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Research Article

A Generalized Sum-Difference Inequality and Applications to Partial Difference Equations

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We establish a general form of sum-difference inequality in two variables, which includes both two distinct nonlinear sums without an assumption of monotonicity and a nonconstant term outside the sums. We employ a technique of monotonization and use a property of stronger monotonicity to give an estimate for the unknown function. Our result enables us to solve those discrete inequalities considered by Cheung and Ren (2006). Furthermore, we apply our result to a boundary value problem of a partial difference equation for boundedness, uniqueness, and continuous dependence.

1. Introduction

Being an important tool in the study of differential equations and integral equation, various generalizations of Gronwall inequality [1, 2] and their applications have attracted great interests of many mathematicians (see [3–5]). Some recent works can be found, for example, in [6–9] and some references therein. Along with the development of the theory of integral inequalities and the theory of difference equations, more attentions are paid to some discrete versions of Gronwall-type inequalities (see, e.g., [10–12] for some early works). Found in [13], the unknown function u in the fundamental form of sum-difference inequality

$$u(n) \le a(n) + \sum_{s=0}^{n-1} f(s)u(s) \tag{1.1}$$

can be estimated by $u(n) \le a(n)\prod_{s=0}^{n-1}(1+f(s))$. Pang and Agarwal [14] considered the inequality

$$u^2(n) \le P^2u^2(0) + 2\sum_{s=0}^{n-1}\left[\alpha u^2(s) + Qg(s)u(s)\right], \tag{1.2}$$

where α, P, and Q are nonnegative constants and u and g are nonnegative functions defined on $\{1,2,\ldots,T\}$ and $\{1,2,\ldots,T-1\}$, and they estimated that $u(n) \le (1+\alpha)^n[Pu(0)+\sum_{s=0}^{n-1}Qg(s)]$, for all $0 \le n \le T$. Another form of sum-difference inequality,

$$u^2(n) \le c^2 + 2\sum_{s=0}^{n-1}\left[f_1(s)u(s)w\big(u(s)\big) + f_2(s)u(s)\right], \tag{1.3}$$

where c is a constant, f_1 and f_2 are both real-valued nonnegative functions defined on $\mathbb{N}_0 = \{0,1,2,\ldots\}$, and w is a continuous nondecreasing function defined on $[u_0,\infty)$ such that $w(u) > 0$ on (u_0,∞) and $w(u_0) = 0$, for a real constant u_0, was estimated by Pachpatte [15] as $u(n) \le \Omega^{-1}[\Omega(c+\sum_{s=0}^{n-1}f_2(s)) + \sum_{s=0}^{n-1}f_1(s)]$, where $\Omega(u) := \int_{u_0}^{u} ds/w(s)$. Recently, discretization (see [16, 17]) was also made for Ou-Yang's inequality [18]. In [16], the inequality of two variables,

$$u^2(m,n) \le c^2 + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}a(s,t)u(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b(s,t)u(s,t)w\big(u(s,t)\big), \tag{1.4}$$

was discussed. Later, this result was generalized in [17] to the inequality

$$u^p(m,n) \le c + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}a(s,t)u^q(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b(s,t)u^q(s,t)w\big(u(s,t)\big), \tag{1.5}$$

where $c \ge 0$ and $p > q > 0$ are all constant, a and b are both nonnegative real-valued functions defined on a lattice in $\mathbb{Z}_+^2$, and w is a continuous nondecreasing function satisfying $w(u) > 0$, for all $u > 0$.

In this paper, we establish a more general form of sum-difference inequality

$$\psi\big(u(m,n)\big) \le a(m,n) + \sum_{i=1}^{2}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}f_i(s,t)\varphi_i\big(u(s,t)\big) \tag{1.6}$$

for nonnegative integers m,n. In (1.6), we replace the constant c in (1.5) with a function $a(m,n)$ and replace the functions u^p, u^q, $u^q w(u)$ in (1.5) with the more general form of functions $\psi(u)$, $\varphi_1(u)$, $\varphi_2(u)$, respectively. Moreover, we do not require the monotonicity of φ_1 and φ_2. We employ a technique of monotonization and use a property of stronger monotonicity to overcome the difficulty from nonmonotonicity so as to give an estimate for the unknown function u. Our result enables us to solve the discrete inequality (1.5) and other inequalities considered in [17]. Furthermore, we apply our result to a boundary value problem of a partial difference equation for boundedness, uniqueness, and continuous dependence.

