z_{n-k}, \quad n = 0, 1, \ldots, \tag{1.7}$$

with the characteristic equation either, for $k \geq l$,

$$\lambda^{k+1} + \frac{p(r-1)}{r}\lambda^{k-l} - 1 = 0 \tag{1.8}$$

or, for $k < l$,

$$\lambda^{l+1} - \lambda^{l-k} + \frac{p(r-1)}{r} = 0. \tag{1.9}$$

When $p = 1$, $k, l \in \{0, 1\}$, (1.1) has been investigated in [1–4]. When $k = 1$, $l = 2$, (1.1) reduces to the following form:

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^p}, \quad n = 0, 1, \ldots. \tag{1.10}$$

El-Owaidy et al. [3] investigated the global asymptotical stability of zero equilibrium, the periodic character and the existence of unbounded solutions of (1.10).

On the other hand, when $k = 0$, $p = 1$, (1.1) is just the discrete delay logistic model investigated in [4, P_{75}]. Therefore, it is both theoretically and practically meaningful to study (1.1).

Our aim in this paper is to extend and generalize the work in [3]. That is, we will investigate the global behavior of (1.1), including the global asymptotical stability of zero equilibrium, the existence of unbounded solutions, the existence of period two solutions, the existence of oscillatory solutions, the existence and asymptotic behavior of nonoscillatory solutions of the equation. Our results extend and generalize the corresponding ones of [3].

For the sake of convenience, we now present some definitions and known facts, which will be useful in the sequel.

Consider the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{1.11}$$

where $k \geq 1$ is a positive integer, and the function F has continuous partial derivatives.

A point $\overline{x}$ is called an equilibrium of (1.11) if

$$\overline{x} = F(\overline{x}, \ldots, \overline{x}). \tag{1.12}$$

That is, $x_n = \overline{x}$ for $n > 0$ is a solution of (1.11), or equivalently, $\overline{x}$ is a fixed point of F.

The linearized equation of (1.11) associated with the equilibrium point $\overline{x}$ is

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F}{\partial u_i}(\overline{x}, \ldots, \overline{x}) y_{n-i}, \quad n = 0, 1, \ldots. \tag{1.13}$$

We need the following lemma.

Lemma 1.1 (see [4–6]). (i) *If all the roots of the polynomial equation*

$$\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial F}{\partial u_i}(\overline{x}, \ldots, \overline{x}) \lambda^{k-i} = 0 \tag{1.14}$$

lie in the open unit disk $|\lambda| < 1$, *then the equilibrium* $\overline{x}$ *of* (1.11) *is locally asymptotically stable.*

(ii) *If at least one root of* (1.11) *has absolute value greater than one, then the equilibrium* $\overline{x}$ *of* (1.11) *is unstable.*

For the related investigations for nonlinear difference equations, see also [7–11] and the references cited therein.

2. Global Asymptotic Stability of Zero Equilibrium

In this section, we investigate global asymptotic stability of zero equilibrium of (1.3). We first have the following results.

Lemma 2.1. *The following statements are true.*

(a) *If $r < 1$, then the equilibrium point $\overline{y}_1 = 0$ of* (1.3) *is locally asymptotically stable.*

(b) *If $r > 1$, then the equilibrium point $\overline{y}_1$ of* (1.3) *is unstable. Moreover, for $k < l$,* (1.3) *has a $(l-k)$-dimension local stable manifold and a $(k+1)$-dimension local unstable manifold.*

(c) *If $r > 1$, k is odd and l is even, then the positive equilibrium point $\overline{y}_2 = (r-1)^{1/p}$ of* (1.3) *is unstable.*

Proof. (a) When $r < 1$, it is clear from (1.5) and (1.6) that every characteristic root λ satisfies $|\lambda|^{k+1} = r < 1$ or $|\lambda| = 0$, and so, by Lemma 1.1(i), $\overline{y}_1$ is locally asymptotically stable.

(b) When $r > 1$, if $k \geq l$, then it is clear from (1.5) that every characteristic root λ satisfies $|\lambda|^{k+1} = r > 1$, and so, by Lemma 1.1(ii), $\overline{y}_1$ is unstable. If $k < l$, then (1.6) has $l-k$ characteristic roots λ satisfying $|\lambda|^{l-k} = 0 < 1$, which corresponds to a $(l-k)$-dimension local stable manifold of (1.3), and $k+1$ characteristic roots λ satisfying $|\lambda|^{k+1} = r > 1$, which corresponds to a $(k+1)$-dimension local unstable manifold of (1.3).

