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Research Article

A Functional Equation of Aczél and Chung in Generalized Functions

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We consider an n-dimensional version of the functional equations of Aczél and Chung in the spaces of generalized functions such as the Schwartz distributions and Gelfand generalized functions. As a result, we prove that the solutions of the distributional version of the equation coincide with those of classical functional equation.

1. Introduction

In [1], Aczél and Chung introduced the following functional equation:

$$\sum_{j=1}^{l} f_j(\alpha_j x + \beta_j y) = \sum_{k=1}^{m} g_k(x)h_k(y), \tag{1.1}$$

where $f_j, g_k, h_k : \mathbb{R} \to \mathbb{C}$ and $\alpha_j, \beta_j \in \mathbb{R}$ for $j = 1, \dots, l$, $k = 1, \dots, m$. Under the natural assumptions that $\{g_1, \dots, g_m\}$ and $\{h_1, \dots, h_m\}$ are linearly independent, and $\alpha_j\beta_j \neq 0$, $\alpha_i\beta_j \neq \alpha_j\beta_i$ for all $i \neq j$, $i, j = 1, \dots, l$, it was shown that the locally integrable solutions of (1.1) are *exponential polynomials*, that is, the functions of the form

$$\sum_{k=1}^{q} e^{r_k x} p_k(x), \tag{1.2}$$

where $r_k \in \mathbb{C}$ and p_k's are polynomials for all $k = 1, 2, \dots, q$.

In this paper, we introduce the following n-dimensional version of the functional equation (1.1) in generalized functions:

$$\sum_{j=1}^{l} u_j \circ T_j = \sum_{k=1}^{m} v_k \otimes w_k, \tag{1.3}$$

where $u_j, v_k, w_k \in \mathfrak{D}'(\mathbb{R}^n)$ (resp., $\mathcal{S}'^{1/2}_{1/2}(\mathbb{R}^n)$), and $\circ$ denotes the pullback, $\otimes$ denotes the tensor product of generalized functions, and $T_j(x,y) = \alpha_j x + \beta_j y$, $\alpha_j = (\alpha_{j,1}, \ldots, \alpha_{j,n})$, $\beta_j = (\beta_{j,1}, \ldots, \beta_{j,n})$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $\alpha_j x = (\alpha_{j,1}x_1, \ldots, \alpha_{j,n}x_n)$, $\beta_j y = (\beta_{j,1}y_1, \ldots, \beta_{j,n}y_n)$, $j = 1, \ldots, l$. As in [1], we assume that $\alpha_{j,p}\beta_{j,p} \neq 0$ and $\alpha_{i,p}\beta_{j,p} \neq \alpha_{j,p}\beta_{i,p}$ for all $p = 1, \ldots, n$, $i \neq j$, $i, j = 1, \ldots, l$.

In [2], Baker previously treated (1.3). By making use of differentiation of distributions which is one of the most powerful advantages of the Schwartz theory, and reducing (1.3) to a system of differential equations, he showed that, for the dimension $n = 1$, the solutions of (1.3) are *exponential polynomials*. We refer the reader to [2–6] for more results using this method of reducing given functional equations to differential equations.

In this paper, by employing tensor products of regularizing functions as in [7, 8], we consider the regularity of the solutions of (1.3) and prove in an elementary way that (1.3) can be reduced to the classical equation (1.1) of smooth functions. This method can be applied to prove the Hyers-Ulam stability problem for functional equation in Schwartz distribution [7, 8]. In the last section, we consider the Hyers-Ulam stability of some related functional equations. For some elegant results on the classical Hyers-Ulam stability of functional equations, we refer the reader to [6, 9–21].

2. Generalized functions

In this section, we briefly introduce the spaces of generalized functions such as the Schwartz distributions, Fourier hyperfunctions, and Gelfand generalized functions. Here we use the following notations: $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1!, \ldots, \alpha_n!$, $x^\alpha = x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}$, and $\partial^\alpha = \partial_1^{\alpha_1}, \ldots, \partial_n^{\alpha_n}$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of nonnegative integers and $\partial_j = \partial/\partial x_j$.

Definition 2.1. A distribution u is a linear functional on $C_c^\infty(\mathbb{R}^n)$ of infinitely differentiable functions on $\mathbb{R}^n$ with compact supports such that for every compact set $K \subset \mathbb{R}^n$ there exist constants C and k satisfying

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq k} \sup \left| \partial^\alpha \varphi \right| \tag{2.1}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ with supports contained in K. One denotes by $\mathfrak{D}'(\mathbb{R}^n)$ the space of the Schwartz distributions on $\mathbb{R}^n$.