2. Main result

Throughout this paper, let $\mathbb{R}$ denote the set of all real numbers, $\mathbb{R}_+ = [0,\infty)$, and $\mathbb{N}_0 = \{0,1,2,\ldots\}$. Given $m_0,n_0 \in \mathbb{N}_0$, $M,N \in \mathbb{N}_0 \cup \{\infty\}$, consider two lattices $I = [m_0,M) \cap \mathbb{N}_0$ and $J = [n_0,N) \cap \mathbb{N}_0$ of integer points in $\mathbb{R}$. Let $\Lambda = I \times J \subset \mathbb{N}_0^2$. For any $(s,t) \in \Lambda$, let $\Lambda_{(s,t)}$ denote the sublattice $[m_0,s] \times [n_0,t] \cap \Lambda$ of Λ.

For functions $w(m)$, $z(m,n)$, $m,n \in \mathbb{N}_0$, their first-order differences are defined by $\Delta w(m) = w(m+1) - w(m)$, $\Delta_1 w(m,n) = w(m+1,n) - w(m,n)$, and $\Delta_2 z(m,n) = z(m,n+1) - z(m,n)$. Obviously, the linear difference equation $\Delta x(m) = b(m)$ with the initial condition $x(m_0) = 0$ has the solution $\sum_{s=m_0}^{m-1}b(s)$. For convenience, in the sequel we complementarily define that $\sum_{s=m_0}^{m_0-1}b(s) = 0$.

Our basic assumptions for inequality (1.6) are given in the following.

(H_1) ψ is a strictly increasing continuous function on $\mathbb{R}_+$ satisfying that $\psi(u) > 0$, for all $u > 0$.

(H_2) All φ_i $(i = 1, 2)$ are continuous functions on $\mathbb{R}_+$ and positive on $(0, \infty)$.

(H_3) $a(m, n) \geq 0$ on Λ.

(H_4) All f_i $(i = 1, 2)$ are nonnegative functions on Λ.

With given functions φ_1, φ_2, and ψ, we define

$$w_1(u) := \max_{\tau\in[0,u]} \{\varphi_1(\tau)\}, \tag{2.1}$$

$$w_2(u) := \max_{\tau\in[0,u]} \left\{ \frac{\varphi_2(\tau)}{w_1(\tau)} \right\} w_1(u), \tag{2.2}$$

$$W_1(u, u_1) := \int_{u_1}^{u} \frac{dx}{w_1(\psi^{-1}(x))}, \tag{2.3}$$

$$W_2(u, u_2) := \int_{u_2}^{u} \frac{dx}{w_2(\psi^{-1}(x))}, \tag{2.4}$$

where $u_i > 0$ $(i = 1, 2)$ are given constants. Sometimes we simply let $W_i(u)$ denote $W_i(u, u_i)$ when there is no confusion. Obviously, W_1 and W_2 are both strictly increasing in $u > 0$ and therefore the inverses W_i^{-1} $(i = 1, 2)$ are well defined, continuous, and increasing.

Theorem 2.1. *Suppose that (H_1)–(H_4) hold and $u(m, n)$ is a nonnegative function on Λ satisfying* (1.6)*. Then,*

$$u(m,n) \leq \psi^{-1}\left\{ W_2^{-1}\left[W_2(\Upsilon_2(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_2(s,t) \right] \right\} \tag{2.5}$$

for $(m, n) \in \Lambda_{(m_1,n_1)}$, a sublattice in Λ, where

$$\begin{aligned} \Upsilon_2(m,n) &:= W_1^{-1}\left[W_1(\Upsilon_1(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_1(s,t) \right], \\ \Upsilon_1(m,n) &:= a(m_0,n_0) + \sum_{s=m_0}^{m-1} \left| a(s+1,n_0) - a(s,n_0) \right| + \sum_{t=n_0}^{n-1} \left| a(m,t+1) - a(m,t) \right| \end{aligned} \tag{2.6}$$

and $(m_1, n_1) \in \Lambda$ is arbitrarily given on the boundary of the lattice

$$U := \left\{ (m,n) \in \Lambda : W_i(\Upsilon_i(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_i(s,t) \leq \int_{u_i}^{\infty} \frac{dx}{w_i(\psi^{-1}(x))},\ i = 1,2 \right\}. \tag{2.7}$$

Remark 2.2. Different choices of u_i in W_i $(i = 1, 2)$ do not affect our results. For positive constants $v_i \neq u_i$, $i = 1, 2$, let $\widetilde{W}_i(u) = \int_{v_i}^{u} dx / w_i(\psi^{-1}(x))$. Obviously, $\widetilde{W}_i(u) = W_i(u) + \widetilde{W}_i(u_i)$

and $\widetilde{W}_i^{-1}(v) = W_i^{-1}(v - \widetilde{W}_i(u_i))$. It follows that $\widetilde{W}_i^{-1}[\widetilde{W}_i(\Upsilon_i(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_i(s,t)] = W_i^{-1}[W_i(\Upsilon_i(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_i(s,t)]$, that is, we obtain the same expression in (2.5) if we replace W_i with $\widetilde{W}_i$. Moreover, by replacing W_i with $\widetilde{W}_i$, the condition in the definition of U in our theorem reads