(c) If k is odd and l is even, then, regardless of $k \geq l$ or $k < l$, correspondingly, the characteristic equation (1.8) or (1.9) always has one characteristic root λ lying the interval $(-\infty, -1)$. It follows from Lemma 1.1(ii) that $\overline{y}_2$ is unstable. □

Remark 2.2. Lemma 2.1(a) includes and improves [3, Theorem 3.1(i)]. Lemma 2.1(b) and (c) include and generalize [3, Theorem 3.1(ii) and (iii)], respectively.

Now we state the main results in this section.

Theorem 2.3. *Assume that $r < 1$, then the equilibrium point $\overline{y}_1 = 0$ of* (1.3) *is globally asymptotically stable.*

Proof. We know from Lemma 2.1 that the equilibrium point $\overline{y}_1 = 0$ of (1.3) is locally asymptotically stable. It suffices to show that $\lim_{n\to\infty} y_n = 0$ for any nonnegative solution $\{y_n\}_{n=-\max\{k,l\}}^{\infty}$ of (1.3).

Since

$$0 \leq y_{n+1} = \frac{r y_{n-k}}{1 + y_{n-l}^{p}} \leq r y_{n-k} \leq y_{n-k}, \tag{2.1}$$

$\{y_{-j+(k+1)i}\}_{i=0}^{\infty}$ converges for any $j \in \{0, 1, \ldots, k\}$. Let $\lim_{i\to\infty} y_{-j+(k+1)i} = \alpha_{-j}$, $j \in \{0, 1, \ldots, k\}$, then

$$\alpha_0 = \frac{r\alpha_0}{1 + \alpha_{-1-l}^{p}}, \ldots, \alpha_{l+1-k} = \frac{r\alpha_{l+1-k}}{1 + \alpha_{-k}^{p}}, \alpha_{l-k} = \frac{r\alpha_{l-k}}{1 + \alpha_{0}^{p}}, \ldots, \alpha_{-k} = \frac{r\alpha_{-k}}{1 + \alpha_{-l}^{p}}. \tag{2.2}$$

Thereout, one has

$$\alpha_{-k} = \alpha_{-k+1} = \cdots = \alpha_0 = 0, \tag{2.3}$$

that is,

$$\lim_{i\to\infty} y_{-j+(k+1)i} = 0, \quad \text{for any } j \in \{0,1,\dots,k\}, \tag{2.4}$$

which implies $\lim_{n\to\infty} y_n = 0$. The proof is over. □

Remark 2.4. Theorem 2.3 includes [3, Theorem 3.3].

3. Existence of Eventual Period Two Solution

In this section, one studies the eventual nonnegative prime period two solutions of (1.3). A solution $\{x_n\}_{n=-\max\{k,l\}}^{\infty}$ of (1.3) is said to be eventual prime periodic two solution if there exists an $n_0 \in \{-\max\{k,l\}, -\max\{k,l\}+1,\dots\}$ such that $x_{n+2} = x_n$ for $n \ge n_0$ and $x_{n+1} \neq x_n$ holds for all $n \ge n_0$.

Theorem 3.1. (a) *Assume k is odd and l is even, then* (1.3) *possesses eventual prime period two solutions if and only if $r = 1$.*

(b) *Assume k is odd and l is odd, then* (1.3) *possesses eventual prime period two solutions if and only if $r > 1$.*

(c) *Assume k is even and l is even. Then the necessary condition for* (1.3) *to possess eventual prime period two solutions is $r^2(p-2) > p$ and $r^p > \max\{(p/(p-2))^{p-2}r^2, (p-1)r^2 - p\}$.*

(d) *Assume k is even and l is odd. Then,* (1.3) *has no eventual prime period two solutions.*

Proof. (a) If (1.3) has the eventual nonnegative prime period two solution $\dots,\varphi,\ \psi,\ \varphi,\psi,\dots$, then, we eventually have $\varphi = r\varphi/(1+\psi^p)$ and $\psi = r\psi/(1+\varphi^p)$. Hence,

$$\varphi\left(1 - r + \psi^p\right) = 0, \qquad \psi\left(1 - r + \varphi^p\right) = 0. \tag{3.1}$$

If $r \neq 1$, then we can derive from (3.1) that $\varphi = 0$ if $\psi = 0$ or vice versa, which contradicts the assumption that $\dots,\varphi,\ \psi,\ \varphi,\psi,\dots$ is the eventual prime period two solution of (1.3). So, $\varphi\psi \neq 0$. Accordingly, $1 - r + \psi^p = 0$ and $1 - r + \varphi^p = 0$, which indicate that $\varphi = \psi$ when $r > 1$ or that φ and ψ do not exist when $r < 1$, which are also impossible. Therefore, $r = 1$.