Definition 2.2. For given $r, s \geq 0$, one denotes by $\mathcal{S}_r^s$ or $\mathcal{S}_r^s(\mathbb{R}^n)$ the space of all infinitely differentiable functions $\varphi(x)$ on $\mathbb{R}^n$ such that there exist positive constants h and k satisfying

$$\|\varphi\|_{h,k} := \sup_{x \in \mathbb{R}^n,\, \alpha, \beta \in \mathbb{N}_0^n} \frac{\left| x^\alpha \partial^\beta \varphi(x) \right|}{h^{|\alpha|} k^{|\beta|} \alpha!^r \beta!^s} < \infty. \tag{2.2}$$

The topology on the space $\mathcal{S}_r^s$ is defined by the seminorms $\|\cdot\|_{h,k}$ in the left-hand side of (2.2), and the elements of the dual space $\mathcal{S}'^s_r$ of $\mathcal{S}_r^s$ are called *Gelfand-Shilov generalized functions*. In particular, one denotes $\mathcal{S}'^1_1$ by $\mathcal{F}'$ and calls its elements *Fourier hyperfunctions*.

It is known that if $r > 0$ and $0 \le s < 1$, the space $\mathcal{S}_r^s(\mathbb{R}^n)$ consists of all infinitely differentiable functions $\varphi(x)$ on $\mathbb{R}^n$ that can be continued to an entire function on $\mathbb{C}^n$ satisfying

$$|\varphi(x+iy)| \le C \exp\left(-a|x|^{1/r} + b|y|^{1/(1-s)}\right) \tag{2.3}$$

for some $a, b > 0$.

It is well known that the following topological inclusions hold:

$$\mathcal{S}_{1/2}^{1/2} \hookrightarrow \mathcal{F}, \qquad \mathcal{F}' \hookrightarrow \mathcal{S}'^{1/2}_{1/2}. \tag{2.4}$$

We briefly introduce some basic operations on the spaces of the generalized functions.

Definition 2.3. Let $u \in \mathfrak{D}'(\mathbb{R}^n)$. Then, the kth partial derivative $\partial_k u$ of u is defined by

$$\langle \partial_k u, \varphi \rangle = -\langle u, \partial_k \varphi \rangle \tag{2.5}$$

for $k = 1, \dots, n$. Let $f \in C^\infty(\mathbb{R}^n)$. Then the multiplication fu is defined by

$$\langle fu, \varphi \rangle = \langle u, f\varphi \rangle. \tag{2.6}$$

Definition 2.4. Let $u_j \in \mathfrak{D}'(\mathbb{R}^{n_j})$, $j = 1, 2$. Then, the tensor product $u_1 \otimes u_2$ of u_1 and u_2 is defined by

$$\langle u_1 \otimes u_2, \varphi(x_1, x_2) \rangle = \langle u_1, \langle u_2, \varphi(x_1, x_2) \rangle \rangle, \quad \varphi(x_1, x_2) \in C_c^\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}). \tag{2.7}$$

The tensor product $u_1 \otimes u_2$ belongs to $\mathfrak{D}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

Definition 2.5. Let $u_j \in \mathfrak{D}'(\mathbb{R}^{n_j})$, $j = 1, 2$, and let $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ be a smooth function such that for each $x \in \mathbb{R}^{n_1}$ the derivative $f'(x)$ is surjective. Then there exists a unique continuous linear map $f^* : \mathfrak{D}'(\mathbb{R}^{n_2}) \to \mathfrak{D}'(\mathbb{R}^{n_1})$ such that $f^*u = u \circ f$, when u is a continuous function. One calls f^*u the pullback of u by f and simply is denoted by $u \circ f$.

The differentiations, pullbacks, and tensor products of Fourier hyperfunctions and Gelfand generalized functions are defined in the same way as distributions. For more details of tensor product and pullback of generalized functions, we refer the reader to [9, 22].