$$\widetilde{W}_i(\Upsilon_i(m_1,n_1)) + \sum_{s=m_0}^{m_1-1}\sum_{t=n_0}^{n_1-1} f_i(s,t) \le \int_{v_i}^{\infty} \frac{dx}{w_i(\psi^{-1}(x))}, \tag{2.8}$$

the left-hand side of which is equal to $\widetilde{W}_i(u_i) + W_i(\Upsilon_i(m_1,n_1)) + \sum_{s=m_0}^{m_1-1}\sum_{t=n_0}^{n_1-1} f_i(s,t)$ and the right-hand side of which equals

$$\int_{v_i}^{u_i} \frac{dx}{w_i(\psi^{-1}(x))} + \int_{u_i}^{\infty} \frac{dx}{w_i(\psi^{-1}(x))} = \widetilde{W}_i(u_i) + \int_{u_i}^{\infty} \frac{dx}{w_i(\psi^{-1}(x))}. \tag{2.9}$$

Comparison between both sides implies that (2.8) is equivalent to the condition given in the definition of U in our theorem with $(m,n) = (m_1,n_1)$.

Remark 2.3. If we choose $\psi(u) = u^p$, $\varphi_1(u) = u^q$, $\varphi_2(u) = u^q w(u)$, $f_1(s,t) = a(s,t)$, and $f_2(s,t) = b(s,t)$ with $p > q > 0$ in (1.6) and restrict $a(m,n)$ to be a constant c, then we can apply Theorem 2.1 to inequality (1.5) as discussed in [17].

3. Proof of theorem

First of all, we monotonize some given functions φ_i in the sums. Obviously, $w_1(s)$ and $w_2(s)$, defined by φ_1 and φ_2 in (2.1) and (2.2), are nondecreasing and nonnegative functions and satisfy $w_i(s) \ge \varphi_i(s)$, $i = 1,2$. Moreover, we can check that the ratio $w_2(s)/w_1(s)$ is also nondecreasing. Therefore, from (1.6) we get

$$\psi(u(m,n)) \le a(m,n) + \sum_{i=1}^{2}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_i(s,t) w_i(u(s,t)), \quad \forall (m,n) \in \Lambda. \tag{3.1}$$

We first discuss in the case that $a(m,n) > 0$, for all $(m,n) \in \Lambda$. It means that $\Upsilon_1(m,n) > 0$, for all $(m,n) \in \Lambda$. In such a circumstance, Υ_1 is positive and nondecreasing on Λ and satisfies

$$\Upsilon_1(m,n) = a(m_0,n_0) + \sum_{s=m_0}^{m-1} |a(s+1,n_0) - a(s,n_0)| + \sum_{t=n_0}^{n-1} |a(m,t+1) - a(m,t)| \ge a(m,n). \tag{3.2}$$

Because ψ is strictly increasing, from (3.1) we have

$$\begin{aligned} u(m,n) &\le \psi^{-1}\left[\Upsilon_1(m,n) + \sum_{i=1}^{2}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_i(s,t) w_i(u(s,t))\right] \\ &= \psi^{-1}(\Upsilon_1(m,n) + z(m,n)), \quad \forall (m,n) \in \Lambda, \end{aligned} \tag{3.3}$$

where

$$z(m,n) = \sum_{i=1}^{2}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_i(s,t) w_i(u(s,t)). \tag{3.4}$$

From the properties of f_i and w_i, we see that z is nonnegative and nondecreasing in each variable on Λ. Since Υ_1 is nondecreasing, for arbitrarily fixed pair of integers $(K, L) \in \Lambda_{(m_1,n_1)}$, we observe from (3.3) that

$$u(m,n) \leq \varphi^{-1}\big(\Upsilon_1(K,L) + z(m,n)\big), \quad \forall (m,n) \in \Lambda_{(K,L)}. \tag{3.5}$$

Moreover, we note that w_i is nondecreasing and satisfies $w_i(u) > 0$, for $u > 0$ $(i = 1, 2)$, and that $\Upsilon_1(K,L) + z(m,n) > 0$. It implies by (3.5) that

$$\begin{aligned}\frac{\Delta_1\big(\Upsilon_1(K,L)+z(m,n)\big)}{w_1\big(\varphi^{-1}\big(\Upsilon_1(K,L)+z(m,n)\big)\big)} &= \frac{\sum_{t=n_0}^{n-1} f_1(m,t) w_1\big(u(m,t)\big)}{w_1\big(\varphi^{-1}\big(\Upsilon_1(K,L)+z(m,n)\big)\big)} + \frac{\sum_{t=n_0}^{n-1} f_2(m,t) w_2\big(u(m,t)\big)}{w_1\big(\varphi^{-1}\big(\Upsilon_1(K,L)+z(m,n)\big)\big)} \\ &\leq \sum_{t=n_0}^{n-1} f_1(m,t) + \sum_{t=n_0}^{n-1} f_2(m,t)\theta\big(\varphi^{-1}\big(\Upsilon_1(K,L)+z(m,t)\big)\big),\end{aligned} \tag{3.6}$$