Conversely, if $r = 1$, then choose the initial conditions such as $y_{-k} = y_{-k+2} = \cdots = 0$ and $y_{-k+1} = y_{-k+3} = \cdots = y_0 = \varphi > 0$, or such as $y_{-k} = y_{-k+2} = \cdots = \varphi > 0$ and $y_{-k+1} = y_{-k+3} = \cdots = y_0 = 0$. We can see by induction that $\dots,0,\varphi,0,\varphi,\dots$ is the prime period two solution of (1.3).

(b) Let $\dots,\varphi,\ \psi,\ \varphi,\ \psi,\dots$ be the eventual prime period two solution of (1.3), then, it holds eventually that $\varphi = r\varphi/(1+\varphi^p)$ and $\psi = r\psi/(1+\psi^p)$. Hence,

$$\varphi\left(1 - r + \varphi^p\right) = 0, \qquad \psi\left(1 - r + \psi^p\right) = 0. \tag{3.2}$$

If $r \le 1$, then $\varphi = \psi = 0$. This is impossible. So $r > 1$. Moreover, $\varphi = 0$ and $\psi = (r-1)^{1/p}$ or $\psi = 0$ and $\varphi = (r-1)^{1/p}$, that is, $\dots,0,(r-1)^{1/p},0,(r-1)^{1/p}\cdots$ is the prime period two solution of (1.3).

(c) Assume that (1.3) has the eventual nonnegative prime period two solution $\ldots, \varphi, \psi, \varphi, \psi, \ldots$, then eventually

$$\varphi = \frac{r\psi}{1+\psi^p}, \qquad \psi = \frac{r\varphi}{1+\varphi^p}. \tag{3.3}$$

Obviously, $\varphi = 0$ implies $\psi = 0$ or vice versa. This is impossible. So $\varphi\psi > 0$. It is easy to see from (3.3) that φ and ψ satisfy the equation

$$g(y) = (1+y^p)^{p-1}\left[1 - r^2 + y^p\right] + r^p y^p = 0, \tag{3.4}$$

that is, φ and ψ are two distinct positive roots of $g(y) = 0$. From (3.4) we can see that $g(y) = 0$ does not have two distinct positive roots at all when $r \le 1$, alternatively, (1.3) does not have the nonnegative prime period two solution at all when $r \le 1$. Therefore, we assume $r > 1$ in the following. □

Let $1 + y^p = x$ in (3.4), then the equation $f(x) = x^p - r^2 x^{p-1} + r^p(x-1) = 0,\ x > 1$, has at least two distinct positive roots.

By simple calculation, one has

$$f'(x) = px^{p-2}\left[x - \frac{(p-1)r^2}{p}\right] + r^p, \qquad f''(x) = p(p-1)x^{p-3}\left[x - \frac{(p-2)r^2}{p}\right]. \tag{3.5}$$

If $(p-1)r^2/p \le 1$, we can see $f'(x) > 0$ for all $x \in (1,\infty)$. This means that $f(x)$ is strictly increasing in the interval $(1,\infty)$ and hence the equation, $f(x) = x^p - r^2 x^{p-1} + r^p(x-1) = 0, x > 1$, cannot have two distinct positive roots. So, next we consider $(p-1)r^2/p > 1$, which implies $p > 1$. Denote $x_0 = (p-2)r^2/p$. We need to discuss several cases, respectively, as follows.

Case 1. It holds that $x_0 \le 1$. Then $f''(x) > 0$ for all $x \in (1,\infty)$, hence, $f(x)$ is convex. Again, $f(1) = 1 - r^2 < 0$. So it is impossible for $f(x)$ to have two distinct positive roots.

Case 2. It holds that $x_0 > 1$ and $f'(x_0) = r^p[1 - ((p-2)/p)^{p-2} r^{p-2}] \ge 0$. Then, for $x > x_0$, $f''(x) > 0$ and so $f'(x) > f'(x_0) \ge 0$; for $1 < x < x_0$, $f''(x) < 0$ and so $f'(x) > f'(x_0) \ge 0$. At this time, one always has $f'(x) \ge f'(x_0) \ge 0$. Then $f(x)$ cannot have two distinct positive roots.