3. Main result

We employ a function $\psi \in C^{\infty}(\mathbb{R}^n)$ such that

$$\begin{gathered}\psi(x) \geq 0 \quad \forall x \in \mathbb{R}^n, \\ \operatorname{supp} \psi \subset \{x \in \mathbb{R}^n : |x| \leq 1\}, \\ \int_{\mathbb{R}^n} \psi(x)\, dx = 1.\end{gathered} \tag{3.1}$$

Let $u \in \mathfrak{D}'(\mathbb{R}^n)$ and $\psi_t(x) := t^{-n}\psi(x/t)$, $t > 0$. Then, for each $t > 0$, $(u * \psi_t)(x) := \langle u_y, \psi_t(x-y)\rangle$ is well defined. We call $(u*\psi_t)(x)$ a regularizing function of the distribution u, since $(u * \psi_t)(x)$ is a smooth function of x satisfying $(u * \psi_t)(x) \to u$ as $t \to 0^+$ in the sense of distributions, that is, for every $\varphi \in C_c^{\infty}(\mathbb{R}^n)$,

$$\langle u, \varphi\rangle = \lim_{t\to 0^+}\int (u * \psi_t)(x)\varphi(x)\, dx. \tag{3.2}$$

Theorem 3.1. *Let $u_j, v_k, w_k \in \mathfrak{D}'(\mathbb{R}^n)$, $j = 1,\ldots,l$, $k = 1,\ldots,m$, be a solution of* (1.3)*, and both $\{v_1,\ldots,v_m\}$ and $\{w_1,\ldots,w_m\}$ are linearly independent. Then, $u_j = f_j$, $v_k = g_k$, $w_k = h_k$, $j = 1,\ldots,l$, $k = 1,\ldots,m$, where $f_j, g_k, h_k : \mathbb{R}^n \to \mathbb{C}$, $j = 1,\ldots,l$, $k = 1,\ldots,m$, a smooth solution of* (1.1).

Proof. By convolving the tensor product $\psi_t(x)\psi_s(y)$ in each side of (1.3), we have, for $j = 1,\ldots,l$,

$$\begin{aligned}\left[(u_j \circ T_j) * (\psi_t(x)\psi_s(y))\right](\xi,\eta) &= \langle u_j \circ T_j, \psi_t(\xi - x)\psi_s(\eta - y)\rangle \\ &= \left\langle u_j, \int \left|\alpha_j\right|^{-1}\psi_t(\alpha_j^{-1}(\alpha_j\xi - x + y))\left|\beta_j\right|^{-1}\psi_s(\beta_j^{-1}(\beta_j\eta - y))\, dy\right\rangle \\ &= \left\langle u_j, \int \psi_{t,\alpha_j}(\alpha_j\xi - x + y)\psi_{s,\beta_j}(\beta_j\eta - y)\, dy\right\rangle \\ &= \langle u_j, (\psi_{t,\alpha_j} * \psi_{s,\beta_j})(\alpha_j\xi + \beta_j\eta - x)\rangle \\ &= (u_j * \psi_{t,\alpha_j} * \psi_{s,\beta_j})(\alpha_j\xi + \beta_j\eta),\end{aligned} \tag{3.3}$$

where $|\alpha_j| = \alpha_{j,1},\ldots,\alpha_{j,n}$, $\alpha_j^{-1} = (\alpha_{j,1}^{-1},\ldots,\alpha_{j,n}^{-1})$, $\psi_{t,\alpha_j}(x) = |\alpha_j|^{-1}\psi_t(\alpha_j^{-1}x)$. Similarly we have for $k = 1,\ldots,m$,

$$\left[(v_k \otimes w_k) * (\psi_t(x)\psi_s(y))\right](\xi,\eta) = (v_k * \psi_t)(\xi)(w_k * \psi_s)(\eta). \tag{3.4}$$

Thus (1.3) is converted to the following functional equation:

$$\sum_{j=1}^{l} F_j(x,y,t,s) = \sum_{k=1}^{m} G_k(x,t)H_k(y,s), \tag{3.5}$$

where

$$F_j(x,y,t,s) = \left(u_j * \psi_{t,\alpha_j} * \psi_{s,\beta_j}\right)(\alpha_j x + \beta_j y),$$
$$G_k(x,t) = (v_k * \psi_t)(x), \qquad H_k(y,s) = (w_k * \psi_s)(y), \tag{3.6}$$

for $j = 1,\dots,l$, $k = 1,\dots,m$. We first prove that $\lim_{t\to 0^+} G_k(x,t)$ are smooth functions and equal to v_k for all $k = 1,\dots,m$. Let