where

$$\theta(u) := \frac{w_2(u)}{w_1(u)}. \tag{3.7}$$

On the other hand, by the mean-value theorem for integrals, for arbitrarily given $(m,n), (m+1,n) \in \Lambda_{(K,L)}$ there exists ξ in the open interval $(\Upsilon_1(K,L)+z(m,n), \Upsilon_1(K,L)+z(m+1,n))$ such that

$$\begin{aligned}&W_1\big(\Upsilon_1(K,L)+z(m+1,n)\big) - W_1\big(\Upsilon_1(K,L)+z(m,n)\big) \\ &\quad = \int_{z(m,n)+\Upsilon_1(K,L)}^{z(m+1,n)+\Upsilon_1(K,L)} \frac{du}{w_1\big(\varphi^{-1}(u)\big)} = \frac{\Delta_1\big(\Upsilon_1(K,L)+z(m,n)\big)}{w_1\big(\varphi^{-1}(\xi)\big)} \leq \frac{\Delta_1\big(\Upsilon_1(K,L)+z(m,n)\big)}{w_1\big(\varphi^{-1}\big(\Upsilon_1(K,L)+z(m,n)\big)\big)}\end{aligned} \tag{3.8}$$

by the monotonicity of w_1 and φ. It follows from (3.6) and (3.8) that

$$\begin{aligned}&W_1\big(\Upsilon_1(K,L)+z(m+1,n)\big) - W_1\big(\Upsilon_1(K,L)+z(m,n)\big) \\ &\quad \leq \sum_{t=n_0}^{n-1} f_1(m,t) + \sum_{t=n_0}^{n-1} f_2(m,t)\theta\big(\varphi^{-1}\big(\Upsilon_1(K,L)+z(m,t)\big)\big).\end{aligned} \tag{3.9}$$

Keep n fixed and substitute m with s in (3.9). Then, taking the sum on both sides of (3.9) over $s = m_0, m_0+1, m_0+2, \ldots, m-1$, we get

$$\begin{aligned}&W_1\big(\Upsilon_1(K,L)+z(m,n)\big) \\ &\quad \leq W_1\big(\Upsilon_1(K,L)\big) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_1(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_2(s,t)\theta\big(\varphi^{-1}\big(\Upsilon_1(K,L)+z(s,t)\big)\big),\end{aligned} \tag{3.10}$$

for all $(m,n) \in \Lambda_{(K,L)}$, where we note from the definition of $z(m,n)$ in (3.3) and the remark about $\sum_{s=m_0}^{m_0-1}$ in the second paragraph of Section 2 that $z(m_0,n) = 0$. For convenience, let

$$\Xi(m,n) := W_1\big(\Upsilon_1(K,L) + z(m,n)\big), \tag{3.11}$$

$$\sigma(m,n) := W_1\big(\Upsilon_1(K,L)\big) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_1(s,t). \tag{3.12}$$

Then, (3.10) can be rewritten as

$$\Xi(m,n) \le \sigma(K,L) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_2(s,t)\theta\Big(\psi^{-1}\Big(W_1^{-1}\big(\Xi(s,t)\big)\Big)\Big), \tag{3.13}$$

for all $(m,n) \in \Lambda_{(K,L)}$, where we note that $\sigma(K,L) \ge \sigma(m,n)$, for all $(m,n) \in \Lambda_{(K,L)}$. Let $g(m,n)$ denote the function on the right-hand side of (3.13), which is obviously a positive function and nondecreasing in each variable. Since the composition $\theta(\psi^{-1}(W_1^{-1}(u)))$ is also nondecreasing in u, by (3.13), that is the fact that $\Xi(m,n) \le g(m,n)$, we have

$$\begin{aligned}
\frac{\Delta_1\big(g(m,n)\big)}{\theta\Big(\psi^{-1}\Big(W_1^{-1}\big(g(m,n)\big)\Big)\Big)} &= \frac{\sum_{t=n_0}^{n-1} f_2(m,t)\theta\Big(\psi^{-1}\Big(W_1^{-1}\big(\Xi(m,t)\big)\Big)\Big)}{\theta\Big(\psi^{-1}\Big(W_1^{-1}\big(g(m,n)\big)\Big)\Big)} \\
&\le \frac{\sum_{t=n_0}^{n-1} f_2(m,t)\theta\Big(\psi^{-1}\Big(W_1^{-1}\big(g(m,t)\big)\Big)\Big)}{\theta\Big(\psi^{-1}\Big(W_1^{-1}\big(g(m,n)\big)\Big)\Big)} \\
&\le \sum_{t=n_0}^{n-1} f_2(m,t).
\end{aligned} \tag{3.14}$$