Case 3. It holds that $x_0 > 1$, $f'(x_0) < 0$ and $f'(1) = p - (p-1)r^2 + r^p \le 0$. Then, for $1 < x < x_0$, $f''(x) < 0$ and so $f'(x) < f'(1) \le 0$ and hence $f(x_0) < f(x) < f(1) = 1 - r^2 < 0$, that is, $f(x) = 0$ has no solutions for $1 < x < x_0$; for $x > x_0$, $f''(x) > 0$, that is, $f(x)$ is convex for $x > x_0$. Noticing $f(x_0) < 0$, it is also impossible for $f(x)$ to have two distinct positive roots for $x > x_0$.

Case 4. It holds that $x_0 > 1$, $f'(x_0) < 0$ and $f'(1) = p - (p-1)r^2 + r^p > 0$. This is only case where $f(x)$ could have two distinct positive roots, which implies $r^2(p-2) > p$ and $r^p > \max\{(p/(p-2))^{p-2} r^2, (p-1)r^2 - p\}$.

(d) Let $\dots, \varphi, \psi, \varphi, \psi, \dots$ be the eventual nonnegative prime period two solution of (1.3), then, it is eventually true that

$$\varphi = \frac{r\psi}{1+\varphi^p}, \qquad \psi = \frac{r\varphi}{1+\psi^p}. \tag{3.6}$$

It is easy to see that $\varphi > 0$ and $\psi > 0$. So, we have

$$\begin{aligned} &\varphi^p(1+\varphi^p)^{p+1} + r^p\left(1-r^2+\varphi^p\right) = 0,\\ &\psi^p(1+\varphi^p)^{p+1} + r^p\left(1-r^2+\psi^p\right) = 0, \end{aligned} \tag{3.7}$$

that is, φ and ψ are two distinct positive roots of $h(x) = x^p(1+x^p)^{p+1} + r^p(1-r^2+x^p) = 0$. Obviously, when $r \le 1$, the $h(x) = 0$ has no positive roots.

Now let $r > 1$ and set $1+x^p = y$. Then the function, $f(y) = y^{p+2} - y^{p+1} + yr^p - r^{p+2},\ y > 1$, has at least two distinct positive roots. However, $f'(y) = y^p[(p+2)y - (p+1)] + r^p > 0$ for any $y \in (1, \infty)$, which indicates that $f(y)$ is strictly increasing in the interval $(1, \infty)$. This implies that the function $f(y)$ does not have two distinct positive roots at all in the interval $(1, \infty)$. In turn, (1.3) does not have the prime period two solution when $r > 1$.

4. Existence of Oscillatory Solution

For the oscillatory solution of (1.3), we have the following results.

Theorem 4.1. *Assume $r > 1$, k is odd and l is even. Then, there exist solutions $\{y_n\}_{n=-\max\{k,l\}}^{\infty}$ of* (1.3) *which oscillate about $\overline{y}_2 = (r-1)^{1/p}$ with semicycles of length one.*

Proof. We only prove the case where $k \ge l$ (the proof of the case where $k < l$ is similar and will be omitted). Choose the initial values of (1.3) such that

$$y_{-k}, y_{-k+2}, \dots, y_{-1} \le \overline{y}_2, \qquad y_{-k+1}, y_{-k+3}, \dots, y_0 \ge \overline{y}_2, \tag{4.1}$$

or

$$y_{-k}, y_{-k+2}, \dots, y_{-1} \ge \overline{y}_2 \qquad y_{-k+1}, y_{-k+3}, \dots, y_0 \le \overline{y}_2. \tag{4.2}$$

We will only prove the case where (4.1) holds. The case where (4.2) holds is similar and will be omitted. According to (1.3), one can see that

$$\begin{gathered} y_1 = \frac{ry_{-k}}{1+y_{-l}^p} < y_{-k} \le \overline{y}_2, \qquad y_2 = \frac{ry_{-k+1}}{1+y_{1-l}^p} \ge y_{-k+1} \ge \overline{y}_2,\\ \vdots\\ y_k = \frac{ry_{-1}}{1+y_{k-1-l}^p} < y_{-1} \le \overline{y}_2, \qquad y_{k+1} = \frac{ry_0}{1+y_{k-l}^p} \ge y_0 \ge \overline{y}_2. \end{gathered} \tag{4.3}$$

So, the proof follows by induction. □

5. Existence of Unbounded Solution

With respect to the unbounded solutions of (1.3), the following results are derived.