$$F(x,y,t,s) = \sum_{j=1}^{l} F_j(x,y,t,s). \tag{3.7}$$

Then,

$$\lim_{t\to 0^+} F(x,y,t,s) = \sum_{j=1}^{l} \left(u_j * \psi_{s,\beta_j}\right)(\alpha_j x + \beta_j y) \tag{3.8}$$

is a smooth function of x for each $y \in \mathbb{R}^n$, $s > 0$, and $\{H_1,\dots,H_m\}$ is linearly independent. We may choose $y_m \in \mathbb{R}^n$, $s_m > 0$ such that $H_m(y_m,s_m) := b_m^{(0)} \neq 0$. Then, it follows from (3.5) that

$$G_m(x,t) = b_m^{(0)^{-1}} \left(F(x,y_m,t,s_m) - \sum_{k=1}^{m-1} b_k^{(0)} G_k(x,t) \right), \tag{3.9}$$

where $b_k^{(0)} = H_k(y_m,s_m)$, $k = 1,\dots,m-1$. Putting (3.9) in (3.5), we have

$$F^{(1)}(x,y,t,s) = \sum_{k=1}^{m-1} G_k(x,t) H_k^{(1)}(y,s), \tag{3.10}$$

where

$$F^{(1)}(x,y,t,s) = F(x,y,t,s) - b_m^{(0)^{-1}} F(x,y_m,t,s_m) H_m(y,s), \tag{3.11}$$

$$H_k^{(1)}(y,s) = H_k(y,s) - b_m^{(0)^{-1}} b_k^{(0)} H_m(y,s), \quad k = 1,\dots,m-1. \tag{3.12}$$

Since $\lim_{t\to 0^+} F(x,y,t,s)$ is a smooth function of x for each $y \in \mathbb{R}^n$, $s > 0$, it follows from (3.11) that

$$\lim_{t\to 0^+} F^{(1)}(x,y,t,s) \tag{3.13}$$

is a smooth function of x for each $y \in \mathbb{R}^n$, $s > 0$. Also, since $\{H_1,\dots,H_m\}$ is linearly independent, it follows from (3.12) that

$$\left\{H_1^{(1)},\dots,H_{m-1}^{(1)}\right\} \tag{3.14}$$

is linearly independent. Thus we can choose $y_{m-1} \in \mathbb{R}^n$, $s_{m-1} > 0$ such that $H_{m-1}^{(1)}(y_{m-1}, s_{m-1}) := b_{m-1}^{(1)} \neq 0$. Then, it follows from (3.10) that

$$G_{m-1}(x,t) = {b_{m-1}^{(1)}}^{-1}\left(F^{(1)}(x, y_{m-1}, t, s_{m-1}) - \sum_{k=1}^{m-2} b_k^{(1)} G_k(x,t)\right), \tag{3.15}$$

where $b_k^{(1)} = H_k^{(1)}(y_{m-1}, s_{m-1})$, $k = 1, \ldots, m-2$. Putting (3.15) in (3.10), we have

$$F^{(2)}(x,y,t,s) = \sum_{k=1}^{m-2} G_k(x,t) H_k^{(2)}(y,s), \tag{3.16}$$

where

$$\begin{aligned} F^{(2)}(x,y,t,s) &= F^{(1)}(x,y,t,s) - {b_{m-1}^{(1)}}^{-1} F^{(1)}(x, y_{m-1}, t, s_{m-1}) H_{m-1}^{(1)}(y,s), \\ H_k^{(2)}(y,s) &= H_k^{(1)}(y,s) - {b_{m-1}^{(1)}}^{-1} b_k^{(1)} H_{m-1}^{(1)}(y,s), \quad k = 1, \ldots, m-2. \end{aligned} \tag{3.17}$$

By continuing this process, we obtain the following equations:

$$F^{(p)}(x,y,t,s) = \sum_{k=1}^{m-p} G_k(x,t) H_k^{(p)}(y,s), \tag{3.18}$$

for all $p = 0, 1, \ldots, m-1$, where $F^{(0)} = F$, $H_k^{(0)} = H_k$, $k = 1, \ldots, m$,

$$G_{m-p}(x,t) = {b_{m-p}^{(p)}}^{-1}\left(F^{(p)}(x, y_{m-p}, t, s_{m-p}) - \sum_{k=1}^{m-p-1} b_k^{(p)} G_k(x,t)\right), \tag{3.19}$$

for all $p = 0, 1, \ldots, m-2$, and

$$G_1(x,t) = \left(b_1^{(m-1)}\right)^{-1} F^{(m-1)}(x, y_1, t, s_1). \tag{3.20}$$