In order to estimate the left-hand side of (3.14) further, we consider the following integral:

$$\begin{aligned}
\int_{g(m,n)}^{g(m+1,n)} \frac{dx}{\theta\big(\psi^{-1}\big(W_1^{-1}(x)\big)\big)} &= \int_{g(m,n)}^{g(m+1,n)} \frac{w_1\big(\psi^{-1}\big(W_1^{-1}(x)\big)\big)dx}{w_2\big(\psi^{-1}\big(W_1^{-1}(x)\big)\big)} \\
&= \int_{W_1^{-1}(g(m,n))}^{W_1^{-1}(g(m+1,n))} \frac{dx}{w_2\big(\psi^{-1}(x)\big)} \\
&= W_2\Big(W_1^{-1}\big(g(m+1,n)\big)\Big) - W_2\Big(W_1^{-1}\big(g(m,n)\big)\Big),
\end{aligned} \tag{3.15}$$

where we note the definitions of W_1, W_2, and θ in (2.3), (2.4), and (3.7). Applying the mean-value theorem to (3.15), we see that for arbitrarily given $(m,n), (m+1,n) \in \Lambda_{(K,L)}$, there exists η in the open interval $(g(m,n), g(m+1,n))$ such that

$$\int_{g(m,n)}^{g(m+1,n)} \frac{dx}{\theta\big(\psi^{-1}\big(W_1^{-1}(x)\big)\big)} = \frac{\Delta_1\big(g(m,n)\big)}{\theta\big(\psi^{-1}\big(W_1^{-1}(\eta)\big)\big)} \le \frac{\Delta_1\big(g(m,n)\big)}{\theta\Big(\psi^{-1}\Big(W_1^{-1}\big(g(m,n)\big)\Big)\Big)}. \tag{3.16}$$

Thus, it follows from (3.14), (3.15), and (3.16) that

$$W_2\Big(W_1^{-1}\big(g(m+1,n)\big)\Big) - W_2\Big(W_1^{-1}\big(g(m,n)\big)\Big) \le \sum_{t=n_0}^{n-1} f_2(m,t), \tag{3.17}$$

for all $(m,n) \in \Lambda_{(K,L)}$. Furthermore, using the same procedure as done for (3.9), we keep n fixed and setting $m = s$ in (3.17). Then, summing up both sides of (3.17) over $s = m_0, m_0 + 1, m_0 + 2, \ldots, m-1$, we get

$$W_2\left(W_1^{-1}\left(g(m,n)\right)\right) \leq W_2\left(W_1^{-1}\left(\sigma(K,L)\right)\right) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_2(s,t), \tag{3.18}$$

for all $(m,n) \in \Lambda_{(K,L)}$, where we note the fact that $g(m_0,n) = \sigma(K,L)$ and the definition of σ in (3.12). By the monotonicity of W_1 and ψ, the fact that $\Xi(m,n) \leq g(m,n)$, given in (3.13), and inequality (3.18), we obtain from (3.5) that

$$\begin{aligned} u(m,n) &\leq \psi^{-1}\left(\Upsilon_1(K,L) + z(m,n)\right) \\ &= \psi^{-1}\left(W_1^{-1}\left(\Xi(m,n)\right) \leq \psi^{-1}\left(W_1^{-1}\left(g(m,n)\right)\right)\right. \\ &\leq \psi^{-1}\left\{W_2^{-1}\left[W_2\left(W_1^{-1}\left(\sigma(K,L)\right)\right) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_2(s,t)\right]\right\}, \end{aligned} \tag{3.19}$$

for all $(m,n) \in \Lambda_{(K,L)}$, where we note the definitions of Ξ in (3.11) and g just after (3.13). This result also implies the particular case that

$$u(K,L) \leq \psi^{-1}\left\{W_2^{-1}\left[W_2\left(W_1^{-1}\left(W_1\left(\Upsilon_1(K,L)\right) + \sum_{s=m_0}^{K-1}\sum_{t=n_0}^{L-1} f_1(s,t)\right)\right) + \sum_{s=m_0}^{K-1}\sum_{t=n_0}^{L-1} f_2(s,t)\right]\right\}. \tag{3.20}$$

For the arbitrary choice of $(K,L) \in \Lambda_{(m_1,n_1)}$, it also implies that (2.5) holds for all $(m,n) \in \Lambda_{(m_1,n_1)}$.