Theorem 5.1. *Assume $r > 1$, k is odd, and l is even, then* (1.3) *possesses unbounded solutions.*

Proof. We only prove the case where $k \geq l$ (the proof of the case where $k < l$ is similar and will be omitted). Choose the initial values of (1.3) such that

$$0 < y_{-k}, y_{-k+2}, \dots, y_{-1} < \overline{y}_2, \qquad y_{-k+1}, y_{-k+3}, \dots, y_0 > \overline{y}_2. \tag{5.1}$$

In the following, assume $j \geq -k$. From the proof of Theorem 4.1, one can see that $y_j < \overline{y}_2$ when j is odd and that $y_j > \overline{y}_2$ for j even. Together with

$$y_{j+(k+1)(i+1)} = \frac{r y_{j+(k+1)i}}{1 + y^p_{k-l+j+(k+1)i}}, \tag{5.2}$$

It is derived that

$$\begin{gathered} 0 < y_{j+(k+1)(i+1)} < y_{j+(k+1)i} < \overline{y}_2 \quad \text{for } j \text{ odd}, \\ y_{j+(k+1)(i+1)} > y_{j+(k+1)i} > \overline{y}_2 \quad \text{for } j \text{ even}. \end{gathered} \tag{5.3}$$

So, $\{y_{j+(k+1)i}\}_{i=0}^{\infty}$ is decreasing for j odd whereas $\{y_{j+(k+1)i}\}_{i=0}^{\infty}$ is increasing for j even. Let

$$\lim_{i \to \infty} y_{j+(k+1)i} = \alpha_j, \quad \forall j \geq -k, \tag{5.4}$$

then one has

(1) $0 \leq \alpha_j < \overline{y}_2$ for j odd and $\overline{y}_2 < \alpha_j \leq \infty$ for j even,

(2) $\alpha_j = \alpha_{j+(k+1)i}$, $j \in \{-k, -k+1, \dots\}$, $i \in \{0, 1, \dots\}$.

Now, either $\alpha_j = \infty$ for some even j in which case the proof is complete, or $\alpha_j < \infty$ for all even j. We shall prove that this latter is impossible. In fact, we prove that $\alpha_j = \infty$ for all even j.

Assume $\alpha_j < \infty$ for some even $j \geq -k$, then one has, by (5.2), $\alpha_j = (r\alpha_j)/(1 + \alpha^p_{j+k-l})$. Noticing (1), one hence further gets $\alpha_{j+k-l} = \overline{y}_2$. However $j + k - l$ is odd, according to (1), $\alpha_{j+k-l} < \overline{y}_2$. This is a contradiction.

Therefore, $\alpha_j = \infty$ for any even j. Accordingly, $\{y_{j+(k+1)(i+1)}\}$ are unbounded subsequences of this solution $\{y_n\}$ of (1.3) for even j. Simultaneously, for odd j, we get

$$\alpha_j = \lim_{i \to \infty} y_{j+(k+1)i} = \lim_{i \to \infty} y_{k+1+j+(k+1)i} = \lim_{i \to \infty} \frac{r y_{j+(k+1)i}}{1 + y^p_{k-l+j+(k+1)i}} = 0. \tag{5.5}$$

The proof is complete. □

Remark 5.2. Theorem 5.1 includes and generalizes [3, Theorem 3.5].

6. Existence and Asymptotic Behavior of Nonoscillatory Solution

In this section, we consider the existence and asymptotic behavior of nonoscillatory solution of (1.3). Because all solutions of (1.3) are nonnegative, relative to the zero equilibrium point $\overline{y}_1$, every solution of (1.3) is a positive semicycle, a trivial nonoscillatory solution! Thus, it suffices to consider the positive equilibrium $\overline{y}_2$ when studying the nonoscillatory solutions of (1.3). At this time, $r > 1$.

Firstly, we have the following results.

Theorem 6.1. *Every nonoscillatory solution of* (1.3) *with respect to* $\overline{y}_2$ *approaches* $\overline{y}_2$.