By the induction argument, we have for each $p = 0, 1, \ldots, m-1$,

$$\lim_{t \to 0^+} F^{(p)}(x,y,t,s) \tag{3.21}$$

is a smooth function of x for each $y \in \mathbb{R}^n$, $s > 0$. Thus, in view of (3.20),

$$g_1(x) := \lim_{t \to 0^+} G_1(x,t) \tag{3.22}$$

is a smooth function. Furthermore, $G_1(x,t)$ converges to $g_1(x)$ locally uniformly, which implies that $v_1 = g_1$ in the sense of distributions, that is, for every $\varphi(x) \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\langle v_1, \varphi\rangle &= \lim_{t\to 0^+}\int G_1(x,t)\varphi(x)\,dx \\ &= \int g_1(x)\varphi(x)\,dx.\end{aligned} \tag{3.23}$$

In view of (3.19) and the induction argument, for each $k = 2,\ldots,m$, we have

$$g_k(x) := \lim_{t\to 0^+} G_k(x,t) \tag{3.24}$$

is a smooth function and $v_k = g_k$ for all $k = 2,3,\ldots,m$. Changing the roles of G_k and H_k for $k = 1,2,\ldots,m$, we obtain, for each $k = 1,2,\ldots,m$,

$$h_k(x) := \lim_{t\to 0^+} H_k(x,t) \tag{3.25}$$

is a smooth function and $w_k = h_k$. Finally, we show that for each $j = 1,2,\ldots,l$, u_j is equal to a smooth function. Letting $s \to 0^+$ in (3.5), we have

$$\sum_{j=1}^{l}\left(u_j * \psi_{t,\alpha_j}\right)\left(\alpha_j x + \beta_j y\right) = \sum_{k=1}^{m} G_k(x,t)h_k(y). \tag{3.26}$$

For each fixed i, $1 \le i \le l$, replacing x by $\alpha_i^{-1}(x - \beta_i y)$, multiplying $\psi_s(y)$ and integrating with respect to y, we have

$$\left(u_i * \psi_{t,\alpha_i}\right)(x) = -\sum_{j\neq i}\left(u_j * \psi_{t,\alpha_j} * \psi_{s,\gamma_j}\right)(x) + \sum_{k=1}^{m}\int G_k\left(\alpha_i^{-1}x - \alpha_i^{-1}\beta_i y, t\right)h_k(y)\psi_s(y)\,dy, \tag{3.27}$$

where $\gamma_j = \alpha_i^{-1}(\beta_i\alpha_j - \alpha_i\beta_j)$ for all $1 \le j \le l$, $j \neq i$. Letting $t \to 0^+$ in (3.27), we have

$$u_i = -\sum_{j\neq i}\left(u_j * \psi_{s,\gamma_j}\right)(x) + \sum_{k=1}^{m}\int g_k\left(\alpha_i^{-1}x - \alpha_i^{-1}\beta_i y\right)h_k(y)\psi_s(y)\,dy := f_i(x). \tag{3.28}$$

It is obvious that f_i is a smooth function. Also it follows from (3.27) that each $(u_i * \psi_t)(x)$, $i = 1,\ldots,l$, converges locally and uniformly to the function $f_i(x)$ as $t \to 0^+$, which implies that the equality (3.28) holds in the sense of distributions. Finally, letting $s \to 0^+$ and $t \to 0^+$ in (3.5) we see that $f_j, g_k, h_k, j = 1,\ldots,l, k = 1,\ldots,m$ are smooth solutions of (1.1). This completes the proof. □

Combined with the result of Aczél and Chung [1], we have the following corollary as a consequence of the above result.

Corollary 3.2. *Every solution* $u_j, v_k, w_k \in \mathfrak{D}'(\mathbb{R})$, $j = 1, \ldots, l$, $k = 1, \ldots, m$, *of* (1.3) *for the dimension* $n = 1$ *has the form of exponential polynomials.*

The result of Theorem 3.1 holds for $u_j, v_k, w_k \in \mathcal{S}'^{1/2}_{1/2}(\mathbb{R}^n)$, $j = 1, \ldots, l$, $k = 1, \ldots, m$. Using the following n-dimensional heat kernel,

$$E_t(x) = (4\pi t)^{-n/2} \exp\left(\frac{-|x|^2}{4t}\right), \quad t > 0. \tag{3.29}$$

Applying the proof of Theorem 3.1, we get the result for the space of Gelfand generalized functions.