The remainder case is that $a(m,n) = 0$, for some $(m,n) \in \Lambda$. Let

$$\Upsilon_{1,\varepsilon}(m,n) = \Upsilon_1(m,n) + \varepsilon, \tag{3.21}$$

where $\varepsilon > 0$ is an arbitrary small number. Obviously, $\Upsilon_{1,\varepsilon}(m,n) > 0$, for all $(m,n) \in \Lambda$. Using the same arguments as above, where $\Upsilon_1(m,n)$ is replaced with $\Upsilon_{1,\varepsilon}(m,n)$, we get

$$u(m,n) \leq \psi^{-1}\left\{W_2^{-1}\left[W_2\left(W_1^{-1}\left(W_1\left(\Upsilon_{1,\varepsilon}(m,n)\right) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_1(s,t)\right)\right) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_2(s,t)\right]\right\}, \tag{3.22}$$

for all $(m,n) \in \Lambda_{(m_1,n_1)}$. Letting $\varepsilon \to 0_+$, we obtain (2.5) because of continuity of $\Upsilon_{i,\varepsilon}$ in ε and continuity of W_i and W_i^{-1}, for $i = 1,2$. This completes the proof.

Remark that m_1 and n_1 lie on the boundary of the lattice U. In particular, (2.5) is true for all $(m,n) \in \Lambda$ when all w_is $(i = 1,2)$ satisfy $\int_{u_i}^{\infty} dx/w_i(\psi^{-1}(x)) = \infty$, so we may take $m_1 = M, n_1 = N$.

4. Applications to a difference equation

In this section, we apply our result to the following boundary value problem (simply called BVP) for the partial difference equation:

$$\begin{aligned} &\Delta_1\Delta_2\psi(z(m,n)) = F(m,n,z(m,n)), \quad (m,n)\in\Lambda,\\ &z(m,n_0) = f(m), \quad z(m_0,n) = g(n), \quad (m,n)\in\Lambda, \end{aligned} \tag{4.1}$$

where $\Lambda := I\times J$ is defined as in the beginning of Section 2, $\psi\in C^0(\mathbb{R},\mathbb{R})$ is a strictly increasing odd function satisfying $\psi(u)>0$, for $u>0$, $F:\Lambda\times\mathbb{R}\to\mathbb{R}$ satisfies

$$|F(m,n,u)| \le h_1(m,n)\varphi_1(|u|) + h_2(m,n)\varphi_2(|u|) \tag{4.2}$$

for given functions $h_1,h_2:\Lambda\to\mathbb{R}_+$ and $\varphi_i\in C^0(\mathbb{R}_+,\mathbb{R}_+)$ $(i=1,2)$ satisfying $\varphi_i(u)>0$, for $u>0$, and functions $f:I\to\mathbb{R}$ and $g:J\to\mathbb{R}$ satisfy $f(m_0)=g(n_0)=0$. Obviously, (4.1) is a generalization of the BVP problem considered in [17, Section 3]. So the results of [17] cannot be applied immediately. In what follows we first apply our main result to discuss boundedness of solutions of (4.1).

Corollary 4.1. *All solutions $z(m,n)$ of BVP* (4.1) *have the estimate*

$$|z(m,n)| \le \psi^{-1}\left\{W_2^{-1}\left[W_2(\Upsilon_2(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}h_2(s,t)\right]\right\}, \tag{4.3}$$

for all $(m,n)\in\Lambda_{(m_1,n_1)}$, where m_1,n_1 are given as in Theorem 2.1 and

$$\begin{aligned} W_2(u) &= \int_1^u dx\Big/\left\{\max_{\tau\in[0,x]}\left\{\frac{\varphi_2(\psi^{-1}(\tau))}{\max_{\tau_1\in[0,\tau]}\{\varphi_1(\psi^{-1}(\tau_1))\}}\right\}\max_{\tau\in[0,x]}\{\varphi_1(\psi^{-1}(\tau))\}\right\},\\ W_1(u) &= \int_1^u dx\Big/\max_{\tau\in[0,x]}\{\varphi_1(\psi^{-1}(\tau))\},\\ \Upsilon_2(m,n) &= W_1^{-1}\left[W_1(\Upsilon_1(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}h_1(t,s)\right],\\ \Upsilon_1(m,n) &\le \sum_{s=m_0}^{m-1}|\psi(f(s+1))-\psi(f(s))| + \sum_{t=n_0}^{n-1}|\psi(g(t+1))-\psi(g(t))|. \end{aligned} \tag{4.4}$$

Proof. Clearly, the difference equation of BVP (4.1) is equivalent to

$$\psi(z(m,n)) = \psi(f(m)) + \psi(g(n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}F(s,t,z(s,t)). \tag{4.5}$$

It follows that

$$|\psi(z(m,n))| \le |\psi(f(m))+\psi(g(n))| + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}h_1(s,t)\varphi_1(|z(s,t)|) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}h_2(s,t)\varphi_2(|z(s,t)|) \tag{4.6}$$

by (4.2). Let $a(m,n)=|\psi(f(m))+\psi(g(n))|$. Since $|\psi(z(m,n))|=\psi(|z(m,n)|)$, (4.6) is of the form (1.6). Applying Theorem 2.1 to inequality (4.6), we obtain the estimate of $z(m,n)$ as given in this corollary. □

Corollary 4.1 gives a condition of boundedness for solutions. Concretely, if

$$\Upsilon_1(m,n) < \infty, \quad \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_1(s,t) < \infty, \quad \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_2(s,t) < \infty, \tag{4.7}$$

for all $(m,n) \in \Lambda_{(m_1,n_1)}$, then every solution $z(m,n)$ of BVP (4.1) is bounded on $\Lambda_{(m_1,n_1)}$.