Proof. Let $\{y_n\}_{n=-\max\{k,l\}}^{\infty}$ be any one nonoscillatory solution of (1.3) with respect to $\overline{y}_2$. Then, there exists an $n_0 \in \{-\max\{k,l\}, -\max\{k,l\}+1, \dots\}$ such that

$$y_n \geq \overline{y}_2 \quad \text{for } n \geq n_0 \tag{6.1}$$

or

$$y_n < \overline{y}_2 \quad \text{for } n \geq n_0. \tag{6.2}$$

We only prove the case where (6.1) holds. The proof for the case where (6.2) holds is similar and will be omitted. According to (6.1), for $n \geq n_0 + \max\{k,l\}$, one has

$$y_{n+1} = \frac{r y_{n-k}}{1 + y_{n-l}^{p}} \leq y_{n-k}. \tag{6.3}$$

So, $\{y_{j+(k+1)i}\}_{i=0}^{\infty}$ is decreasing for $j \in \{-\max\{k,l\}, -\max\{k,l\}+1, \dots, -1, 0\}$ with upper bound $\overline{y}_2$. Hence, $\lim_{i\to\infty} y_{j+(k+1)i}$ exists and is finite. Denote

$$\lim_{i\to\infty} y_{j+(k+1)i} = \alpha_j, \quad j \in \{-\max\{k,l\}, -\max\{k,l\}+1, \dots, -1, 0\}. \tag{6.4}$$

Then $\alpha_j \geq \overline{y}_2$. Taking limits on both sides of (1.3), we can derive

$$\alpha_j = \overline{y}_2 \quad \text{for } j \in \{-\max\{k,l\}, -\max\{k,l\}+1, \dots, -1, 0\}, \tag{6.5}$$

which shows $\lim_{n\to\infty} y_n = \overline{y}_2$ and completes this proof. □

A problem naturally arises: are there nonoscillatory solutions of (1.3)? Next, we will positively answer this question. Our result is as follows.

Theorem 6.2. *However* (1.3) *possesses asymptotic solutions with a single semicycle (positive semicycle or negative semicycle).*

The main tool to prove this theorem is to make use of Berg' inclusion theorem [12]. Now, for the sake of convenience of statement, we first state some preliminaries. For this, refer also to [13]. Consider a general real nonlinear difference equation of order $m \geq 1$ with the form

$$F(x_n, x_{n+1}, \dots, x_{n+m}) = 0, \tag{6.6}$$

where $F : \mathbb{R}^{m+1} \mapsto \mathbb{R},\ n \in \mathbb{N}_0$. Let φ_n and ψ_n be two sequences satisfying $\psi_n > 0$ and $\psi_n = o(\varphi_n)$ as $n \to \infty$. Then (maybe under certain additional conditions), for any given $\epsilon > 0$, there exist a solution $\{x_n\}_{n=-1}^{\infty}$ of (6.6) and an $n_0(\epsilon) \in \mathbb{N}$ such

$$\varphi_n - \epsilon\psi_n \leq x_n \leq \varphi_n + \epsilon\psi_n, \quad n \geq n_0(\epsilon). \tag{6.7}$$

Denote

$$X(\epsilon) = \{x_n : \varphi_n - \epsilon\psi_n \leq x_n \leq \varphi_n + \epsilon\psi_n,\ n \geq n_0(\epsilon)\}, \tag{6.8}$$

which is called an asymptotic stripe. So, if $x_n \in X(\epsilon)$, then it is implied that there exists a real sequence C_n such that $x_n = \varphi_n + C_n\psi_n$ and $|C_n| \leq \epsilon$ for $n \geq n_0(\epsilon)$.

We now state the inclusion theorem [12].

Lemma 6.3. *Let $F(\omega_0, \omega_1, \dots, \omega_m)$ be continuously differentiable when $\omega_i = y_{n+i}$, for $i = 0, 1, \dots, m$, and $y_n \in X(1)$. Let the partial derivatives of F satisfy*

$$F_{\omega_i}(y_n, y_{n+1}, \dots, y_{n+m}) \sim F_{\omega_i}(\varphi_n, \varphi_{n+1}, \dots, \varphi_{n+m}) \tag{6.9}$$

as $n \to \infty$ uniformly in C_j for $|C_j| \leq 1$, $n \leq j \leq n+m$, as far as $F_{\omega_i} \not\equiv 0$. Assume that there exist a sequence $f_n > 0$ and constants $A_0, A_1, \dots, A_m$ such that

$$\begin{gathered} F(\varphi_n, \dots, \varphi_{n+m}) = o(f_n), \\ \varphi_{n+i} F_{w_i}(\varphi_n, \dots, \varphi_{n+m}) \sim A_i f_n \end{gathered} \tag{6.10}$$

for $i = 0, 1, \dots, m$ as $n \to \infty$, and suppose there exists an integer s, with $0 \leq s \leq m$, such that

$$|A_0| + \cdots + |A_{s-1}| + |A_{s+1}| + \cdots + |A_m| < |A_s|. \tag{6.11}$$

Then, for sufficiently large n, there exists a solution $\{x_n\}_{n=-1}^{\infty}$ of (6.6) satisfying (6.7).