4. Hyers-Ulam stability of related functional equations

The well-known Cauchy equation, Pexider equation, Jensen equation, quadratic functional equation, and d'Alembert functional equation are typical examples of the form (1.1). For the distributional version of these equations and their stabilities, we refer the reader to [7, 8]. In this section, as well-known examples of (1.1), we introduce the following trigonometric differences:

$$\begin{aligned} T_1(f,g) &:= f(x+y) - f(x)g(y) - g(x)f(y), \\ T_2(f,g) &:= g(x+y) - g(x)g(y) + f(x)f(y), \\ T_3(f,g) &:= f(x-y) - f(x)g(y) + g(x)f(y), \\ T_4(f,g) &:= g(x-y) - g(x)g(y) - f(x)f(y), \end{aligned} \tag{4.1}$$

where $f, g : \mathbb{R}^n \to \mathbb{C}$. In 1990, Székelyhidi [23] has developed his idea of using invariant subspaces of functions defined on a group or semigroup in connection with stability questions for the sine and cosine functional equations. As the results, he proved that if $T_j(f,g)$, $j = 1,2,3,4$, is a bounded function on $\mathbb{R}^{2n}$, then either there exist $\lambda, \mu \in \mathbb{C}$, not both zero, such that $\lambda f - \mu g$ is a bounded function on $\mathbb{R}^n$, or else $T_j(f,g) = 0$, $j = 1,2,3,4$, respectively. For some other elegant Hyers-Ulam stability theorems, we refer the reader to [6, 9–21].

By generalizing the differences (4.1), we consider the differences

$$\begin{aligned} G_1(u,v) &:= u \circ A - u \otimes v - v \otimes u, \\ G_2(u,v) &:= v \circ A - v \otimes v + u \otimes u, \\ G_3(u,v) &:= u \circ S - u \otimes v + v \otimes u, \\ G_4(u,v) &:= v \circ S - v \otimes v - u \otimes u, \end{aligned} \tag{4.2}$$

and investigate the behavior of $u, v \in \mathcal{S}'^{1/2}_{1/2}(\mathbb{R}^n)$ satisfying the inequality $\|G_j(u,v)\| \le M$ for each $j = 1,2,3,4$, where $A(x,y) = x + y$, $S(x,y) = x - y$, $x, y \in \mathbb{R}^n$, $\circ$ denotes the pullback, $\otimes$ denotes the tensor product of generalized functions as in Theorem 3.1, and $\|G_j(u,v)\| \le M$ means that $|\langle G_j(u,v), \varphi\rangle| \le \|\varphi\|_{L^1}$ for all $\varphi \in S^{1/2}_{1/2}(\mathbb{R}^n)$.

As a result, we obtain the following theorems.

Theorem 4.1. *Let* $u, v \in \mathcal{S}'^{1/2}_{1/2}$ *satisfy* $\|G_1(u,v)\| \le M$. *Then,* u *and* v *satisfy one of the following items:*

(i) $u = 0$, v*: arbitrary,*

(ii) u *and* v *are bounded measurable functions,*

(iii) $u = c \cdot x e^{ia\cdot x} + B(x)$, $v = e^{ia\cdot x}$,

(iv) $u = \lambda(e^{c\cdot x} - B(x))$, $v = (1/2)(e^{c\cdot x} + B(x))$,

(v) $u = \lambda(e^{b\cdot x} - e^{c\cdot x})$, $v = (1/2)(e^{b\cdot x} + e^{c\cdot x})$,

(vi) $u = b \cdot x e^{c\cdot x}$, $v = e^{c\cdot x}$,

where $a \in \mathbb{R}^n$, $b, c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$, *and* B *is a bounded measurable function.*

Theorem 4.2. *Let* $u, v \in \mathcal{S}'^{1/2}_{1/2}$ *satisfy* $\|G_2(u,v)\| \le M$. *Then,* u *and* v *satisfy one of the following items:*

(i) u *and* v *are bounded measurable functions,*

(ii) $v = e^{c\cdot x}$ *and* u *is a bounded measurable function,*

(iii) $v = c \cdot x e^{ia\cdot x} + B(x)$, $u = \pm[(1 - c\cdot x)e^{ia\cdot x} - B(x)]$,