Next, we discuss the uniqueness of solutions for BVP (4.1).

Corollary 4.2. *Suppose additionally that*

$$\left|F(m,n,u_1) - F(m,n,u_2)\right| \le h_1(m,n)\varphi_1\left(\left|\psi(u_1) - \psi(u_2)\right|\right) + h_2(m,n)\varphi_2\left(\left|\psi(u_1) - \psi(u_2)\right|\right), \tag{4.8}$$

for $u_1, u_2 \in \mathbb{R}$ and $(m,n) \in \Lambda := I \times J$, where $I = [m_0, M) \cap \mathbb{N}_0$, $J = [n_0, N) \cap \mathbb{N}_0$ as assumed in the beginning of Section 2 with natural numbers M and N, h_1, h_2 are both nonnegative functions defined on the lattice Λ, $\varphi_1, \varphi_2 \in C^0(\mathbb{R}_+, \mathbb{R}_+)$ are both nondecreasing with the nondecreasing ratio φ_2/φ_1 such that $\varphi_i(0) = 0$, $\varphi_i(u) > 0$, for all $u > 0$ and $\int_0^1 ds/\varphi_i(s) = +\infty$, for $i = 1, 2$, and $\psi \in C^0(\mathbb{R}, \mathbb{R})$ is a strictly increasing odd function satisfying $\psi(u) > 0$, for $u > 0$. Then, BVP (4.1) has at most one solution on Λ.

Proof. Assume that both $z(m,n)$ and $\tilde{z}(m,n)$ are solutions of BVP (4.1). From the equivalent form (4.5) of (4.1), we have

$$\begin{aligned} \left|\psi(z(m,n)) - \psi(\tilde{z}(m,n))\right| \le{}& \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_1(s,t)\varphi_1\left(\left|\psi(z(s,t)) - \psi(\tilde{z}(s,t))\right|\right) \\ &+ \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_2(s,t)\varphi_2\left(\left|\psi(z(s,t)) - \psi(\tilde{z}(s,t))\right|\right), \end{aligned} \tag{4.9}$$

for all $(m,n) \in \Lambda$, which is an inequality of the form (1.6), where $a(m,n) \equiv 0$. Applying Theorem 2.1 with the choice that $u_1 = u_2 = 1$, we obtain an estimate of the difference $|\psi(z(m,n)) - \psi(\tilde{z}(m,n))|$ in the form (2.5), where $\Upsilon_1(m,n) \equiv 0$, because $a(m,n) \equiv 0$. Furthermore, by the definition of W_i we see that

$$\lim_{u\to 0} W_i(u) = -\infty, \quad \lim_{u\to -\infty} W_i^{-1}(u) = 0, \quad i = 1, 2. \tag{4.10}$$

It follows that

$$W_1\left(\Upsilon_1(m,n)\right) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_1(s,t) = -\infty \tag{4.11}$$

since $m < M, n < N$. Thus, by (4.10)

$$\Upsilon_2(m,n) = W_1^{-1}\left[W_1\left(\Upsilon_1(m,n)\right) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_1(s,t)\right] = 0. \tag{4.12}$$

Similarly, we get $W_2(\Upsilon_2(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_2(s,t) = -\infty$ and therefore

$$W_2^{-1}\left[W_2\left(\Upsilon_2(m,n)\right) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_2(s,t)\right] = 0. \tag{4.13}$$

Thus, we conclude from (2.5) that $|\psi(z(m,n))-\psi(\widetilde{z}(m,n))| \le 0$, implying that $z(m,n) = \widetilde{z}(m,n)$, for all $(m,n) \in \Lambda$ since ψ is strictly increasing. It proves the uniqueness. □

Remark 4.3. If $h_1 \equiv 0$ or $h_2 \equiv 0$ in (4.8), the conclusion of Corollary 4.2 also can be obtained.