Proof of Theorem 6.2. We only prove the case where $k \geq l$ (the proof of the case where $k < l$ is similar and will be omitted). Put $x_n = y_n - \overline{y}$ ($\overline{y}_2$ is denoted into $\overline{y}$ for short). Then (1.3) is transformed into

$$(x_{n+k+1} + \overline{y})(1 + (x_{n+k-l} + \overline{y})^p) - r(x_n + \overline{y}) = 0, \quad n = -k, -k+1, \dots. \tag{6.12}$$

An approximate equation of (6.12) is

$$z_{n+k+1}(1 + \overline{y}^p) + p\overline{y}^p z_{n+k-l} - r z_n = 0, \quad n = -k, -k+1, \dots, \tag{6.13}$$

provided that $x_n \to 0$ as $n \to \infty$. The general solution of (6.13) is

$$z_n = \sum_{i=0}^{k} c_i t_i^n, \tag{6.14}$$

where $c_i \in \mathbb{C}$ and $t_i,\ i = 0, 1, \ldots, k$, are the $k+1$ roots of the polynomial

$$P(t) = t^{k+1}\left(1 + \overline{y}^p\right) + p\overline{y}^p t^{k-l} - r = rt^{k+1} + p(r-1)t^{k-l} - r. \tag{6.15}$$

Obviously, $P(0)P(1) = -rp(r-1) < 0$. So, $P(t) = 0$ has a positive root t lying in the interval $(0, 1)$. Now, choose the solution $z_n = t^n$ for this $t \in (0,1)$. For some $\lambda \in (1,2)$, define the sequences $\{\varphi_n\}$ and $\{\psi_n\}$, respectively, as follows:

$$\varphi_n = t^n, \qquad \psi_n = t^{\lambda n}. \tag{6.16}$$

Obviously, $\psi_n > 0$ and $\psi_n = o(\varphi_n)$ as $n \to \infty$.

Now, define again the function

$$F(\omega_0, \omega_1, \ldots, \omega_{k-l}, \ldots, \omega_{k+1}) = \left(\omega_{k+1} + \overline{y}\right)\left(1 + \left(\omega_{k-l} + \overline{y}\right)^p\right) - r\left(\omega_0 + \overline{y}\right). \tag{6.17}$$

Then the partial derivatives of F w.r.t. $\omega_0, \omega_1, \ldots, \omega_{k+1}$, respectively, are

$$\begin{gathered} F_{\omega_0} = -r, \\ F_{\omega_{k-l}} = p\left(\omega_{k+1} + \overline{y}\right)\left(\omega_{k-l} + \overline{y}\right)^{p-1}, \\ F_{\omega_{k+1}} = 1 + \left(\omega_{k-l} + \overline{y}\right)^p, \\ F_{\omega_i} = 0, \quad i = 1, \ldots, k,\ i \neq k-l. \end{gathered} \tag{6.18}$$

When $y_n \in X(1)$, $y_n \sim \varphi_n$. So, $F_{\omega_i}(y_n, y_{n+1}, \ldots, y_{n+k+1}) \sim F_{\omega_i}(\varphi_n, \varphi_{n+1}, \ldots, \varphi_{n+k+1})$, $i = 0, 1, \ldots, k+1$, as $n \to \infty$ uniformly in C_j for $|C_j| \leq 1$, $n \leq j \leq n+k+1$.