(iv) $v = (e^{c\cdot x} + \lambda B(x))/(1-\lambda^2)$, $u = (\lambda e^{c\cdot x} + B(x))/(1-\lambda^2)$,

(v) $v = (1 - b\cdot x)e^{c\cdot x}$, $u = \pm b \cdot x e^{c\cdot x}$,

(vi) $v = e^{b\cdot x}[\cos(c\cdot x) + \lambda \sin(c\cdot x)]$, $u = \sqrt{\lambda^2+1}\, e^{b\cdot x} \sin(c\cdot x)$,

where $a \in \mathbb{R}^n$, $b, c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$, *and* B *is a bounded measurable function.*

Theorem 4.3. *Let* $u, v \in \mathcal{S}'^{1/2}_{1/2}$ *satisfy* $\|G_3(u,v)\| \le M$. *Then,* u *and* v *satisfy one of the following items:*

(i) $u \equiv 0$ *and* v *is arbitrary,*

(ii) u *and* v *are bounded measurable functions,*

(iii) $u = c\cdot x + r(x)$, $v = \lambda(c\cdot x + r(x)) + 1$,

(iv) $u = \lambda \sin(c\cdot x)$, $v = \cos(c\cdot x) + \lambda \sin(c\cdot x)$,

for some $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$ *and a bounded measurable function* $r(x)$.

Theorem 4.4. *Let* $u, v \in \mathcal{S}'^{1/2}_{1/2}$ *satisfy* $\|G_4(u,v)\| \le M$. *Then,* u *and* v *satisfy one of the following items:*

(i) u *and* v *are bounded measurable functions,*

(ii) $u = \cos(c\cdot x)$, $v = \sin(c\cdot x)$, $c \in \mathbb{C}^n$.

For the proof of the theorems, we employ the n-dimensional heat kernel

$$E_t(x) = (4\pi t)^{-n/2} \exp\left(\frac{-|x|^2}{4t}\right), \quad t > 0. \tag{4.3}$$

In view of (2.3), it is easy to see that for each $t > 0$, E_t belongs to the Gelfand-Shilov space $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^n)$. Thus the convolution $(u * E_t)(x) := \langle u_y, E_t(x-y)\rangle$ is well defined and is a smooth solution of the heat equation $(\partial/\partial_t - \Delta)U = 0$ in $\{(x,t) : x \in \mathbb{R}^n, t > 0\}$ and $(u * E_t)(x) \to u$ as $t \to 0^+$ in the sense of generalized functions for all $u \in \mathcal{S}'^{1/2}_{1/2}$.

Similarly as in the proof of Theorem 3.1, convolving the tensor product $E_t(x)E_s(y)$ of heat kernels and using the semigroup property

$$(E_t * E_s)(x) = E_{t+s}(x) \tag{4.4}$$

of the heat kernels, we can convert the inequalities $\|G_j(u,v)\| \le M$, $j = 1,2,3,4$, to the classical Hyers-Ulam stability problems, respectively,

$$\begin{aligned}
&\left|U(x+y,t+s) - U(x,t)V(y,s) - V(x,t)U(y,s)\right| \le M,\\
&\left|V(x+y,t+s) - V(x,t)V(y,s) + U(x,t)U(y,s)\right| \le M,\\
&\left|U(x-y,t+s) - U(x,t)V(y,s) + V(x,t)U(y,s)\right| \le M,\\
&\left|V(x-y,t+s) - V(x,t)V(y,s) - U(x,t)U(y,s)\right| \le M,
\end{aligned} \tag{4.5}$$

for the smooth functions $U(x,t) = (u * E_t)(x)$, $V(x,t) = (v * E_t)(x)$. Proving the Hyers-Ulam stability problems for the inequalities (4.5) and taking the initial values of U and V as $t \to 0^+$, we get the results. For the complete proofs of the result, we refer the reader to [24].

Remark 4.5. The referee of the paper has recommended the author to consider the Hyers-Ulam stability of the equations, which will be one of the most interesting problems in this field. However, the author has no idea of solving this question yet. Instead, Baker [25] proved the Hyers-Ulam stability of the equation

$$\sum_{j=1}^{l} f_j\left(\alpha_j x + \beta_j y\right) = 0. \tag{4.6}$$

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