Finally, we discuss the continuous dependence of solutions of BVP (4.1) on the given functions F, f, and g. Consider a variation of BVP (4.1)

$$\begin{gathered} \Delta_1 \Delta_2 \psi(z(m,n)) = \widetilde{F}(m,n,z(m,n)), \quad (m,n) \in \Lambda, \\ z(m,n_0) = \widetilde{f}(m), \quad z(m_0,n) = \widetilde{g}(n), \quad (m,n) \in \Lambda, \end{gathered} \tag{4.14}$$

where $\psi \in C^0(\mathbb{R},\mathbb{R})$ is a strictly increasing odd function satisfying $\psi(u) > 0$ for $u > 0$, $\widetilde{F} \in C^0(\Lambda \times \mathbb{R},\mathbb{R})$, and $\widetilde{f} : I \to \mathbb{R}$, $\widetilde{g} : J \to \mathbb{R}$ are functions satisfying $\widetilde{f}(m_0) = \widetilde{g}(n_0) = 0$.

Corollary 4.4. *Let F be a function as assumed in the beginning of Section 4 and satisfy* (4.2) *and* (4.8) *on the same lattice Λ as assumed in* Corollary 4.2. *Suppose that the three differences*

$$\max_{m\in I}\left|\widetilde{f} - f\right|, \qquad \max_{n\in J}\left|\widetilde{g} - g\right|, \qquad \max_{(s,t,u)\in\Lambda\times\mathbb{R}}\left|\widetilde{F}(s,t,u) - F(s,t,u)\right| \tag{4.15}$$

are all sufficiently small. Then, solution $\widetilde{z}(m,n)$ of BVP (4.14) *is sufficiently close to the solution $z(m,n)$ of BVP* (4.1).

Proof. By Corollary 4.2, the solution $z(m,n)$ is unique. By the continuity and the strict monotonicity of ψ, we suppose that

$$\begin{gathered} \max_{m\in I}\left|\psi(\widetilde{f}(m)) - \psi(f(m))\right| < \epsilon, \qquad \max_{n\in J}\left|\psi(\widetilde{g}(n)) - \psi(g(n))\right| < \epsilon, \\ \max_{(s,t,u)\in I\times J\times\mathbb{R}}\left|\widetilde{F}(s,t,u) - F(s,t,u)\right| < \epsilon, \end{gathered} \tag{4.16}$$

where $\epsilon > 0$ is a small number. By the equivalent difference equation (4.5) and inequality (4.8), we get

$$\begin{aligned} &\left|\psi(\widetilde{z}(m,n)) - \psi(z(m,n))\right| \\ &\quad\le \left|\psi(\widetilde{f}(m)) - \psi(f(m)) + \psi(\widetilde{g}(n)) - \psi(g(n))\right| + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|\widetilde{F}(s,t,\widetilde{z}(s,t)) - F(s,t,z(s,t))\right| \\ &\quad\le 2\epsilon + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|\widetilde{F}(s,t,\widetilde{z}(s,t)) - F(s,t,\widetilde{z}(s,t))\right| + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|F(s,t,\widetilde{z}(s,t)) - F(s,t,z(s,t))\right| \\ &\quad\le \{2 + (m_1 - m_0)(n_1 - n_0)\}\epsilon + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_1(s,t)\varphi_1(\left|\psi(\widetilde{z}(s,t)) - \psi(z(s,t))\right|) \\ &\qquad + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_2(s,t)\varphi_2(\left|\psi(\widetilde{z}(s,t)) - \psi(z(s,t))\right|), \end{aligned} \tag{4.17}$$

that is, an inequality of the form (1.6). Applying Theorem 2.1 to (4.17), we obtain

$$\left|\psi\left(\tilde{z}(m,n)\right)-\psi\left(z(m,n)\right)\right| \le W_2^{-1}\left[W_2\left(\Upsilon_2(m,n)\right)+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}h_2(s,t)\right], \tag{4.18}$$

for all $(m,n) \in \Lambda_{(m_1,n_1)}$, where m_1, n_1 are given as in Theorem 2.1,

$$\begin{aligned}\Upsilon_2(m,n) &= W_1^{-1}\left[W_1\left(\Upsilon_1(m,n)\right)+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}h_1(t,s)\right],\\ \Upsilon_1(m,n) &= \left\{2+\left(m_1-m_0\right)\left(n_1-n_0\right)\right\}\epsilon.\end{aligned} \tag{4.19}$$

By (4.10) we see that $\Upsilon_i(m,n) \to 0$ $(i = 1,2)$ as $\epsilon \to 0$. It follows from (4.18) that $\lim_{\epsilon\to 0}|\psi(\tilde{z}(m,n) - \psi(z(m,n))| = 0$, and hence $z(m,n)$ depends continuously on F, f, and g since ψ is strictly increasing. □

Our requirement on the small difference, $\tilde{F} - F$ in Corollary 4.4, is stronger than the condition (iii) in [17, Theorem 3.3], but ours may be easier to check because one has to verify the inequality in his condition (iii) for each solution $\tilde{z}(m,n)$ of BVP (4.14).

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