Moreover, from the definition of the function F and (6.17) and (6.18), after some calculation, we find

$$\begin{gathered} F\left(\varphi_n, \varphi_{n+1}, \ldots, \varphi_{n+k+1}\right) = \left(t^{n+k+1} + \overline{y}\right)\left(1 + \left(t^{n+k-l} + \overline{y}\right)^p\right) - r\left(t^n + \overline{y}\right), \\ \psi_n F_{\omega_0}\left(\varphi_n, \varphi_{n+1}, \ldots, \varphi_{n+k+1}\right) = -rt^{\lambda n}, \\ \psi_{n+k-l} F_{\omega_{k-l}}\left(\varphi_n, \varphi_{n+1}, \ldots, \varphi_{n+k+1}\right) = t^{\lambda(n+k-l)}\left(p\left(t^{n+k+1} + \overline{y}\right)\left(t^{n+k-l} + \overline{y}\right)^{p-1}\right), \\ \psi_{n+k+1} F_{\omega_{k+1}}\left(\varphi_n, \varphi_{n+1}, \ldots, \varphi_{n+k+1}\right) = t^{\lambda(n+k+1)}\left(1 + \left(t^{n+k-l} + \overline{y}\right)^p\right). \end{gathered} \tag{6.19}$$

Now choose $f_n = t^{\lambda n}$. Noting

$$\begin{aligned} F(\varphi_n, \varphi_{n+1}, \dots, \varphi_{n+k+1}) &= \left(t^{n+k+1} + \overline{y}\right)\left(1 + \left(t^{n+k-l} + \overline{y}\right)^p\right) - r\left(t^n + \overline{y}\right) \\ &= \left(t^{n+k+1} + \overline{y}\right)\left(r + p\overline{y}^{p-1}t^{n+k-l} + O\left(t^{2(n+k-l)}\right)\right) - r\left(t^n + \overline{y}\right) \\ &= t^n\left(rt^{k+1} + p(r-1)t^{k-l} - r\right) + \left(t^{n+k+1} + \overline{y}\right)\left(O\left(t^{2(n+k-l)}\right)\right) \\ &= \left(t^{n+k+1} + \overline{y}\right)\left(O\left(t^{2(n+k-l)}\right)\right), \end{aligned} \tag{6.20}$$

we have $F(\varphi_n, \varphi_{n+1}, \dots, \varphi_{n+k+1}) = o(f_n)$. Again,

$$\varphi_{n+i}F_{\omega_i}(\varphi_n, \varphi_{n+1}, \dots, \varphi_{n+k+1}) \sim A_i f_n, i = 0, \quad k-l, k+1, \tag{6.21}$$

where

$$\begin{gathered} A_0 = -r, \\ A_{k-l} = p(r-1)t^{\lambda(k-l)}, \\ A_{k+1} = rt^{\lambda(k+1)}. \end{gathered} \tag{6.22}$$

Therefore, one has

$$|A_1| + \cdots + |A_{k+1}| = p(r-1)t^{\lambda(k-l)} + rt^{\lambda(k+1)} < p(r-1)t^{k-l} + rt^{k+1} = r = |A_0|. \tag{6.23}$$

Up to here, all conditions of Lemma 6.3 with $m = k+1$ and $s = 0$ are satisfied. Accordingly, we see that, for arbitrary $\epsilon \in (0,1)$ and for sufficiently large n, say $n \geq N_0 \in \mathbb{N}$, (6.12) has a solution $\{x_n\}_{n=-k}^{\infty}$ in the stripe $\varphi_n - \epsilon\psi_n \leq x_n \leq \varphi_n + \epsilon\psi_n$, $n \geq N_0$, where φ_n and ψ_n are as defined in (6.16). Because $\varphi_n - \epsilon\psi_n > \varphi_n - \psi_n = t^n - t^{\lambda n} > 0$, $x_n > 0$ for $n \geq N_0$. Thus, (1.3) has a solution $\{y_n\}_{n=-k}^{\infty}$ satisfying $y_n = x_n + \overline{y} > \overline{y}$ for $n \geq N_0$. Since (1.3) is an autonomous equation, $\{y_{n+N_0+k}\}_{n=-k}^{\infty}$ still is its solution, which evidently satisfies $y_{n+N_0+k} > \overline{y}$ for $n \geq -k$. Therefore, the proof is complete. □

Remark 6.4. If we take $\varphi_n = -t^n$ in (6.16), then $\varphi_n + \epsilon\psi_n < -t^n + t^{\lambda n} < 0$. At this time, (1.3) possesses solutions $\{y_n\}_{n=-k}^{\infty}$ which remain below its equilibrium for all $n \geq -k$, that is, (1.3) has solutions with a single negative semicycle.

Remark 6.5. The appropriate equation (6.12) is just the linearized equation of (1.3) associated with $\overline{y}_2$.

Remark 6.6. The existence and asymptotic behavior of nonoscillatory solution of special cases of (1.3) has not been found to be considered in the known literatures.

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